15.053/8 **March 5, 2013**

Game Theory 2

Quotes of the Day

New Quotes

From Marilyn Vos Savant's column.

"Say you're in a public library, and a beautiful stranger strikes up a conversation with you. She says: 'Let's show pennies to each other, either heads or tails. If we both show heads, I pay you $3. If we both show tails, I pay you $1. If they don't match, you pay me $2.'

At this point, she is shushed. You think: 'With both heads 1/4 of the time, I get $3. And with both tails 1/4 of the time, I get $1. So 1/2 of the time, I get $4. And with no matches 1/2 of the time, she gets $4. So it's a fair game.' As the game is quiet, you can play in the library.

But should you? Should she?"

submitted by Edward Spellman to Ask Marilyn on 3/31/02

Marilyn Vos Savant has a weekly column in Parade. She has the highest recorded IQ on record.

Payoff (Reward) Matrix for Vos Savant's Game

You (the Row Player)choose heads or tails

The beautiful stranger chooses heads or tails

Beautiful Stranger

	C_1 Heads	C_2 Tails
R_1: Heads	3	-2
R_2: Tails	-2	1

This matrix is the payoff matrix for you, and the beautiful stranger gets the negative.

The Linear Program

	C_1	C_2	
	3	-2	p
	-2	1	1-p

What is the linear program
for the row player?

Key Observation

- When there are only two rows, the only variables for the LP are z and p.
  - One can create a two dimensional drawing of the LP. There is an equivalent but more standard approach.
  - technique: write z as a minimization of two linear functions. Graph z as a function of p.
- A similar approach works for the column player.

Determining the optimal strategy

	B. S. H	B. S. T	Prob
H	3	-2	p
T	-2	1	1-p

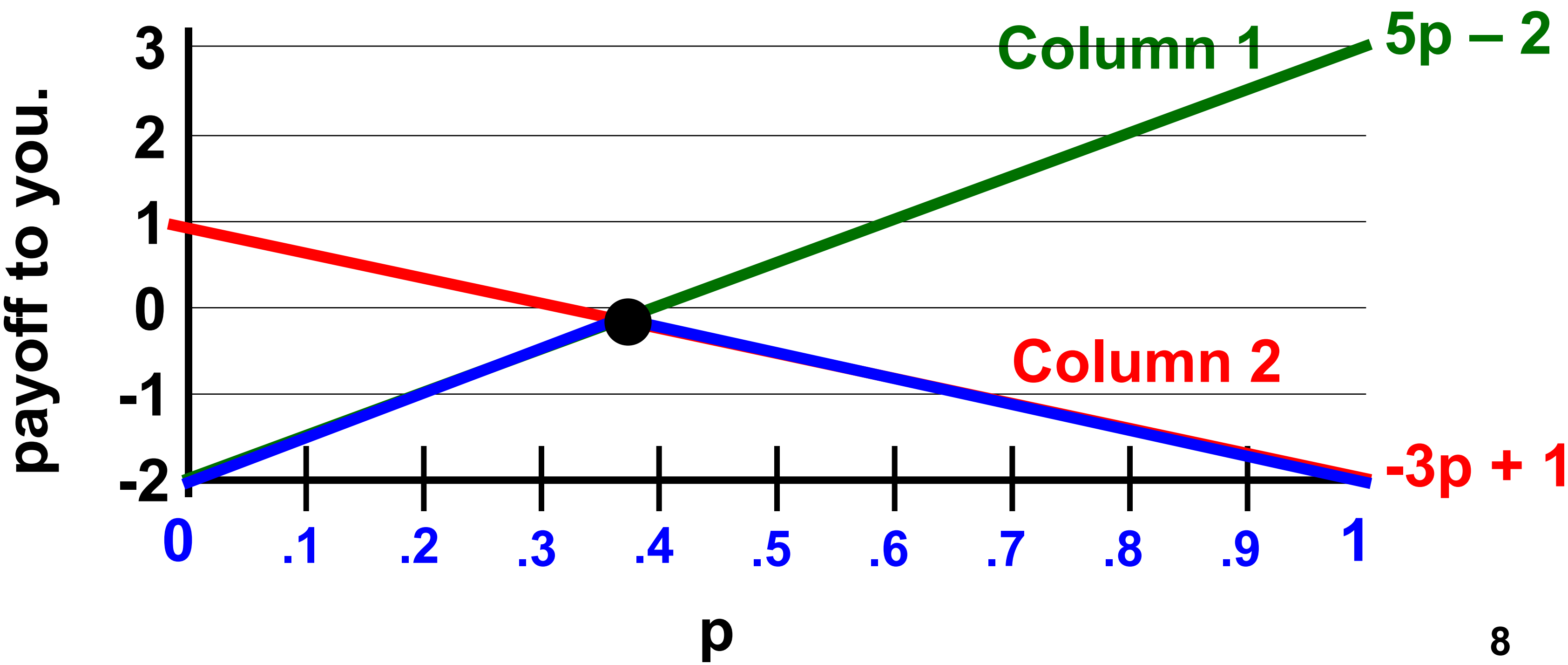
Choose the value of p that maximizes the minimum payoff.

