MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.901: Astrophysics I **Spring Term 2006**

PROBLEM SET 1

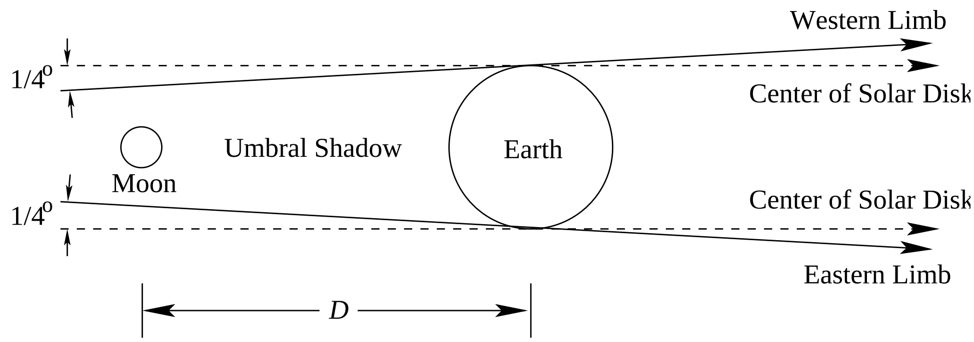
Due: Thursday, February 16 in class

Reading: Hansen, Kawaler, & Trimble, Chapter 1. (Note: This refers to the 2nd edition of 2004, *not* the 1st edition of 1994.) You may also find Carroll & Ostlie, Chapter 7 useful for useful background on stellar binaries.

1. HK&T, Problem 1.1
2. HK&T, Problem 1.2
3. HK&T, Problem 1.3
4. HK&T, Problem 1.6
5. **Historical astronomy: Fundamental length scales.** Accurately determining distances and sizes in astrophysics remains a fundamental and challenging problem to this day. However, a number of surprisingly accurate measurements can be made simply with the naked eye. The ancient Greek philosopher Aristotle (c.384–322 B.C.) was able to deduce that the Earth was spherical from observations like the shape of the Earth's shadow during lunar eclipses and the changing view of stars in the sky during travel from north to south. A number of fundamental length scales in our solar system were also correctly deduced by the ancient Greeks.

   (a) **Size of the Earth.** Eratosthenes (c.276–196 B.C.) deduced the size of the spherical Earth using the following facts: (1) On a particular summer day each year, the Sun penetrated to the bottom of a very deep well (and thus was directly overhead – this point is called the *zenith*) in the town of Syene (now Aswan); (2) On the same day in Alexandria, the Sun at mid-day was 7° south of the zenith; (3) Alexandria was north of Syene by a distance of just under 5000 stadia, where 1 stadium is about 160 meters. Eratosthenes assumed that the Sun is sufficiently distant that its rays can be treated as parallel. Use these facts to reproduce Eratosthenes's inference of the radius of the Earth. (You may use the value of π.) Compare this to the modern value of 6378 km.

   (b) **Size and distance of the Moon.** Aristarchus (c.310–233 B.C.) calculated these using information from a lunar eclipse. Use Timothy Ferris's composite photograph of a lunar eclipse (on the web at `http://web.mit.edu/8.901/images/moon.png`) to make a similar calculation. The diagram below indicates the relevant geometry. You should make use of small angle approximations where appropriate.

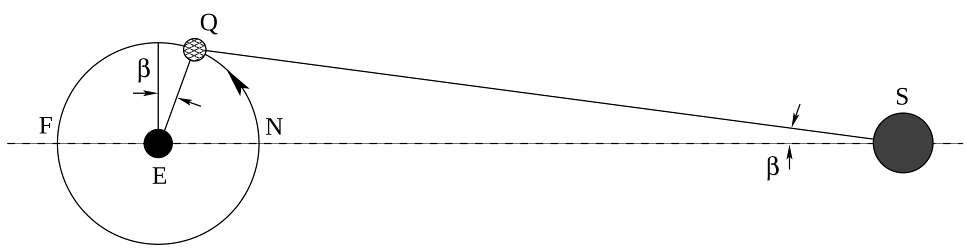


i. Assume that only the darkest part of the Earth's shadow (the *umbra*) corresponds to total eclipse. Estimate the diameter of the circle roughly corresponding to the umbral shadow on the composite image, and also the diameter of one of the lunar images. Note that the center of the shadow does not lie on the line connecting the path of Moon's center — why not?

