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Converting a Linear Program to Standard Form

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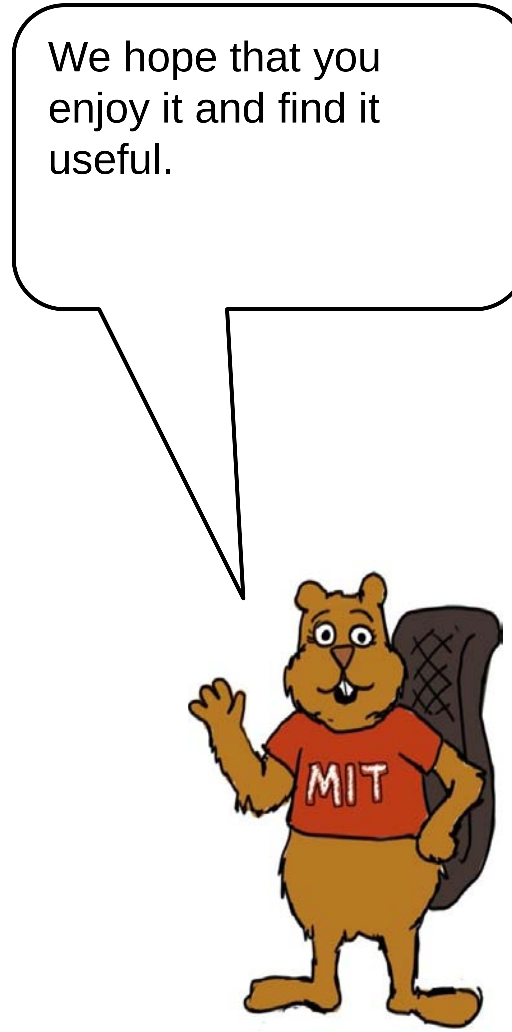
Amit, an MIT Beaver



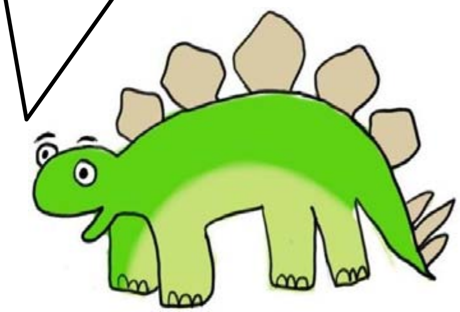
Mita, an MIT Beaver

Linear Programs in Standard Form

We say that a linear program is *in standard form* if the following are all true:

1. Non-negativity constraints for all variables.
2. All remaining constraints are expressed as equality constraints.
3. The right hand side vector, b, is non-negative.

I think it is really cool that when Ella speaks, some of her words are in red, and some are underlined. I wish I could do that.



Stan

Ella

An LP not in Standard Form

max $z = 3x_1 + 2x_2 - x_3 + x_4$

$x_1 + 2x_2 + x_3 - x_4 \leq 5$; **not equality**

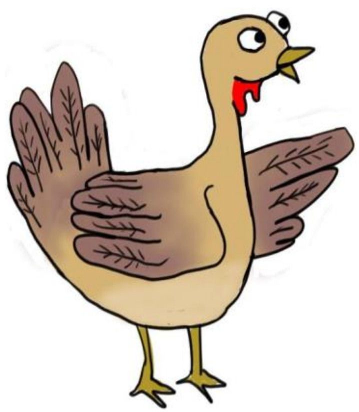
$-2x_1 - 4x_2 + x_3 + x_4 \leq -1$; **not equality, and negative RHS**

$x_1 \geq 0, x_2 \leq 0$ **x_2 is required to be nonpositive; x_3 and x_4 may be positive or negative.**

Why do students need to know how to convert a linear program to standard form? What's so special about standard form?

The main reason that we care about standard form is that this form is the starting point for the simplex method, which is the primary method for solving linear programs. Students will learn about the simplex algorithm very soon.

In addition, it is good practice for students to think about transformations, which is one of the key techniques used in mathematical modeling.



Tom

Next we will show some techniques (or tricks) for transforming an LP into standard form.

