

**15.057 Systems Optimization
Spring 2003
Mid Term Exam**

Instructions

You have 90 minutes to complete the exam from the time you first begin it. Watch your time. Note too the different points assigned to each question. Don't get bogged down on a problem worth few points until you have answered the problems that are worth more. If you are having difficulty with a question, you might wish to pass over it and return to it later if you have enough time at the end of the exam.

The exam is open book: you may use any notes or text material.

The exam is to be done individually, without collaboration with anyone else.

You may use a computer to view your notes, but you may NOT use it to formulate or solve models.

Good luck!!!

Problem 1 (25 points)

Consider our model for scheduling postal workers to 5 day shifts:

$$\begin{aligned} \text{Minimize } z &= MF + TS + WSu + ThM + FT + SW + SuTh \\ \text{Subject to } & MF + ThM + FT + SW + SuTh \geq 17 \\ & MF + TS + FT + SW + SuTh \geq 13 \\ & MF + TS + WSu + SW + SuTh \geq 15 \\ & MF + TS + WSu + ThM + SuTh \geq 19 \\ & MF + TS + WSu + ThM + FT \geq 14 \\ & TS + WSu + ThM + FT + SW \geq 16 \\ & WSu + ThM + FT + SW + SuTh \geq 11 \\ & \text{Non-negativity} \end{aligned}$$

Here, MF for example, is the number of workers assigned to the shift that starts on Monday and runs through Friday. Note that this model allows a worker to be assigned to a shift and only dedicate a fraction of his efforts to the tasks that make up the requirements. Workers like these dedicate the remainder of their efforts to, say, and strategic projects.

- A. (10 points) After solving this problem, the postmaster sees that he will need a staff of at least 23 full time equivalents (22 just isn't enough). He argues that the only way to reduce this is to find ways to reduce the demand on the peak day, Thursday. Is this really the only way to reduce the total staffing required? To be specific, can we reduce the total staffing requirements by reducing the requirements of any single day EXCEPT Thursday by a single worker?

No. The shadow prices and allowable decreases for the constraints on Monday, Wednesday, Thursday and Saturday all indicate that we can reduce total requirements by 0.333 workers if we reduce the requirements on any one of these days by 1.

- B. (10 points) One very senior worker insists on having Fridays and Saturdays off. Provide an estimate of the impact on the total workforce required if we accommodate his request? Discuss briefly the thinking behind your estimate.

We don't have any workers scheduled for the Sunday-Thursday shift, so we can't just swap this worker onto that shift. Instead, to accommodate him, we must put him on this shift. The reduced cost of 0 on the Sunday-Thursday shift suggests doing this will not increase the total staffing requirements, but we may not be able to increase our assignments to this shift by one full unit.

- C. (5 points) Is the solution shown below the unique (i.e., the only) optimal solution to this problem?

No. The non-basic variable Sunday-Thursday has reduced cost 0 and the current solution is non-degenerate (no basic variable is zero). So we can shift some staff to Sunday-Thursday at no cost. [The basic variables are: The 5 shifts we assign staff to

together with the slack variables on the constraints for staffing on Tuesday and Friday.

The Spreadsheet model and sensitivity information are shown below.

Shift	Mon-Fri	Tues-Sat	Wed-Sun	Thu-Mon	Fri-Tue	Sat-Wed	Sun-Thu	Total	
Assignments	6.333333	5	0.333333	7.333333	0	3.333333	0	22.33333	
Day of Week	Workers							Required	
Monday	1			1	1	1	1	17 >=	17
Tuesday	1	1			1	1	1	14.66667 >=	13
Wednesday	1	1	1			1	1	15 >=	15
Thursday	1	1	1	1			1	19 >=	19
Friday	1	1	1	1	1			19 >=	14
Saturday		1	1	1	1	1		16 >=	16
Sunday			1	1	1	1	1	11 >=	11

Microsoft Excel 9.0 Sensitivity Report
Worksheet: [PostalSchedule.xls]Sheet1
Report Created: 2/26/2003 3:20:37 PM

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$B\$3	Assignments Mon-Fri	6.333333333	0	1	0	1
\$C\$3	Assignments Tues-Sat	5	0	1	0	0
\$D\$3	Assignments Wed-Sun	0.333333333	0	1	0	0
\$E\$3	Assignments Thu-Mon	7.333333333	0	1	0.5	1
\$F\$3	Assignments Fri-Tue	0	0.333333333	1	1E+30	0.333333333
\$G\$3	Assignments Sat-Wed	3.333333333	0	1	0.5	1
\$H\$3	Assignments Sun-Thu	0	0	1	1E+30	0

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$I\$6	Monday Workers	17	0.333333333	17	0.5	2.5
\$I\$7	Tuesday Workers	14.66666667	0	13	1.666666667	1E+30
\$I\$8	Wednesday Workers	15	0.333333333	15	11	1

\$I\$9 Thursday Workers	190.3333333333		19	5	1
\$I\$10 Friday Workers	19	0	14	5	1E+30
\$I\$11 Saturday Workers	160.3333333333		16	0.5	2.5
\$I\$12 Sunday Workers	11	0	11	1.666666667	0.3333333333

Problem 2 (30 points)

- A. (10 points) Describe the differences, if any, between the concepts of Shadow Prices and Reduced Costs.

