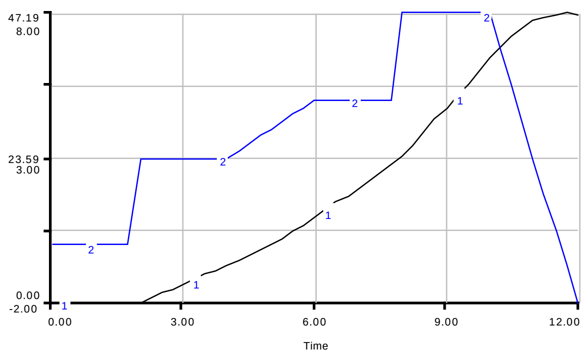
Graphical Integration Exercises

Part One: Exogenous Rates



Prepared for the
MIT System Dynamics in Education Project
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1. Abstract

This is a first paper in the graphical integrations exercises series. The series is intended to teach its readers the basic principles and applications of graphical integration. This paper discusses graphical integration methods for systems with exogenous rates.

2. Introduction

Let us consider a simple system of a bathtub with a constant inflow through a faucet but no drainage of water. Can we predict the behavior of this system? We know that if we turn on the faucet to fill a tub, go outside for five minutes and come back to find the tub one-fourth full, then it will take ten minutes to fill up halfway, fifteen minutes to be three-fourths full, and in a total of twenty minutes from the start of the inflow, the tub will be completely full of water.

What we just did was graphical integration without explicitly drawing graphs. We were able to do this because the system in question was very simple. However, many systems, in real-life situations are much more complex, and it becomes difficult to predict how they will behave. Enough practice, however, will make it easier to use graphical integration to understand the behavior of many systems.

Although we have access to sophisticated computer programs that simulate many complex systems, it is important that we can intuitively predict and understand what we see on the graph after running a simulation. This series of graphical integration papers will enable us to do this. In this first paper of the series, we will explore the most basic case, that of exogenous rates.

An exogenous rate to a system is a rate variable that cannot be affected by that system. In other words, its value does not depend on any level of the system. Although many of the systems we see in the real world are affected by endogenous rates (the opposite of exogenous rates), learning the behavior of simple systems will set the ground for more complicated systems to come later in the series.

3. Systems with One Exogenous Rate

In this section, we will consider a system with a single exogenous rate. We will look at the system's behavior when the rate is constant and when the rate varies with time.

3.1 Constant Rate

If we turn on the faucet and leave it as it is, the rate of water flow into the tub is constant. The behavior of this system can easily be predicted.

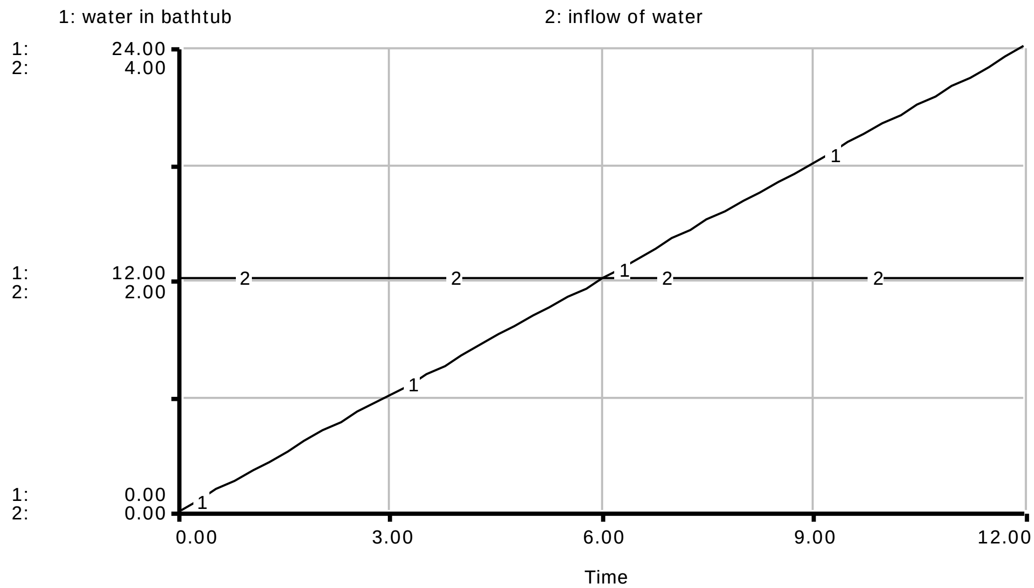


Figure 1: Water Inflow and Bathtub Level
This is an illustration of a bathtub filling with water. If the inflow is constant and positive, the water level increases linearly.

Figure 1 is a graphical analysis of what we predicted in the introduction. Notice that *the slope of the level, measured in amount of water per unit time, is equal to the value of the constant rate, also measured in amount of water per unit time.* In the above example, the water level rises to 12 units in 6 units of time, so the slope of the level is equal to 2 units of water per unit time, which is the value of the inflow.