Converting a “≤” constraint into standard form

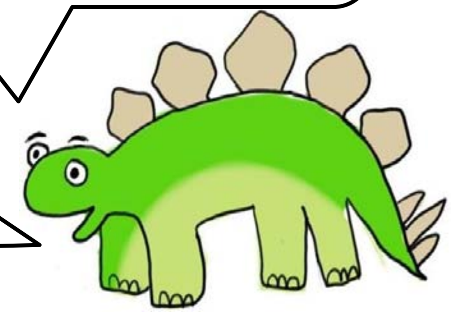
We first consider a simple inequality constraint. The first inequality constraint of the previous LP is

$$x_1 + 2x_2 + x_3 - x_4 \leq 5$$

To convert a “≤” constraint to an equality, add a slack variable. In this case, the inequality constraint becomes the equality constraint:

$$x_1 + 2x_2 + x_3 - x_4 + s_1 = 5.$$

We also require that the slack variable is non-negative. That is $s_1 \geq 0$.



Nooz can speak in red, just like Ella. **How does he do that?**

Wow! I just spoke in boldface. Cool!

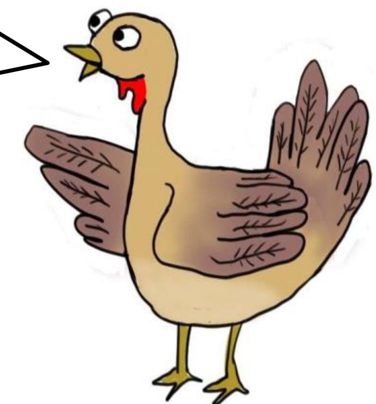
s_1 is called a *slack variable*, which measures the amount of “unused resource.” Note that

$s_1 = 5 - x_1 - 2x_2 - x_3 + x_4$.

Converting a "≥" constraint into standard form, and converting inequalities with a negative RHS.

We next consider the constraint
$-2x_1 - 4x_2 + x_3 + x_4 \leq -1$

I know how to do that one. Just add a slack variable, like we did on the last slide.



Nice try, Tom, but incorrect. First we have to multiply the inequality by -1 in order to obtain a positive RHS. Then we get
$2x_1 + 4x_2 - x_3 - x_4 \geq 1.$
Then we add a surplus variable and get
$2x_1 + 4x_2 - x_3 - x_4 - s_2 = 1.$

s_2 is called a *surplus variable*, which measures the amount by which the LHS exceeds the RHS. Note that

$s_2 = 2x_1 + 4x_2 - x_3 - x_4 - 1$

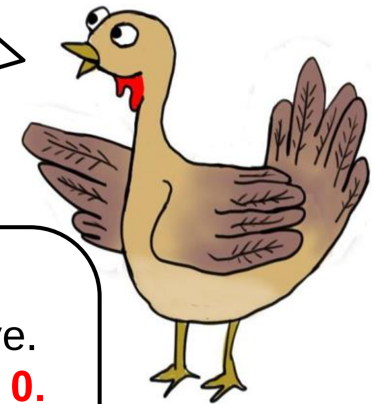
To convert a "≤" constraint to an equality, add a slack variable.

To convert a "≥" constraint to an equality, add a surplus variable.

Getting Rid of Negative Variables

Next, I'll show you how to transform the constraint constraint: $x_2 \leq 0$ into standard form.

Can't we just write: $x_2 + s_3 = 0$ and $s_3 \geq 0$?



Tom, what you wrote is correct, but it doesn't help. Standard form requires all variables to be non-negative. But after your proposed change, it is still true that $x_2 \leq 0$. The solution in this case is a substitution of variables. We let $y_2 = -x_2$. Then $y_2 \geq 0$. And we substitute $-y_2$ for x_2 wherever x_2 appears in the LP. The resulting LP is given below. (after you click.)

$$
\begin{aligned}
\max \quad z = \; & 3x_1 - 2y_2 - x_3 + x_4 \\
& x_1 - 2y_2 + x_3 - x_4 + s_1 = 5; \\
& 2x_1 - 4y_2 - x_3 - x_4 - s_2 = 1; \\
& x_1 \geq 0, \; y_2 \geq 0 \quad s_1 \geq 0, \; s_2 \geq 0
\end{aligned}
$$

Getting Rid of Variables that are Unconstrained in Sign

Next, we' ll show you how to get rid of a variable that is unconstrained in sign. That is, it can be positive or negative.