$$\begin{array}{lll} & & \qquad\quad \text{Col 1} \qquad\qquad\qquad \text{Col 2} \\ \text{maximize} & z = & \min\{3p + -2(1-p),\ -2p + 1(1-p)\} \\ \text{subject to} & & 0 \le p \le 1 \end{array}$$

$$\begin{array}{lll} & & \qquad\ \text{Col 1} \qquad \text{Col 2} \\ \text{maximize} & z = & \min\{5p - 2,\ -3p + 1\} \\ \text{subject to} & & 0 \le p \le 1 \end{array}$$

Determining the optimal strategy

		Col 1	Col 2
maximize	$z = \min\{$	$5p - 2,$	$-3p + 1\}$
subject to		$0 \le p \le 1$	



The beautiful stranger's viewpoint

	B. S. H	B. S. T
H	3	-2
T	-2	1
Prob	y	1-y

Choose the value of y that minimizes that maximum payoff.

$$\text{minimize} \quad z = \max\{\underbrace{3y + -2(1-y)}_{\text{Row 1}},\ \underbrace{-2y + 1(1-y)}_{\text{Row 2}}\}$$

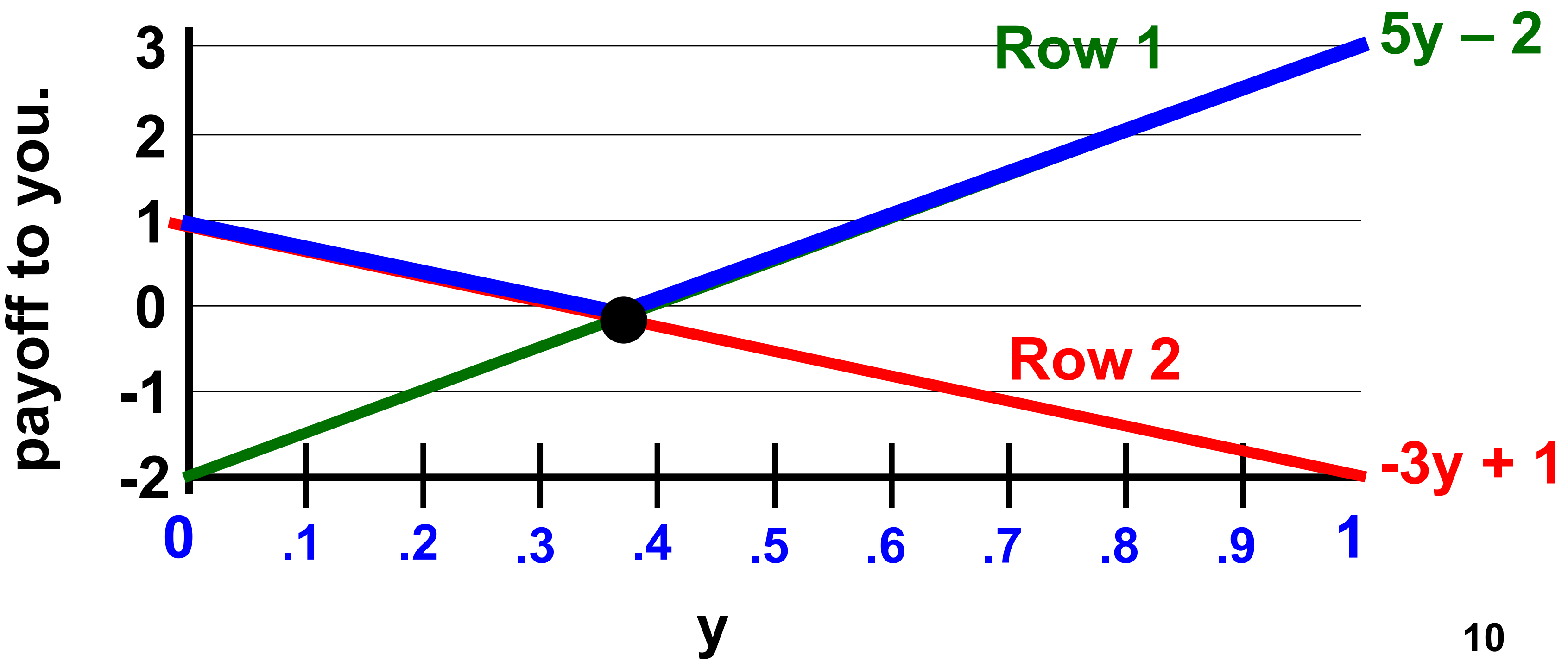
$$\text{subject to} \quad 0 \le y \le 1$$

$$\text{minimize} \quad z = \max\{\underbrace{5y - 2}_{\text{Row 1}},\ \underbrace{-3y + 1}_{\text{Row 2}}\}$$

