1.4   Potential theory

The central result of Newton's gravitational theory is the *inverse square law* for the force between two point masses m_1 and m_2 separated by $\vec{\mathbf{r}} = \vec{\mathbf{x}}_1 - \vec{\mathbf{x}}_2$:

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}) = -\frac{Gm_1 m_2}{r^2}\hat{\mathbf{r}}. \tag{1.32}$$

Since the inverse square law can be added linearly for multiple point masses, we can write the force per unit mass caused by a continuum mass density distribution $\rho(\vec{\mathbf{x}})$ as

$$\vec{\mathbf{F}}(\vec{\mathbf{x}}) = G\int \frac{\vec{\mathbf{x}}' - \vec{\mathbf{x}}}{|\vec{\mathbf{x}}' - \vec{\mathbf{x}}|^3}\rho(\vec{\mathbf{x}}')d^3\vec{x}', \tag{1.33}$$

giving the corresponding gravitational potential

$$\Phi(\vec{\mathbf{x}}) = -G\int \frac{\rho(\vec{\mathbf{x}}')d^3\vec{x}'}{|\vec{\mathbf{x}}' - \vec{\mathbf{x}}|}, \tag{1.34}$$

(Continued on next page.)

from the force-potential relation for conservative forces, $\vec{F} = -\vec{\nabla}\Phi$. Integrating the Poisson equation (which can be derived by taking the divergence of the force equation) and applying the divergence theorem, we can derive Gauss' Theorem for localized mass distributions:

$$\int \vec{\nabla}\Phi \cdot d\vec{S} = 4\pi G \int \rho(\vec{x})d^3\vec{x} = 4\pi G M_{\text{enclosed}}. \tag{1.35}$$

The potential energy of a system (as distinct from the gravitational potential) is defined as the work done against gravitational forces in the assembly of a mass distribution. Calculation of the total work proceeds by examining the change in the gravitational potential $\delta\Phi(\vec{x})$, due to a change in the density $\delta\rho(\vec{x})$. After integration by parts and invocation of the divergence theorem, one finds

$$W = -\frac{1}{8\pi G}\int |\nabla\Phi(\vec{x})|^2 d^3\vec{x} = \frac{1}{2}\int \rho(\vec{x})\Phi(\vec{x})d^3\vec{x}, \tag{1.36}$$

where the minus sign indicates that negative work was done in assembling the mass distribution.

While these results have the advantage of being completely general, in practice the integrals can be solved analytically only for a few special cases. One of the most important simplifications results from spherical symmetry and the first two of *Newton's Theorems*:

a) A body inside a uniform density spherical shell experiences no net gravitational force from that shell.

b) The gravitational force on a body outside such a shell is the same as it would be if the shell's mass were concentrated at its center.

Both theorems can be proved through direct integration of equation (1.33) or by geometric arguments[4]. So for spherical symmetry, the potential can be divided into two pieces, that caused by the interior mass and that caused by the mass exterior to each radius r:

$$\Phi(r) = -4\pi G\left[\underbrace{\frac{1}{r}\int_0^r \rho(r')r'^2 dr'}_{\text{interior}} + \underbrace{\int_r^\infty \rho(r')r' dr'}_{\text{exterior}}\right]. \tag{1.37}$$

As we saw above in section (1.2), one result of spherical symmetry is the relation between the circular orbital velocity v_c and the potential $\Phi(r)$

$$v_c^2(r) = r\frac{d\Phi}{dr}. \tag{1.38}$$

The escape speed is defined as the velocity for which a particle has positive total energy (kinetic plus negative potential) and is thus not gravitationally bound to the system. For spherical symmetry, the escape speed v_e is a function only of radius

$$v_e^2(r) = -2\Phi(r), \tag{1.39}$$

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[4]For proofs of these theorems as well as the more robust *third theorem*, see Binney and Tremaine.

where the convention adopted is that $\Phi(\infty) \to 0$. The Virial Theorem [see below, sec. (1.6)] gives another useful result relating the kinetic energy to the work done by gravity, averaged over an entire orbit:

$$\langle v^2 \rangle_{\text{orbit}} = \langle \vec{\mathbf{r}} \cdot \vec{\nabla}\Phi \rangle_{\text{orbit}}. \tag{1.40}$$

Note that all three of the above equations have a "typical" or "characteristic" velocity squared on the left hand side and the gravitational potential on the right. In two of them the potential is differentiated with respect to length and then multiplied by a length that brings us back to the units of potential – velocity squared.