Actually, we' ll show you two ways. The first way is substitution. For example, x_3 below is **unconstrained in sign**. (Sometimes we call this a **free** variable.) Notice that the second constraint can be rewritten as:
$\mathbf{x_3 = 2x_1 - 4y_2 - x_4 - s_2 - 1}$.

Now substitute $2x_1 - 4y_2 - x_4 - s_2 - 1$ for x_3 into the current linear program. Notice that you get an equivalent linear program without x_3. You can see it on the next slide.

$$\max \quad z = 3x_1 - 2y_2 - x_3 + x_4$$

$$x_1 - 2y_2 + x_3 - x_4 + s_1 = 5;$$

$$2x_1 - 4y_2 - x_3 - x_4 - s_2 = 1;$$

$$x_1 \geq 0,\ y_2 \geq 0, \quad s_1 \geq 0,\ s_2 \geq 0$$

Getting Rid of Free Variables by Substitution

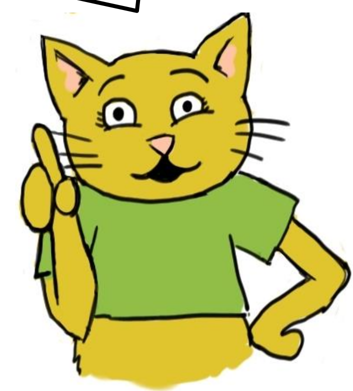
When we substitute $2x_1 - 4y_2 - x_4 - s_2 - 1$ for x_3 here is what we get. (Click now.)

The variable x_4 is also unconstrained in sign. You can substitute for it as well. After this substitution, all that will remain is an objective function and non-negativity constraints for x_1, y_2, s_1 and s_2.

This trick only works for variables that are unconstrained in sign. If you tried eliminating x_1 instead of x_3 by substitution, the optimal solution for the resulting LP would not necessarily satisfy the original constraint $x_1 \geq 0$. So eliminating x_1 in this manner would not create an equivalent LP.

$$\begin{aligned} \max \quad z = \quad & 3x_1 - 2y_2 - x_3 + x_4 \\ & x_1 - 2y_2 + x_3 - x_4 + s_1 = 5 ; \\ & 2x_1 - 4y_2 - x_3 - x_4 - s_2 = 1; \\ & x_1 \geq 0,\ y_2 \geq 0, \quad s_1 \geq 0,\ s_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad z = \quad & 1x_1 + 2y_2 + 2x_4 + s_2 + 1 \\ & 3x_1 - 6y_2 - 2x_4 + s_1 + s_2 = 5 ; \\ & x_1 \geq 0,\ y_2 \geq 0, \quad s_1 \geq 0,\ s_2 \geq 0 \end{aligned}$$



Cathy

Getting Rid of Free Variables: Version 2

There is an even simpler way of getting rid of free variables. We replace a free variable by the difference of two non-negative variables. For example, we replace x_3 by $y_3 – w_3$, and require y_3 and w_3 to be non-negative. (Click now.) You can then substitute $y_4 – w_4$ for x_4.

After solving this new linear program, we can find the solution to the original linear program. For example,
$x_3 = y_3 – w_3$ and
$x_4 = y_4 – w_4$.