$$\text{subject to} \quad 0 \le y \le 1$$

The Beautiful Stranger's Viewpoint

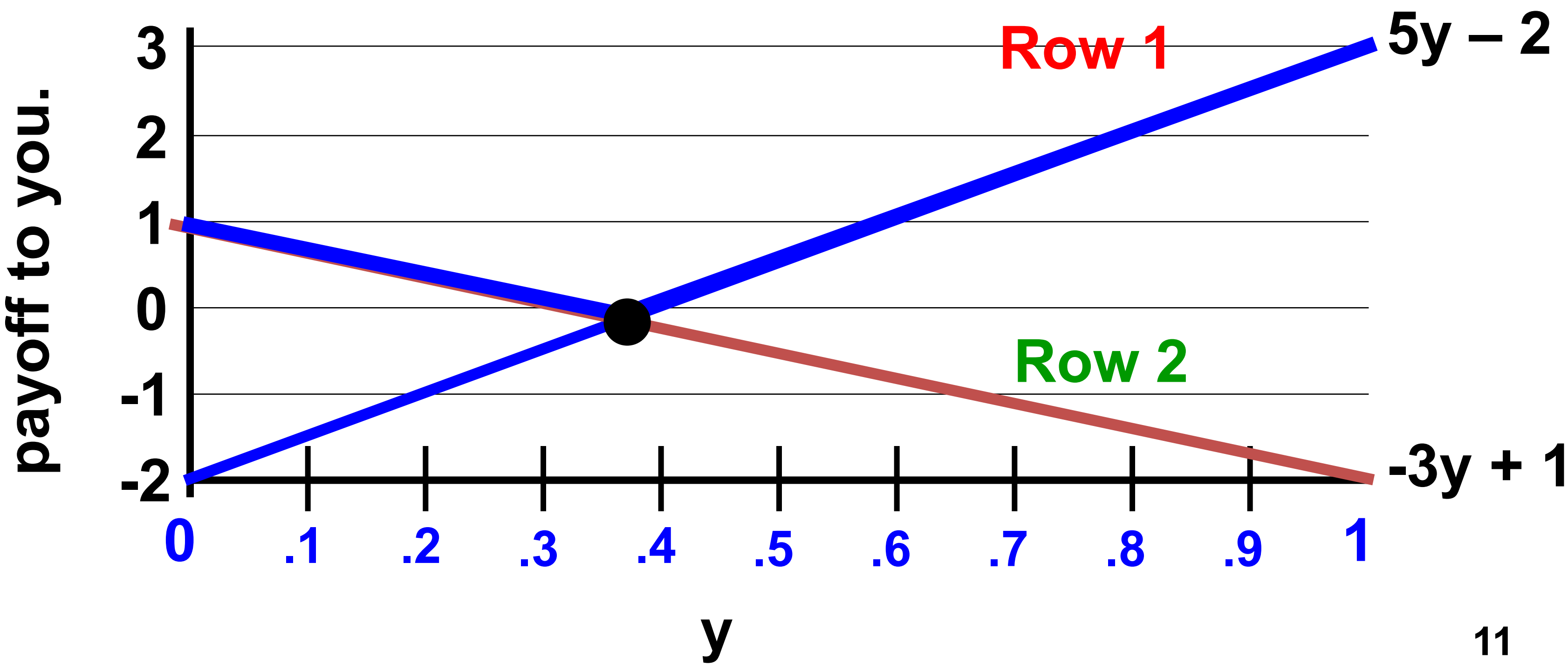
		Row 1 Row 2
minimize	$w = \max\{\overset{\text{Row 1}}{5y - 2}, \overset{\text{Row 2}}{-3y + 1}\}$	
subject to	$0 \leq y \leq 1$	



The payoffs are the same when y = 3/8

optimal payoff to row player = -1/8

Marilyn vos Savant chose y = 1/3, which would given the B.S. a payoff of 0.



