6.265-15.070
Midterm Exam

Date: October 30, 2013

6pm-8pm

80 points total

Problem 1 (30 points) . *Suppose a random variable X is such that $\mathbb{P}(X > 1) = 0$ and $\mathbb{P}(X > 1 - \epsilon) > 0$ for every $\epsilon > 0$. Recall that the large deviations rate function is defined to be $I(x) = \sup_\theta(\theta x - \log M(\theta))$ for every real value x, where $M(\theta) = \mathbb{E}[\exp(\theta X)]$, for every real value θ.*

(a) Show that $I(x) = \infty$ for every $x > 1$.

(b) Show that $I(x) < \infty$ for every $\mathbb{E}[X] \leq x < 1$.

(c) Suppose $\lim_{\epsilon\to 0} \mathbb{P}(1 - \epsilon \leq X \leq 1) = 0$. Show that $I(1) = \infty$.

Problem 2 (20 points) *Recall the following one-dimensional version of the Large Deviations Principle for finite state Markov chains. Given an N-state Markov chain $X_n, n \geq 0$ with transition matrix $P_{i,j}, 1 \leq i,j \leq N$ and a function $f : \{1, \ldots, N\} \to \mathbb{R}$, the sequence $\frac{S_n}{n} = \frac{\sum_{1\leq i\leq n} f(X_i)}{n}$ satisfies the Large Deviations Principle with the rate function $I(x) = \sup_\theta(\theta x - \log \rho(P_\theta))$, where $\rho(P_\theta)$ is the Perron-Frobenius eigenvalue of the matrix $P_\theta = (e^{\theta f(j)} P_{i,j}, 1 \leq i, j \leq N)$.*

Suppose $P_{i,j} = \pi_j$ for some probability vector $\pi_j \geq 0, 1 \leq j \leq N$, $\sum_j \pi_j = 1$. Namely, the observations X_n for $n \geq 1$ are i.i.d. with the probability mass function given by π. In this case we know that the large deviations rate function for the i.i.d. sequence $f(X_n), n \geq 1$ is described by the moment generating function of $f(X_n), n \geq 1$. Establish that the two large deviations rate functions are identical, and thus the LDP for Markov chains in this case is consistent with the LDP for i.i.d. processes.

HINT: *consider the left-eigenvector of P_θ corresponding to the eigenvalue $\rho(P_\theta)$.*

Problem 3 (30 points) *The following two parts can be done independently.*

(a) Suppose, $X_n, n \geq 0$ is a martingale such that the distribution of X_n is identical for all n and the second moment of X_n is finite. Establish that $X_n = X_0$ almost surely for all n.

(b) An urn contains two white balls and one black ball at time zero. At each time $t = 1, 2, \ldots$ *exactly one ball is added to the urn. Specifically, if at time* $t \geq 0$ *there are* W_t *white balls and* B_t *black balls, the ball added at time* $t+1$ *is white with probability* $W_t/(W_t + B_t)$ *and is black with the remaining probability* $B_t/(W_t + B_t)$*. In particular, since there were three balls at the beginning, and at every time* $t \geq 1$ *exactly one ball is added, then* $W_t + B_t = t + 3, t \geq 0$*. Let* T *be the first time when the proportion of white balls is exactly* 50% *if such a time exists, and* $T = \infty$ *if this is never the case. Namely* $T = \min\{t : W_t/(W_t+B_t) = 1/2\}$ *if the set of such* t *is non-empty, and* $T = \infty$ *otherwise. Establish an upper bound* $\mathbb{P}(T < \infty) \leq 2/3$*.*

Hint: *Show that* $W_t/(W_t+B_t)$ *is a martingale and use the Optional Stopping Theorem.*

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