What if there is no inflow but only an outflow? An example would be a full bathtub being drained at a constant rate. In that case, the value of the inflow is 0 and the value of the outflow is a positive number. The net constant flow, defined as *net flow* = *inflow* - *outflow*, is *net flow* = 0 - (positive number) = (negative number), thus the slope of the level is also negative. How would the system behave in response to a negative net flow? This is illustrated in Figure 2.

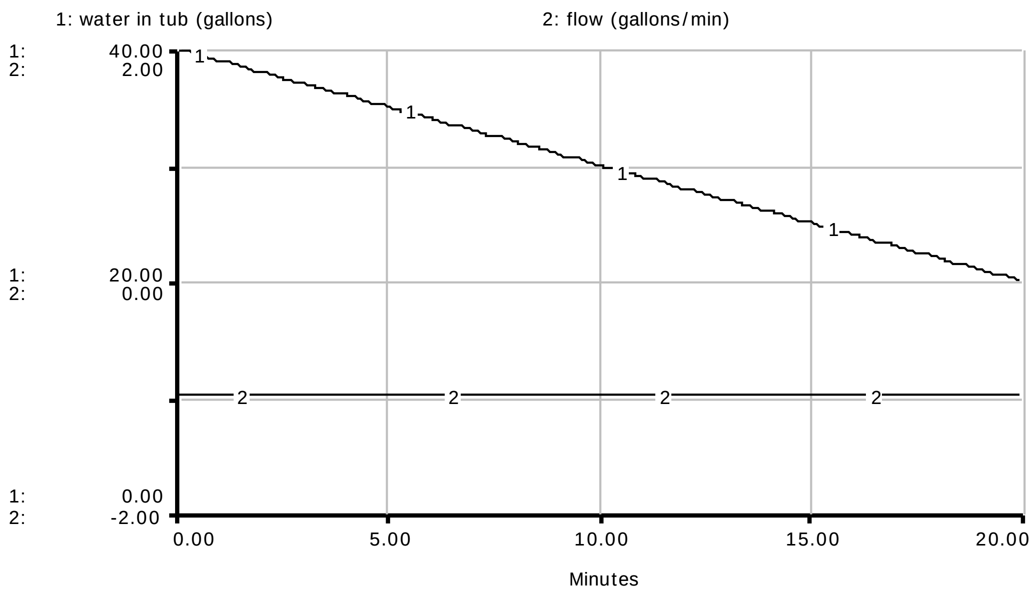


Figure 2: Water Outflow and Amount of Water in Tub
The inflow of water is equal to 0 gallon/min, and the outflow of water is equal to 1 gallon/min. Consequently, the net flow of the system is -1 gallon/min, so the slope of the level is –1 gallon/min.

As you may have expected, with a negative net constant flow, the water level *drops* at a constant rate, in this case at a rate of 1 gallon per minute.

3.2 Step Function

Now, let us look at what happens when we delay turning on the faucet.

What you will see on the next page in Figure 3 is called a *step function*. As the name suggests, a step function starts off at a constant value, in this case 0, and steps either up or down to another constant value. In this case, we step up to +3 at *Time* = 5. A function may have several steps, and we will see examples of that later in this paper.

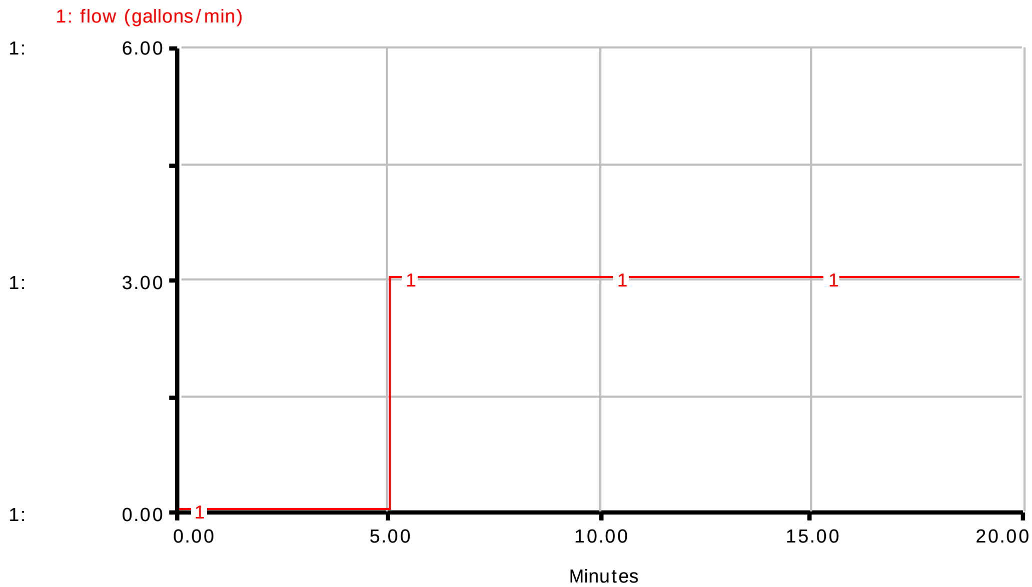


Figure 3: Step Function
Instead of starting the flow at time 0, we start the flow after waiting five minutes.