A difficulty with mixed strategies in practice

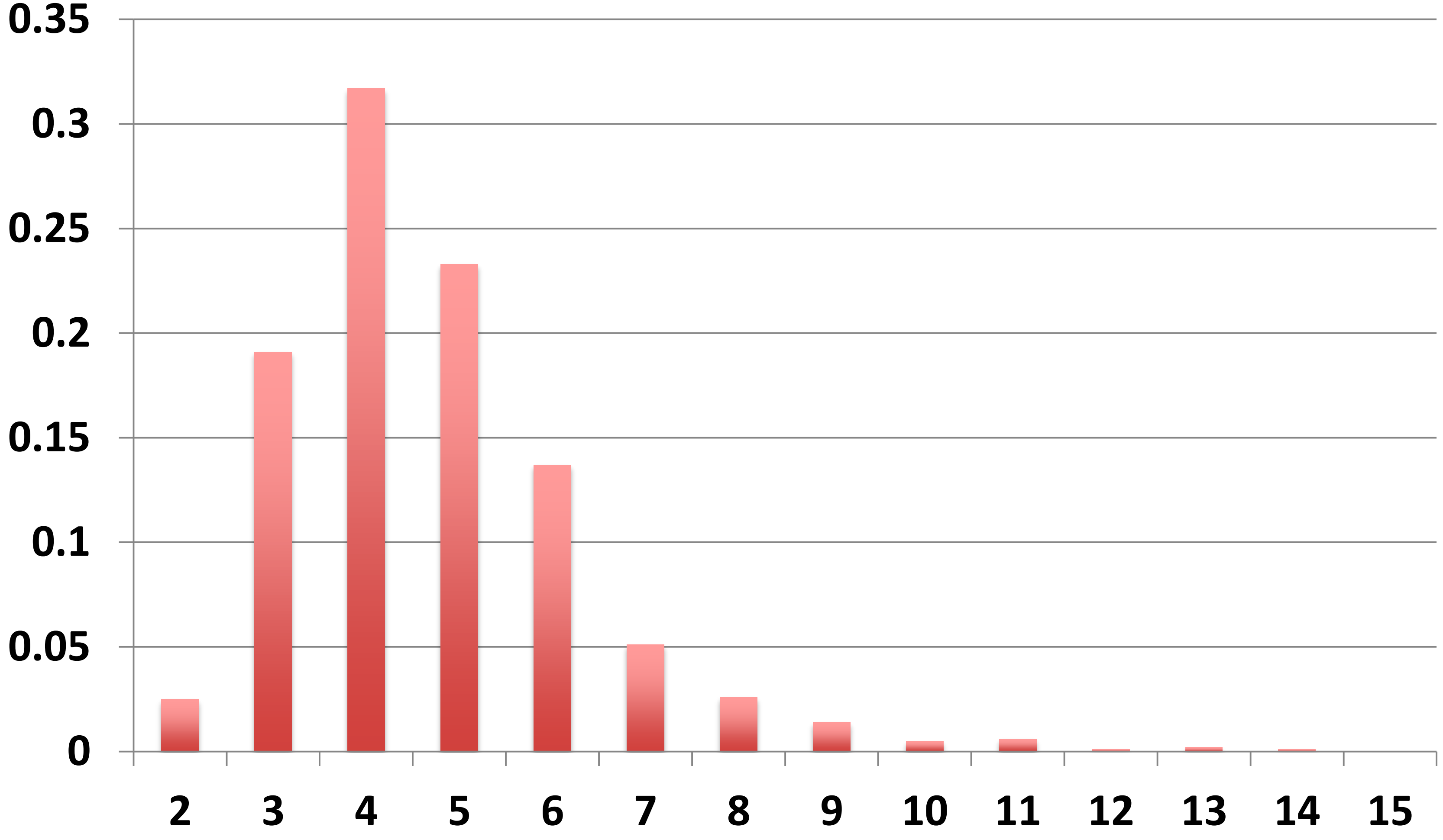
- Do any of you think that you are better than average in playing Rock-Paper-Scissors?

- It is difficult for a person to implement a strategy in which he randomly and independently selects each symbol 1/3 of the time.