ii. Compute the radius of the Moon compared to that of the Earth. Be sure to account for the proper geometry of the umbral shadow at the distance of the Moon; for this purpose, you may take the angular diameter of both the Sun and the Moon to be $0.5°$. Estimate the uncertainty in your answer, given the uncertainty in your estimate of the diameter of the umbral shadow. Compare to the modern value of 6378/1738=3.67.

iii. Taking the angular diameter of the Moon to be $0.5°$, calculate the Earth-Moon distance D in terms of the Earth's radius.

(c) **Distance to the Sun.** Aristarchus also estimated this. In the diagram below, the Moon at Q is at first quarter, so that the angle EQS is $90°$. (Note that EQ is *not* perpendicular to ES.) The interval from new Moon (at position N) to first quarter (at Q) is 35 min shorter than that from first quarter to full Moon (at F). Given that the lunar synodic period (the interval between two identical lunar phases) is 29.53 d, estimate the Earth-Sun distance (ES) in terms of the Earth-Moon distance.



6. **The Kepler problem: hyperbolic motion.** In class, we derived the Kepler equation for elliptical motion, $\omega t = \psi - e \sin \psi$, which relates the eccentric anomaly ψ to the time t since pericenter passage, the eccentricity $e \leq 1$, and the orbital angular frequency ω. For *hyperbolic* motion, the analog of the eccentric anomaly is the angle ψ', defined such that the trajectory can be written as $r = a(e \cosh \psi' - 1)$, where $e > 1$ and $a(e-1)$ is the distance of closest approach to the focus (the pericenter distance). Find an analog to the Kepler equation for hyperbolic motion, which specifies the time t since closest approach as a function of ψ'.

7. **Tidal evolution of the Earth-Moon system.** In this problem, you will compute the evolution of the Earth-Moon system by considering the tidal coupling between the Moon's orbit and the Earth's rotation. Angular momentum may be exchanged between these two components but must be conserved overall. Energy may be lost from the system via the heat generated by tidal friction. You should neglect any effects due to the rotation of the Moon.

   (a) Write down expressions for the total energy E and total angular momentum J of the Earth-Moon system. Some useful symbols will be the Earth's angular rotation frequency ω; the Moon's (Keplerian) orbital frequency Ω; the masses of the Earth and Moon, M_e and M_m; the Earth's moment of inertia I; and the mean separation of the Earth and Moon, a.

   (b) Use the equation for J to eliminate ω from the energy equation.

   (c) Show that the energy equation can be cast into the dimensionless form

   $$\epsilon = -\frac{1}{s} + \alpha(j - s^{1/2})^2,$$

   where ϵ is the total energy in units of $(Gm_e m_m/2a_0)$, j is the total angular momentum in units of $(\mu a_0^2 \Omega_0)$, $s = a/a_0$ is the dimensionless separation, μ is the reduced mass, and the subscript "0" refers to values at the present epoch in history.

   (d) Find numerical values for α and j. Look up the masses of the Earth and Moon, and take the Earth's moment of inertia to be $(2/5)M_e R_e^2$ and $a_0 = 3.84 \times 10^5$ km.

   (e) Graph the dimensionless energy equation to find the two values of s for which ϵ is an extremum.

   (f) Find the same two values of s quantitatively by differentiating the energy equation and solving the resulting nonlinear equation numerically by the Newton-Raphson method or some other scheme. Show that $\omega = \Omega$ at these orbital separations. Find the corresponding orbital period of the Moon and rotation period of the Earth.

   (g) Find the difference in energy ΔE between the current epoch and the time in the future when the Earth's rotation and the Moon's orbit will be synchronous.

   (h) Estimate the rate of energy dissipation due to tidal friction by assuming that, twice per day, the top 1-m layer of the oceans is lifted by 1-m and then lowered. Further assume that a few percent of this mechanical energy is dissipated as heat.

   (i) From the energy dissipation rate and total energy ΔE that must be lost in order for the Earth to come into rotational equilibrium with the Moon's orbit, estimate the time (from the current epoch) when this equilibrium configuration will be reached.

