Massachusetts Institute of Technology
Department of Physics

Physics 8.942 Fall 2001

Problem Set #5
Due in class Tuesday, October 23, 2001.

1. **Thomson optical depth**

   a) Peacock equation (18.38) gives the Thomson optical depth along the line of sight from the observer out to redshift z for a matter-dominated model with no cosmological constant. Derive the correct expression for a general model including matter, vacuum energy, radiation, and curvature. Assume that helium is neutral but hydrogen is partially ionized with ionization fraction x_e. Your answer should depend on z, x_e, Ω_m, Ω_B, Ω_Λ, $a_{\rm eq}$, h, and Y.

   b) Numerically integrate the result of part a) for the flat ΛCDM model with $\Omega_\Lambda = 0.65$, $\Omega_m = 0.35$, $\Omega_B h^2 = 0.02$, and $h = 0.72$, for $a \gg a_{\rm eq}$. At what redshift would the Thomson optical depth equal unity if recombination never occurred, i.e. $x_e = 1$? Compare your answer with that for the OCDM model ($\Omega_k = 0.65$ instead of $\Omega_\Lambda = 0.65$) using Peacock's equation (18.41).

2. **Kompaneets Equation and Inverse Compton Cooling**

   a) Given Peacock equations (12.75)–(12.77), complete the steps to derive the Kompaneets equation (Peacock eq. 12.79) for the modification of a photon spectrum by Compton scattering with a uniform gas of nonrelativistic electrons.

   b) Consider a gas of blackbody radiation at temperature T_γ which is scattered by thermal electrons with temperature $T_e > T_\gamma$. From the Kompaneets equation derive the rate of energy transfer per unit volume between the electrons and photons, $\dot{\rho}_\gamma$. Your result should be proportional to $(T_e - T_\gamma)\rho_\gamma$.

   c) Energy conservation implies that the photon heating is accompanied by electron cooling. Suppose that helium is neutral and that particle collisions maintain thermal equilibrium among electrons, protons, hydrogen atoms, and helium atoms. One can then define an inverse Compton cooling time

$t_{\rm IC} = -\rho_B/\dot{\rho}_e$ where ρ_B is the *thermal* energy density of the baryons (not the rest mass density) and $\dot{\rho}_e = -\dot{\rho}_\gamma$ is the electron energy loss rate. Derive an expression for $t_{\rm IC}$ and evaluate it using the parameters of the CMB and the ΛCDM model. If the gas is fully ionized, the inverse Compton cooling time is shorter than the Hubble time H^{-1} for $z > z_{\rm IC}$. Determine $z_{\rm IC}$.

3. **CMB Spectral Distortions**

a) Starting from the Kompaneets equation, show that the equilibrium solution for the photon occupation number n is a Bose-Einstein distribution with an arbitrary nonzero chemical potential. (The equilibrium solution corresponds to the limit of large Thomson optical depth.)

b) Under what conditions could the CMB have a nonzero μ? (Hint: Compton scattering conserves photon number but not photon energy, as energy is exchanged with electrons.)

c) In the limit of small Thomson optical depth, fill in the steps Peacock omits to derive his equation (12.82) for the Sunyaev-Zel'dovich effect. Check that in the Rayleigh-Jeans regime ($h\nu \ll kT$), the brightness temperature change is $\Delta T/T = -2y$ for $y \ll 1$. Why is the photon temperature *decreased* by scattering from *hotter* electrons?

d) In the limit of small chemical potential, $\mu \ll kT$, linearize the Bose-Einstein phase space density. What is the predicted $\Delta T/T$ in the Rayleigh-Jeans regime? Referring to Figure 9.1 of Peacock, explain qualitatively how the measured CMB brightness temperature versus frequency constrains models of cosmological energy input (Wright et al. 1994, ApJ, 420, 450).

4. **Gravitational perturbations on photons**

Ignoring gravitational radiation and gravitomagnetism, the metric of a weakly-perturbed Robertson-Walker spacetime may be written

$$ds^2 = a^2(\tau)\left[-(1+2\phi)d\tau^2 + (1-2\psi)dl^2\right] \tag{1}$$

where $\phi(x^i,\tau)$ and $\psi(x^i,\tau)$ are small gravitational potentials ($\phi^2 \ll 1$ and $\psi^2 \ll 1$) and dl^2 is the spatial line element of a Robertson-Walker model (e.g. $dl^2 = d\chi^2 + r^2 d\Omega^2$). Throughout this problem we drop terms quadratic in ϕ and ψ.

a) Let E and $p^i = En^i$ be the proper energy and momentum of a photon measured by an observer at fixed spatial coordinates. In other words, the photon 4-momentum is $\vec{P} = E\vec{e}_{\hat{0}} + p^i\vec{e}_{\hat{i}}$ where $\{\vec{e}_{\hat{\mu}}\}$ is an orthonormal basis. From the geodesic equation, show that the photon energy changes with conformal time along its trajectory according to

$$\frac{d\ln(aE)}{d\tau} = -n^i\partial_i\phi + \partial_\tau\psi \ . \tag{2}$$

Identify the terms corresponding to the standard cosmological and static gravitational redshifts. Show also that $dn^i/d\tau$ is independent of E and therefore the photon trajectory is unaffected by its energy.

b) The CMB anisotropy is fully described (in the absence of polarization) by the brightness temperature $T_{\rm br}(x^\mu, E, n^i)$ for photons of proper energy E traveling in direction n^i at spacetime point x^μ. Recall that the brightness temperature is defined by $I_\nu = B_\nu(T_{\rm br})$ where $B_\nu(T)$ is the Planck function. Defining the anisotropy Δ by $T_B = a^{-1}(\tau)T_0(1+\Delta)$, show that equation (2) implies that gravitational perturbations multiply the energy of every photon traveling along a given ray by the same factor and therefore Δ is independent of E.