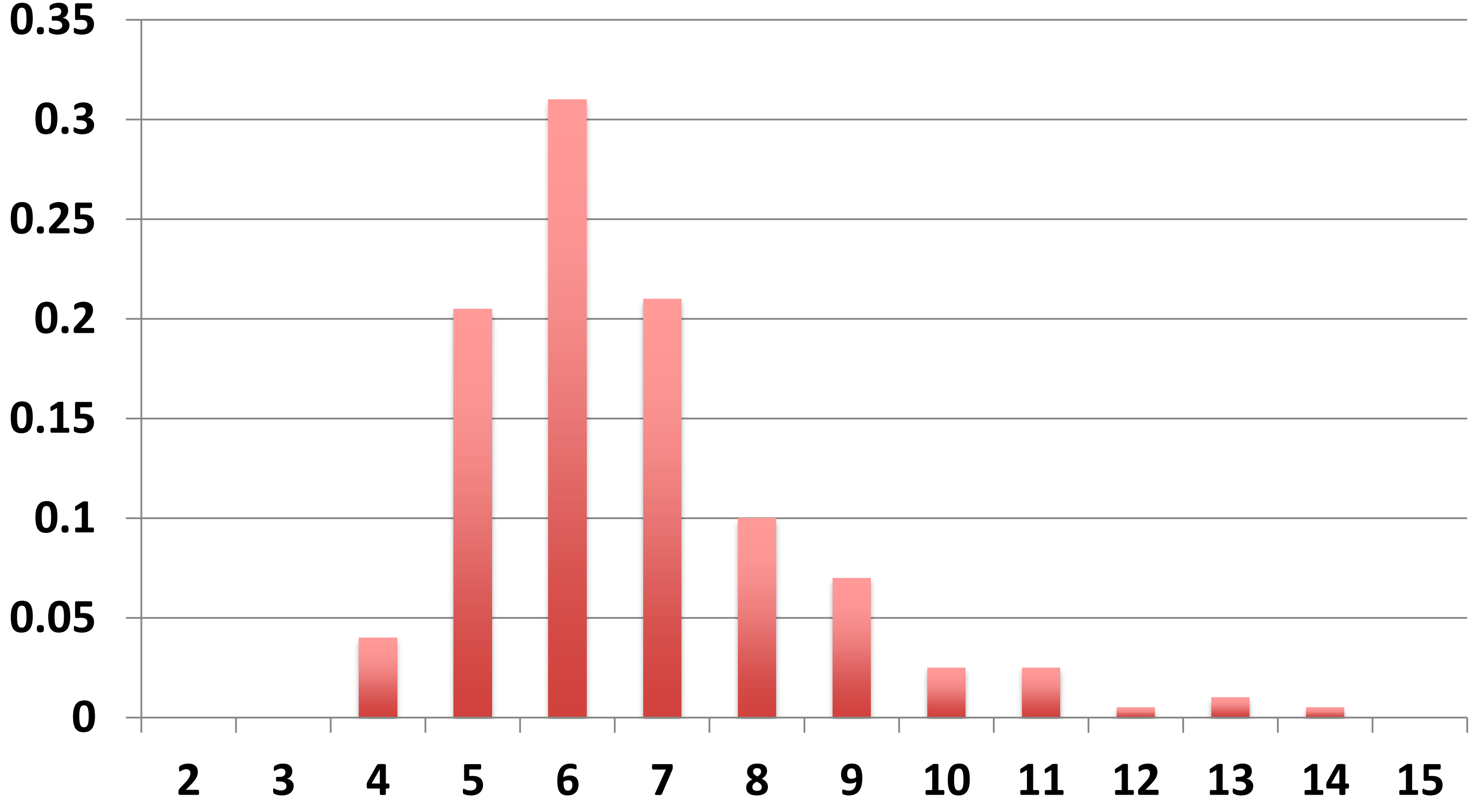
On generating random values

- It is challenging to generate random values.
- Try it yourself.
- Take 80 seconds to generate random 100 values that are \ or ∘. Each should be 50% likely at each step.

∘	\	\																	



Histogram of percentages in 1000 trials



Histogram of 200 trials

Gambler's fallacy

A gambler is playing craps at a Casino.

The probability of winning is 49.3% each time.

The gambler has lost 4 times in a row.

What is the probability of his winning the next time?

In gambler's fallacy, the gambler thinks it is more than 50%.

Count the number of instances that you have \\\\.

- **Ignore cases where it ends a group of 20.**
- **If you have \\\\\, then this is two instances.**

What % of the time is the next symbol \ ?

A. Less than 25%

B. 25% to 40%

C. 40% or higher

D. There were no instances of \\\\.

Mental break

Which answer is True?

- Trivia about MIT Course Numbers

A game involving bluffing (and asymmetric information)

Next: an example based on bluffing in poker.

Version 1 with no bluffing: A coin is tossed.

- **If it comes up heads you win $100.**
- **If it comes up tails, you lose $100.**

Suppose the game is played a lot (say 100 times).

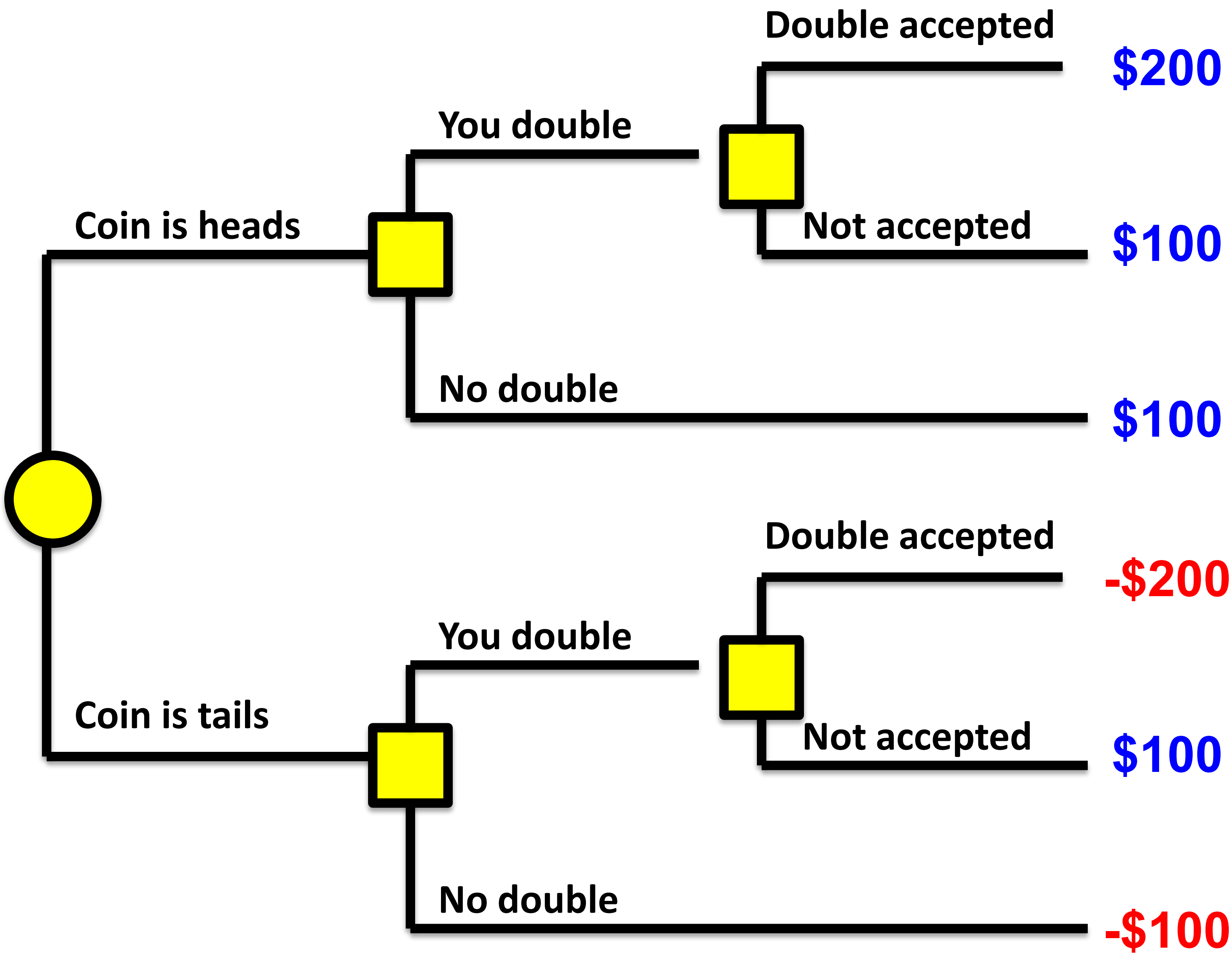
- **On average, you will break even.**
- **Expected value (1/2 × $100 + ½ × -$100.)**