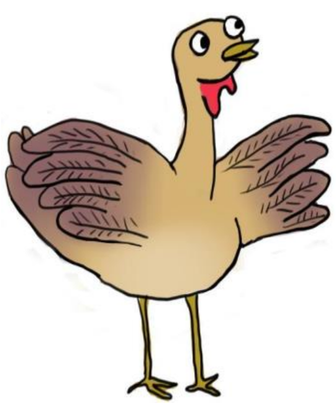
$$
\begin{aligned}
\max \quad z = \; & 3x_1 - 2y_2 - x_3 + x_4 \\
& x_1 - 2y_2 + x_3 - x_4 + s_1 = 5; \\
& 2x_1 - 4y_2 - x_3 - x_4 - s_2 = 1; \\
& x_1 \geq 0, \; y_2 \geq 0, \quad s_1 \geq 0, \; s_2 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\max \quad z = \; & 3x_1 - 2y_2 - y_3 + w_3 + x_4 \\
& x_1 - 2y_2 + y_3 - w_3 - x_4 + s_1 = 5; \\
& 2x_1 - 4y_2 - y_3 + w_3 - x_4 - s_2 = 1; \\
& x_1 \geq 0, \; y_2 \geq 0, \; y_3 \geq 0, \; w_3 \geq 0, \; s_1 \geq 0, \; s_2 \geq 0
\end{aligned}
$$

Getting Rid of Free Variables: Version 2

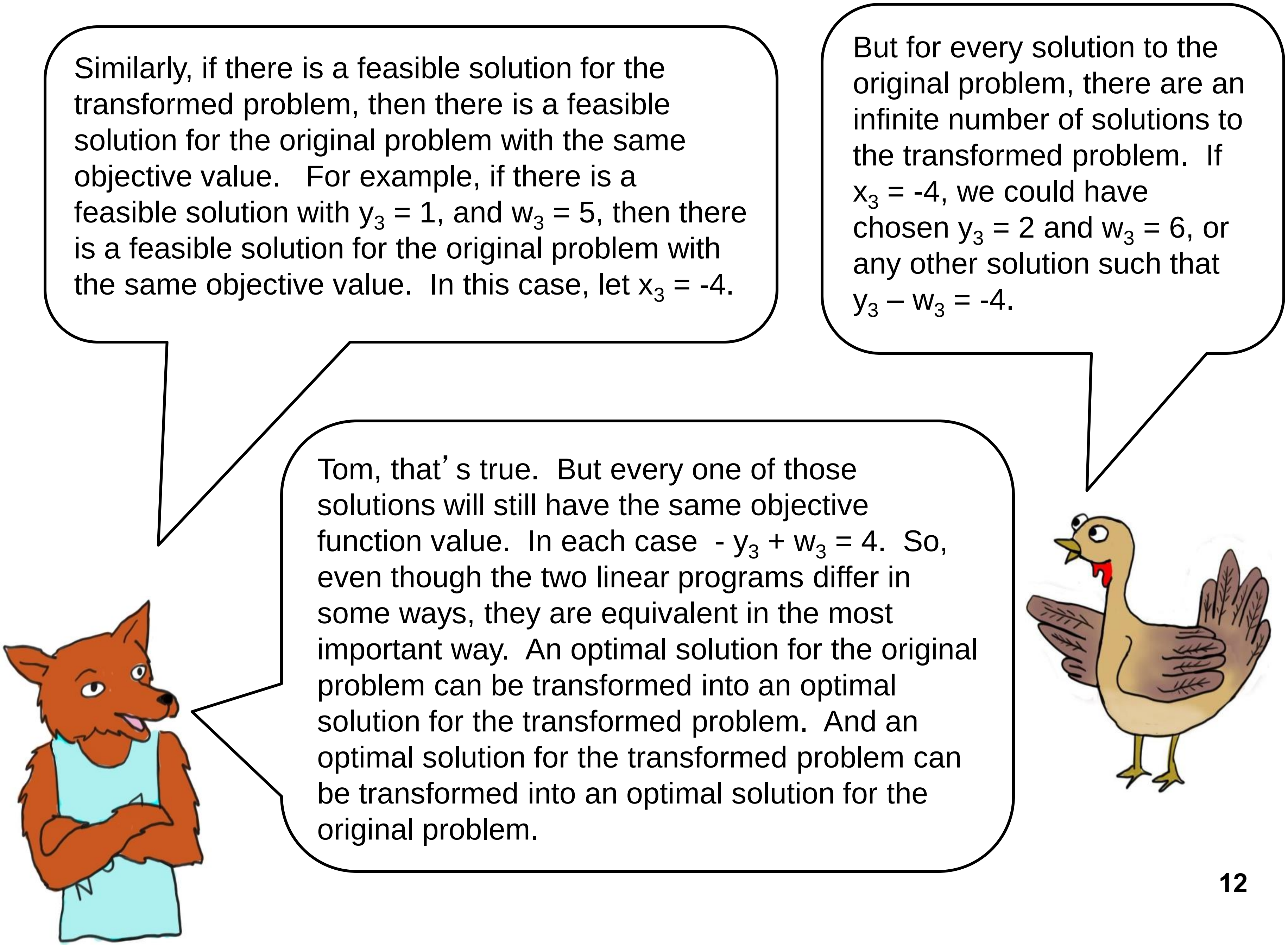
This doesn't make sense to me. Before we had a variable x_3, and now we have two variables y_3 and w_3. How can two variables be the same as a single variable?

