Problem Set 1: Geodesics and Locally Geodesic (Riemann) Coordinates

Fall 2004

1. Consider the expression fro length of arc (or proper time) between two points:

$$\int_A^B dS = \int_A^B \left(\varepsilon g_{p\sigma} dx^p dx^\sigma\right)^{1/2} = \int_A^B \left(\varepsilon g_{p\sigma} \frac{dx^p}{du}\frac{dx^\sigma}{du}\right)^{1/2} du$$

with $\varepsilon = \pm 1$ to make the S real and positive, and u a parameter along the curve.

(a) To get the equation for extremizing this length, regard it as a dynamical system with u as the time. Show that the equation of motion can be written

$$\frac{d^2 x^\mu}{du^2} + T^\mu_{p\sigma}\frac{dx^p}{du}\frac{dx^\sigma}{du} = \frac{1}{2L}\frac{dL}{du}\frac{dx^\mu}{du}$$

with

$$L \frac{dx^\mu}{du}\frac{dx^\nu}{du}^{1/2}$$

(For now, do not worry about the possibility that $L = 0$.)

(b) Show that this can be interpreted in the following way: if I change the tangent vector along the curve by the covariant derivative, it stays parallel to itself. That is, interpret

$$\nabla \frac{d}{du} = \frac{dx^\mu}{du}\nabla_\mu \frac{dx^p}{du} \propto \frac{dx^p}{du}$$

(c) By choosing u as the solution of

$$du = \left(\varepsilon g_{\mu\nu} dx^\mu dx^\nu\right)^{1/2}$$

i.e. $u = S$ the arc-length, derive the nice form

$$\frac{d^2 x^\mu}{dS} + T^\mu_{p\sigma}\frac{dx^p}{dS}\frac{dx^\sigma}{dS} = 0 \tag{1}$$

(d) Now considering (1) on its own, with $S \to \lambda$, show that

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \text{constant}$$

is implied. This works even when that constant is zero, which defines <u>null</u> geodesics. They solve

$$\frac{d^2 x^\mu}{d\lambda^2} + T^\mu_{p\sigma}\frac{dx^p}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

$$g_{p\sigma}\frac{dx^p}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

and define "light rays."

(e) In Part A, λ is an affine parameter. Show that

$$\lambda' = a\lambda + b$$

with constant a, b is still a good affine parameter. What happens to the equations if one chooses instead an arbitrary parameter?

Comment: These "geodesic" concepts are, at this point, not related to physics. The field Lagrangian contains the physics. The emerge in the "geometric optics"– i.e., high-frequency, short wavelength description of solutions of the wave equation. That will appear in the next problem set.

2. At a given point O, pick a set of non-null basis vectors V^0, V^1, V^2, V^3 for the tangent space. Move by distances x^0 along V^0, x^1 along V^1 (parallel transported), etc. to define the point with parameters (x^0, x^1, x^2, x^3). Show that for small x^μ this is a well-defined coordinate system and that in it $T^\mu_{\nu p} = 0$ at O, which also implies $\frac{\partial g_{\alpha p}}{\partial x^\gamma}$ at O.