Coin tossing with "doubling the bet"

A coin is tossed.

- **You are permitted to see the outcome.**
- **Your opponent does not see the outcome.**
- **You may double the bet from $100 to $200.**
- **If you double the bet, your opponent may accept the doubled bet or turn it down. If your opponent turns it down, you win $100. If your opponent accepts the double, then**
  - **If it is heads, you win $200**
  - **If it is tails, you lose $200.**

The six possible outcomes



Is bluffing a good idea?

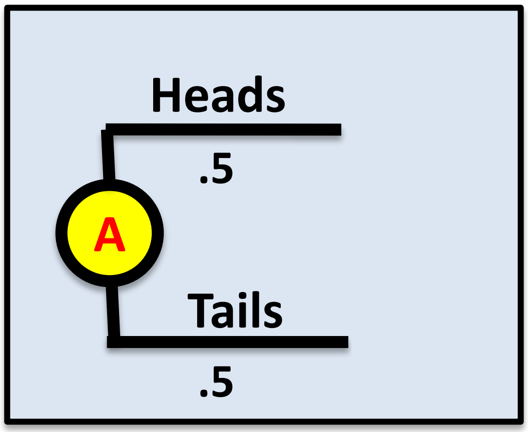
Should we always double when a heads appears?

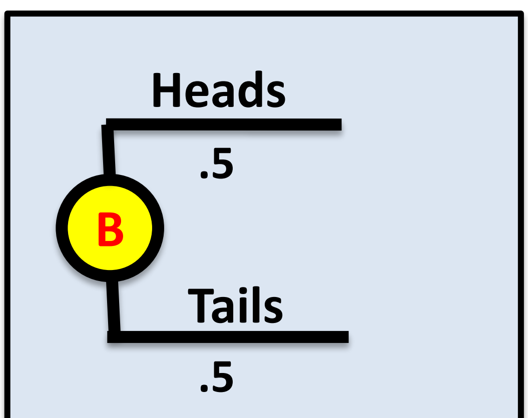
What is a good strategy for when to double the bet if a tails appears?

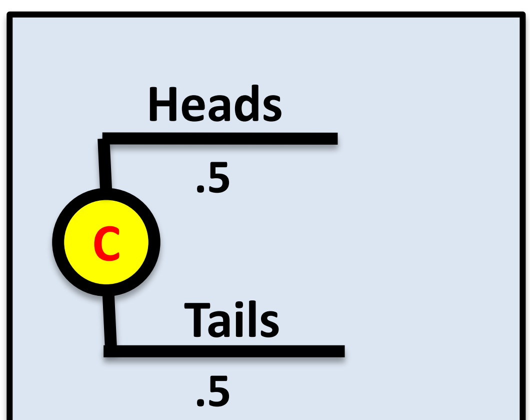
Can we have two volunteers to play 5 rounds of this game?
(no actual money is involved).

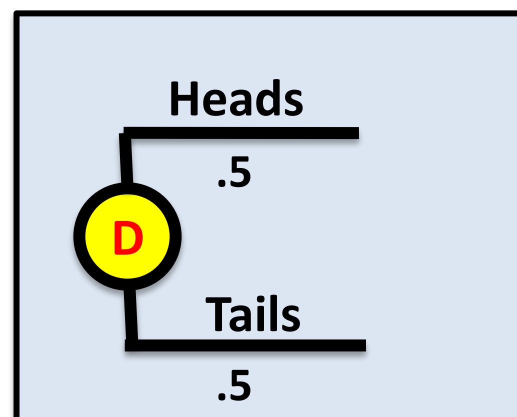
Your opponent

	Accept a double	Do not accept a double.
Double the bet with H or T	A	B
Double with H, not with T	C	D