It depends on what you mean by "the same." Here is what we mean. For every solution to the original LP, there is a solution to the transformed LP with the same objective value. For example, if there is a feasible solution with x_3 = -4, then there is a feasible solution to the transformed problem with the same objective value. In this case, let y_3 = 0 and w_3 = 4.



$$\begin{aligned} \max \quad z = \; & 3x_1 - 2y_2 - x_3 + x_4 \\ & x_1 - 2y_2 + x_3 - x_4 + s_1 = 5; \\ & 2x_1 - 4y_2 - x_3 - x_4 - s_2 = 1; \\ & x_1 \geq 0,\; y_2 \geq 0,\; s_1 \geq 0,\; s_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad z = \; & 3x_1 - 2y_2 - y_3 + w_3 + x_4 \\ & x_1 - 2y_2 + y_3 - w_3 - x_4 + s_1 = 5; \\ & 2x_1 - 4y_2 - y_3 + w_3 - x_4 - s_2 = 1; \\ & x_1 \geq 0,\; y_2 \geq 0,\; y_3 \geq 0,\; w_3 \geq 0,\; s_1 \geq 0,\; s_2 \geq 0 \end{aligned}$$


Similarly, if there is a feasible solution for the transformed problem, then there is a feasible solution for the original problem with the same objective value. For example, if there is a feasible solution with $y_3 = 1$, and $w_3 = 5$, then there is a feasible solution for the original problem with the same objective value. In this case, let $x_3 = -4$.
But for every solution to the original problem, there are an infinite number of solutions to the transformed problem. If $x_3 = -4$, we could have chosen $y_3 = 2$ and $w_3 = 6$, or any other solution such that $y_3 – w_3 = -4$.
Tom, that's true. But every one of those solutions will still have the same objective function value. In each case $- y_3 + w_3 = 4$. So, even though the two linear programs differ in some ways, they are equivalent in the most important way. An optimal solution for the original problem can be transformed into an optimal solution for the transformed problem. And an optimal solution for the transformed problem can be transformed into an optimal solution for the original problem.

Transforming Max to Min

We still have one last pair of transformations. We will show you how to transform a maximization problem into a minimization problem, and how to transform a minimization problem into a maximization problem. This is not part of converting to standard form, but it is still useful.

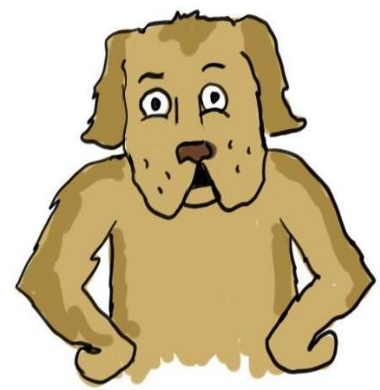
We illustrate with our original linear program, which is given below. All you need to know is that if we maximize z, then we are minimizing –z, and vice versa. See if you can use this hint to figure out how to change the problem to a minimization problem. Then click to see if you are right.