8. **Eclipsing binaries.** Assume that two stars are in circular orbits about a mutual center of mass and are separated by a distance a. Assume also that the binary inclination angle is i (defined as the angle between the line-of-sight and the orbital angular momentum vector, with $0° \le i \le 90°$) and that the two stellar radii are R_1 and R_2. Find an expression for the smallest inclination angle that will just barely produce an eclipse.

9. **Ensemble of binaries.** The table below (from a paper by Hinkle et al. 2003) contains the measured orbital parameters (binary period $P_{\rm orb}$ and radial velocity semi-amplitude K_1) for a group of "single-line" spectroscopic binaries (for which the Doppler radial velocity curve for M_1 is measured, but M_2 is not directly observed). You may assume that all the orbits are circular.

Star	$P_{\rm orb}$ (days)	K_1 (km s^{-1})
EG And	482.6	7.3
Z And	759.0	6.7
T CrB	227.6	23.9
BF Cyg	757.2	6.7
V1329 Cyg	956.5	7.8
CI Cyg	853.8	6.7
AG Dra	548.7	5.9
V443 Her	599.4	2.5
BX Mon	1259	4.6
RS Oph	455.7	16.7
V2116 Oph	1042	16.0
AG Peg	818.2	5.4
AX Per	682.1	7.8
FG Ser	633.5	6.9
V343 Ser	450.5	2.7

(a) Recall from class the definition of the binary mass function,

$$f_1 \equiv \frac{(M_2 \sin i)^3}{(M_1 + M_2)^2} = \frac{4\pi^2 (a_1 \sin i)^3}{G P_{\rm orb}^2}.$$

Derive an expression for the mass function in terms of the observables $P_{\rm orb}$ and K_1. What is the physical significance or interpretation of the mass function?

(b) The systems in the table are a group of *symbiotic binaries*, which consist of a red giant star with a hot, degenerate white dwarf companion. The radial velocity measurements are for the red giant component. Typical component masses are $M_1 = 1.5 M_\odot$ for the red giant and $M_2 = 0.56 M_\odot$ for the white dwarf.

A random ensemble of binaries (i.e., one whose orbital angular momentum vectors are isotropically distributed in direction) has a uniform (flat) probability distribution in $\cos i$. Using the measured distribution of mass functions for the systems in the table, determine if these systems are consistent with being a randomly chosen ensemble of symbiotic binaries with typical component masses.

To do this, first plot the *cumulative distribution function* (CDF) for the set of f_1 measurements. [Recall that for an ensemble of measurements $\{x_i\}$, the function CDF(x_i) is equal to the fraction of ensemble values for which $x < x_i$. The CDF thus increases monotonically between zero and unity.] For comparison, also plot the expected CDF of mass functions for a randomly oriented ensemble of binaries with $M_1 = 1.5 M_\odot$ and $M_2 = 0.56 M_\odot$. What can you say about whether the collection of binaries in the table is "typical" of symbiotic binaries in terms of masses? If there are any individual systems that seem to stand out as different, identify them and indicate what you can deduce about them is you assume only that $M_1 = 1.5 M_\odot$ is correct.

10. **Magnitudes (optical intensity).** Because photon counting detectors are governed by Poisson statistics, when we expect to observe N photons from an astrophysical source in a given time interval, the actual number detected will fluctuate by an amount $\pm\sqrt{N}$.

(a) How many photons need to be collected if the apparent magnitude of a star has to be measured to an accuracy of ± 0.02? How long an exposure would be required with a 1-m (diameter) telescope to measure the B magnitude of an $m_B = 20$ star to this accuracy? You may assume that the telescope+detector combination have 100% detection efficiency. You may also neglect the noise background from the sky.

(b) Now consider an attempt to measure the magnitude of a fainter star ($m_B = 24$) with the same accuracy. Assume that the sky background light has brightness 22.5 mag arcsec^{-2} in B, and that atmospheric effects cause the starlight to be spread over a circle with 1 arcsec diameter. How many photons are required to achieve this measurement? How long would this take with a 1-m telescope? What about with a 4-m telescope? Note that MIT is a consortium partner for the two 6.5-m Magellan telescopes in Chile..