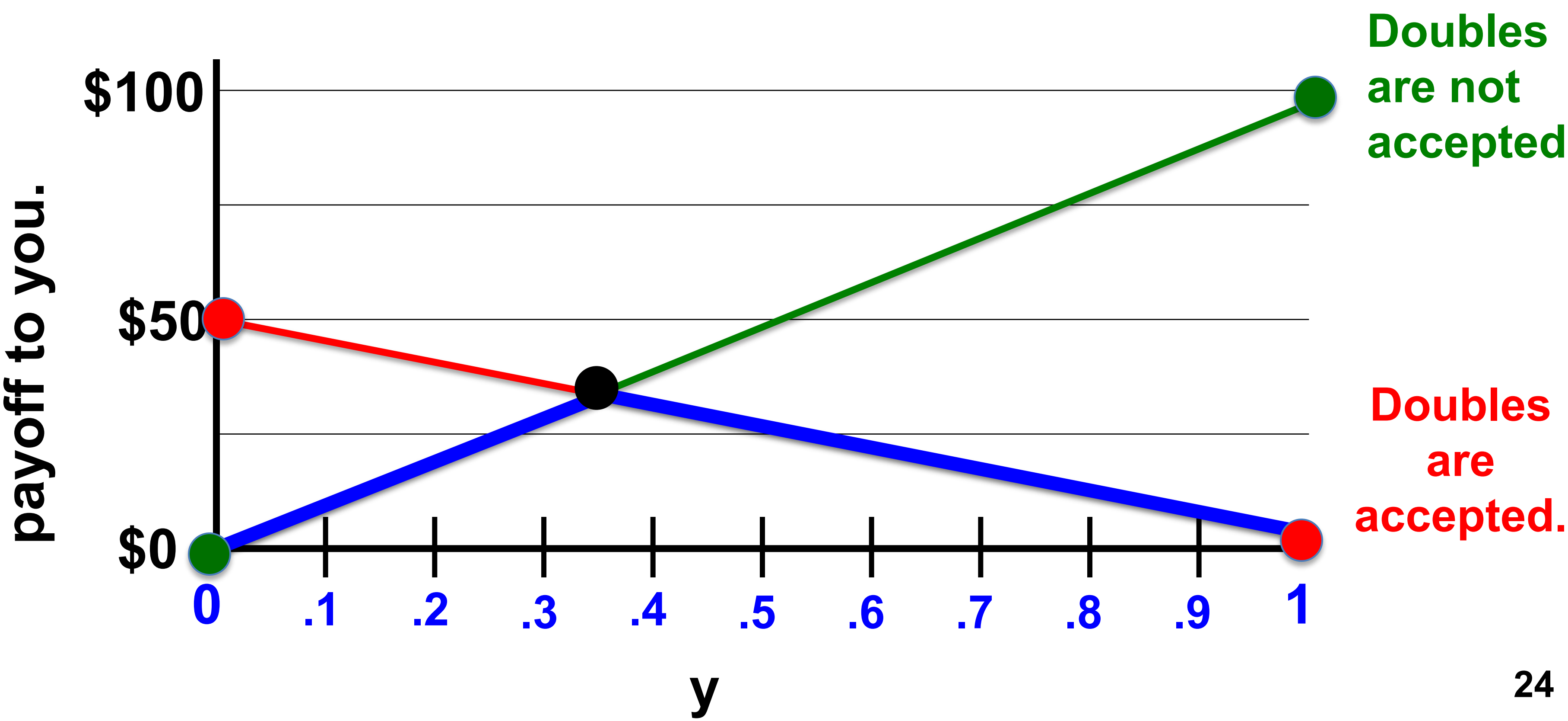




How frequently should you bluff?