Shadow prices give us sensitivity information related to the constraints. In particular, the shadow price for a constraint is the rate of change in the objective as we increase the right-hand-side of the constraint.

Reduced costs give us sensitivity information related to the variables. In particular, the reduced cost of a variable is the amount by which we must improve the objective coefficient before that variable will be attractive. Equivalently, the reduced cost of a non-basic variable is the rate of change in the objective as we increase the activity of that variable and adjust the values of the basic variables to maintain feasibility.

The two are related in that given the shadow prices, the process of pricing out gives us the reduced costs.

- B. (10 points) Describe the differences, if any, between the concepts of Basic Variables and Non-basic Variables.

Non-basic variables are fixed at 0 (or, more generally at a bound) in a basic feasible solution. The values of the basic variables are uniquely determined by the resulting constraints. Thus the basic variables are the only variables whose values can differ from 0 (or more generally a bound) in a basic feasible solution.

Tying this to question A. The shadow prices for a basic feasible solution are determined so that the reduced costs of basic variables are zero. In this way, the reduced costs of non-basic variables represent how attractive they are relative to the basic variables. If none of the non-basic variables is attractive, then the basic feasible solution is an optimal solution.

C. (10 points) Describe the relationships, if any, between the concepts of Network Flow Problems and Linear Programming Problems.

Network flow problems are the special case of Linear Programming Problems in which every variable appears in at most two constraints (not counting bounds). Further, each variable must appear in at most one constraint with a +1 coefficient and at most one constraint with a -1 coefficient. No other coefficients are allowed. The general form of a network flow problem is:

Set NODES;

Set EDGES within NODES cross NODES;

Param C {EDGES};

Param UpperBound {EDGES};

Param LowerBound {EDGES};

Param Supply {NODES};

Var X {EDGES};

Min Cost:

$\text{Sum}\{(f,t) \text{ in EDGES}\} C[f,t]*X[f,t];$

s.t. FlowConservation {node in NODES}:

$\text{sum}\{(f, \text{node}) \text{ in EDGES}\} X[f, \text{node}] - \text{sum}\{(\text{node}, t) \text{ in EDGES}\} X[\text{node}, t] = \text{Supply}[\text{node}];$

s.t. UpperBounds {(f,t) in EDGES}:

$X[f,t] \leq \text{UpperBound}[f,t];$

s.t. LowerBounds {(f,t) in EDGES}:

$X[f,t] \geq \text{LowerBound}[f,t];$

Problem 3 (20 points)

Indicate which of the following are best described as a

1. Network Flow Model,
2. Linear Programming Model,
3. Neither of the above. In this case explain briefly what prevents the formulation from being a Linear Programming Model

A. (10 points)

```
set FUNDS;
set ASSETCLASSES;

param Targets{ASSETCLASSES};
param Ratings{FUNDS, ASSETCLASSES};

var Invest{FUNDS} >= 0;
var AmtInClass{ASSETCLASSES};

minimize Deviation:
  sum{class in ASSETCLASSES} | AmtInClass[class] – Targets[class];
```

3. Neither of the above. The objective is not linear, because absolute value is not a linear function.

```
s.t. DefineAmtInClass{class in ASSETCLASSES}:
  AmtInClass[class] = sum{f in FUNDS} Ratings[f, class]*Invest[f];
```

```
s.t. FullyInvested:
  sum{f in FUNDS} Invest[f] = 1;
```

Comment: Note the use of absolute value ($|x| = \max\{x, -x\}$) in the objective.

B. (10 points)

```
set VPS;
set PLANTS;

param Costs{VPS, PLANTS};

var Assign{VPS, PLANTS} >= 0;

minimize TotalCost: sum{vp in VPS, p in PLANTS} Cost[vp, p]*Assign[vp, p];

s.t. OneVPtoEachPlant{p in PLANTS}:
  sum{vp in VPS} Assign[vp, p] = 1;

s.t. OnePlantForEachVP{vp in VPS}:
  sum{p in PLANTS} Assign[vp, p] = 1;
```

3. Neither of the above. The objective is not linear, because the following constraint includes a non-linear function. We could omit it and obtain the same answers because the assignment problem yields integral solutions.