$$\min \quad -z = -3x_1 - 2x_2 + x_3 - x_4$$

$$x_1 + 2x_2 + x_3 - x_4 \le 5;$$

$$-2x_1 - 4x_2 + x_3 + x_4 \le -1;$$

$$x_1 \ge 0,\ x_2 \le 0$$



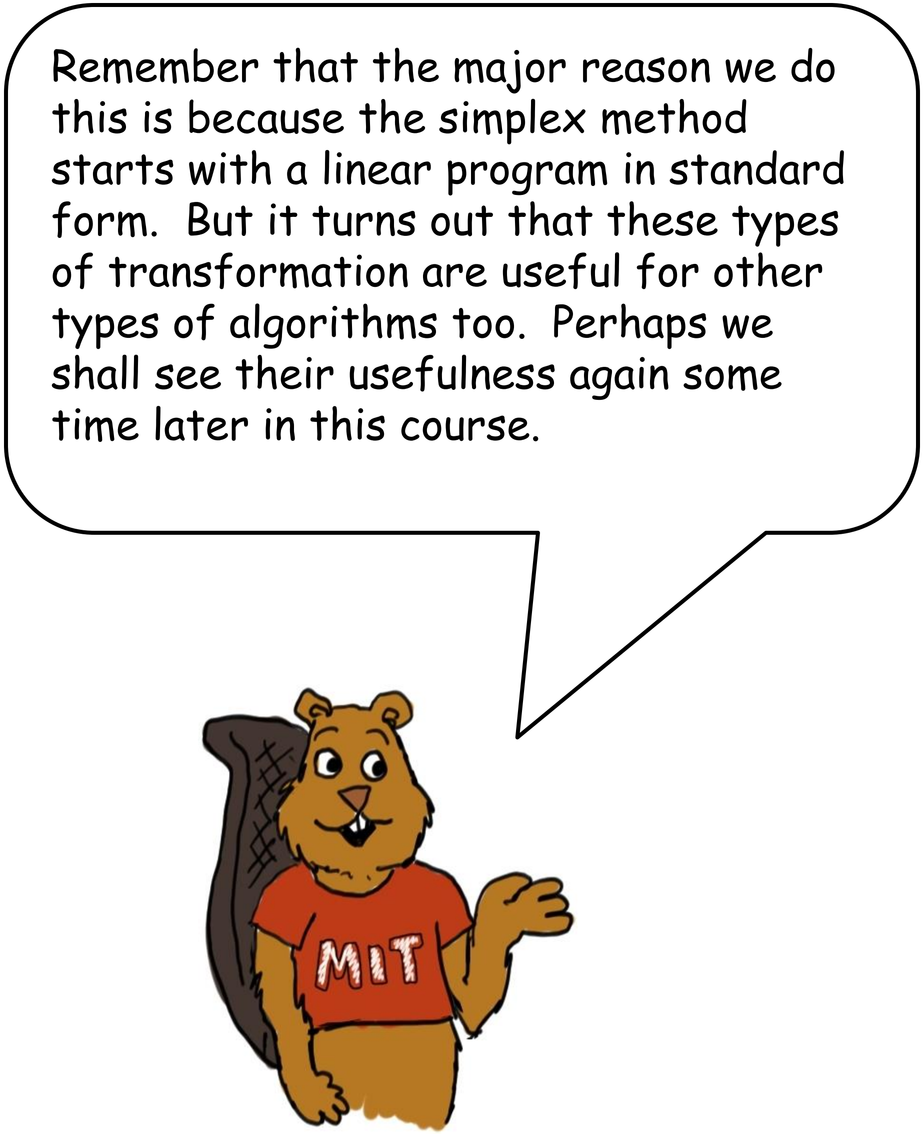
McGraph

$$
\begin{aligned}
\min \quad z = x_1 - x_2 + x_3 & \\
x_1 + 2x_2 - x_3 &\leq 3 \\
-x_1 + x_2 + x_3 &\geq 2 \\
x_1 - x_2 \quad\quad &= 10 \\
x_1 \geq 0,\ x_2 \leq 0 &
\end{aligned}
$$

Here is an example for which you can test out these techniques. Consider the LP to the right. See if you can transform it to standard form, with maximization instead of minimization.

To see the new variables, click once. To see the transformed problem, click again.

Last Slide


Remember that the major reason we do this is because the simplex method starts with a linear program in standard form. But it turns out that these types of transformation are useful for other types of algorithms too. Perhaps we shall see their usefulness again some time later in this course.
MIT


Well, that concludes this tutorial on transforming a linear program into standard form. We hope to see you again soon.
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