accept doubles	do not accept
A $0	B $100
C $50	D $0

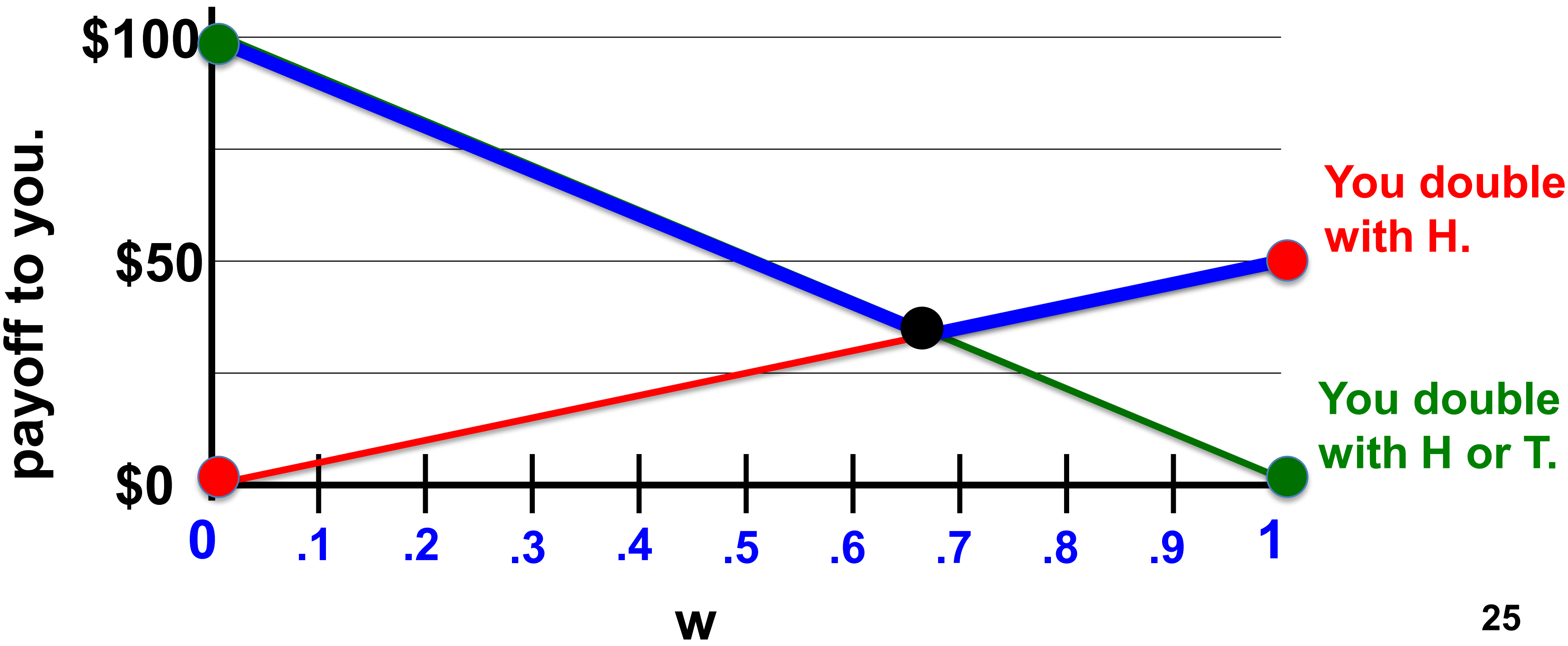
Let y be the probability of doubling when the coin is a tails.



How frequently should your opponent accept doubles?

Let w be the probability of accepting a double, if it is offered.

double with H or T	A $0	B $100
double with H	C $50	D $0



A comment on bluffing

- **With no bluffing, your opponent knows exactly when you have a winning hand.**
- **If bluffing is done optimally as part of a mixed strategy, it guarantees an improved performance regardless of whether the bluffs are accepted or not.**
- **In practice, bluffing works only if your opponent cannot tell if you are bluffing.**
- **The optimal proportion of time to bluff depends on the situation (type of game, number of players, information about the opponents, information about probabilities, etc).**

Optimization under uncertainty

- **When we develop a linear or integer program, it is very rare that we know the data with certainty.**

- **e.g. Recall mc^2 from lecture on sensitivity analysis**
  - **profits from selling A, B, C, D, E**
  - **supplies of chips and drives**
  - **demand forecasts**

- **Approaches for dealing with uncertainty**
  - **sensitivity analysis and running of lots of scenarios**
  - **modeling uncertainty using probability distributions**
  - **robust optimization**

Robust optimization example

- Example: you are on your morning commute, and you have three choices of how to get to work.
- Suppose that for every day, one of four possible scenarios occur.

	Good day	Bad for highway	Bad for local roads	Bad for MBTA
Highway	30	80	30	30
Local roads	40	40	90	40
MBTA	50	50	50	75

In robust optimization, one chooses the decision that is best in the worst case. (One assumes that the worst scenario for you always occurs.)

Robust optimization example

	Good day	Bad for highway	Bad for local roads	Bad for MBTA
Highway	30	80	30	30
Local roads	40	40	90	40
MBTA	50	50	50	75

What is the best choice of commuting in this example if one adopts the robust optimization approach?

1. Highway
2. Local roads
3. MBTA

Robust optimization with mixed strategies

	Good day	Bad for highway	Bad for local roads	Bad for MBTA
Highway	30	80	30	30
Local roads	40	40	90	40
MBTA	50	50	50	75

But perhaps we should not be so pessimistic. Suppose we permitted mixed strategies, and we minimized the average commute time that we can guarantee.

Note: we average with respect to our choices. There are no probabilities for the columns.

Robust optimization with mixed strategies

	Good day	Bad for highway	Bad for local roads	Bad for MBTA
Highway	30	80	30	30
Local roads	40	40	90	40
MBTA	50	50	50	75

Suppose that one finds the optimal mixed strategy. What do you guess is the method with the highest probability?

1. Highway
2. Local roads
3. MBTA
4. All probabilities are 1/3.

An optimal mixed strategy

	Prob	Good day	Bad for highway	Bad for local roads	Bad for MBTA
Highway		30	80	30	30
Local roads		40	40	90	40
MBTA		50	50	50	75
Average					

	Good day	Bad for highway	Bad for local roads	Bad for MBTA	
Probability					
Highway	30	80	30	30	
Local roads	40	40	90	40	
MBTA	50	50	50	75	

Summary

- 2-person 0-sum game theory
  - mixed strategies
  - guaranteed average performance
- Applications to games
- Applications to optimization under uncertainty
- Game theory is an important topic in economics, operations research, and computer science.