s.t. GetZerosOrOnes {vp in VPS, p in PLANTS}:
Assign[vp,p]*Assign[vp,p] = Assign[vp, p];

Problem 4 (25 points)

An aircraft part manufacturer TCI provides Stage 2 low pressure compressor (lpc) blades for Pratt & Whitney's PW2000 aircraft engines. TCI has a certain number of blades in inventory, purchases new blades and reconditions used blades. It can either recondition blades using a rapid, but more expensive process or by a slower, but more economical process. You may assume that TCI takes newly purchased blades out of its supplier's consignment inventory. In other words, newly purchased blades are available immediately. These are the only options:

1. Take blades from existing inventory
2. Purchase new blades
3. Recondition blades via the rapid process
4. Recondition blades via the slower process.

Generally, TCI uses some combination of these methods to ensure it meets its customers' needs on time.

For simplicity, assume that the customers contract with TCI well in advance and so, based on their maintenance schedules, TCI has an accurate forecast of how many worn blades it will receive and how many replacement blades it must provide each day for several weeks into the future.

In general, our objective is to focus on the flow of blades at a macro level to help determine budgets and allocate resources over the medium term. We are not concerned with micro decisions about the handling of individual blades. In fact, to simplify your analysis, assume that TCI cannot track the identity of blades coming back from its customers and so has no idea of a blade's previous maintenance history. Further, assume that every returned blade can be successfully reconditioned.

You have been asked to develop a network flow model to help TCI with its resource planning.

- A. (10 points) Describe the data you will need from the supplier in order to build your model. Be as specific as possible. For example, you will need to know how many days TCI forecasts and the number of blades it expects to receive each day of the forecast. What other information will you want from TCI?

There are several possible answers. All should include (perhaps at different time scales):

- The forecast horizon in days
- The requirements for blades at each customer on each day of the forecast
- The number of worn blades each customer will return each day
- The cost, capacity and duration of the quick reconditioning method
- The cost, capacity and duration of the slow reconditioning method
- The cost of new blades
- The current inventory of blades (and perhaps their status of worn or usable).
- Cost of inventory
- Etc.

- B. (10 points) Describe in words the different types of edges (or variables or flows) you will consider in your model. For example, your model will include a variable for the number of new blades purchased each day.

Again, there are several possible answers. All should include (perhaps at different time scales):

- The number of blades that were returned each day
- The number of blades in inventory each day (we assume these are all usable)
- The number of blades quick reconditioned each day
- The number of blades slow reconditioned each day
- The number of blades purchased new each day

- C. (5 points) Formulate a network flow model to help TCI. It is best if you can use AMPL-like syntax to describe your model as it is unambiguous. Don't worry about the details of the syntax like semicolons, etc. The point is to provide a clear, unambiguous description of the model. If are not able to do this, you may even draw a network flow model instead, but this won't win you full credit naturally.

In PseudoAmpl....

```
Param Ndays;
Set DAYS 0..Ndays;
```

```
Set Methods; /* Quick and slow */
Param Duration{Methods}; /* How long it takes to recondition by each method */
Param Cost{Methods}; /* $/blade to recondition by each method */
Param Capacity{Methods}; /* Number of blades we can start each day on each method */
```

```
Param NewCost; /* $/blade */
Param InvCost; /* $/blade/day */
Param Supply{DAYS}; /* returned blades on each day */
Param Demand{DAYS}; /* required blades on each day */
Param InitInventory;
```

```
Var Recondition{method in Methods, day in DAYS} >= 0, <=Capacity[method]; /* How many blades we recondition by each method each day */
```

```
Var Purchase {DAYS} >= 0; /* How many blades we purchase each day */
```

```
Var Inventory{DAYS}>= 0; /* How many usable blades we hold in inventory */
```

Minimize TotalCost:

```
Sum{day in DAYS} InvCost*Inventory[day] +
Sum{day in DAYS} NewCost*Purchase[day] +
Sum{method in Methods, day in DAYS} Cost[method]*Recondition[method, day];
```

s.t. InitialInventoryBalance:

```
InitialInventory + Purchase[0] = Inventory[0] + Demand[day];
```

s.t. InventoryBalance[day in DAYS: day > 0]:

$$\text{Inventory}[\text{day} - 1] + \text{Purchase}[\text{day}] + \sum\{\text{method in Methods: day} \geq \text{Duration}[\text{method}]\} \text{Recondition}[\text{method}, \text{day} - \text{Duration}[\text{method}]] = \text{Inventory}[\text{day}] + \text{Demand}[\text{day}];$$

s.t. $\text{AllocateReturnsToMethods}\{\text{day in DAYS}\}:$
$$\sum\{\text{method in Methods}\} \text{Recondition}[\text{method}, \text{day}] = \text{Supply}[\text{day}];$$