The result is illustrated in Figure 4 below. Until *Time = 5 minutes*, the stock stays at zero, and when the flow increases to 3, the stock accumulates at the rate of 3 gallons per minute.

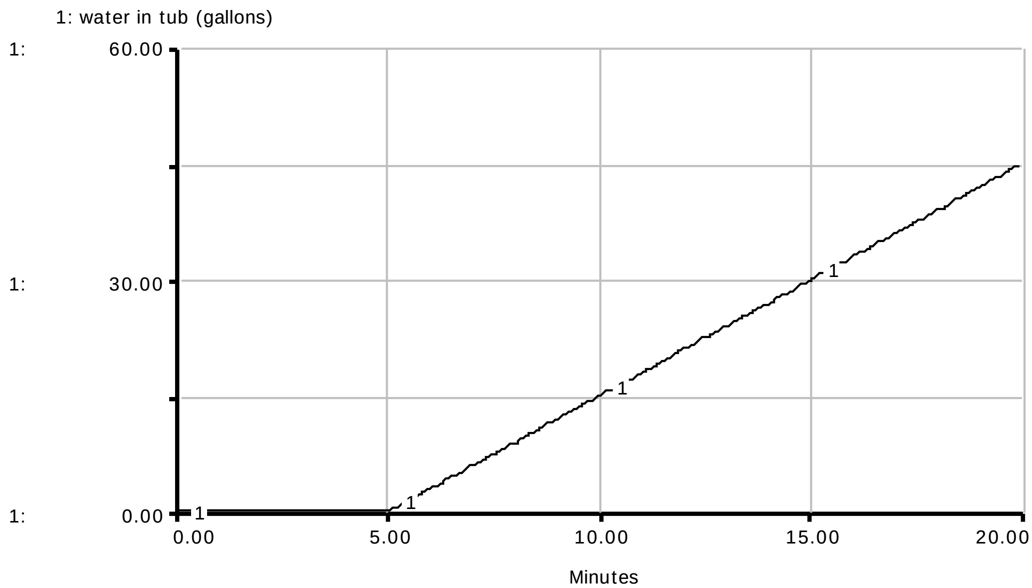


Figure 4: Graph of Stock Resulting from Step Function
Stock remains at zero until the flow increases from 0 to 3 at *Time* = 5.

Here is another example. This one has two steps, one step up at *Time* = 5 and another at *Time* = 15.

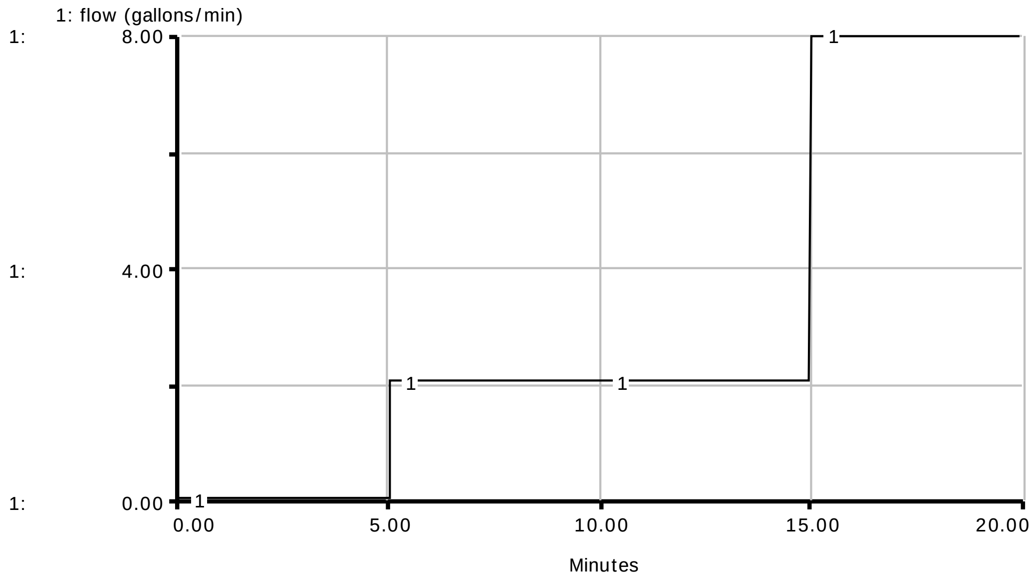


Figure 5: A Step Function Rate

Graphically integrating the rate in Figure 5, we get the following behavior of the stock.

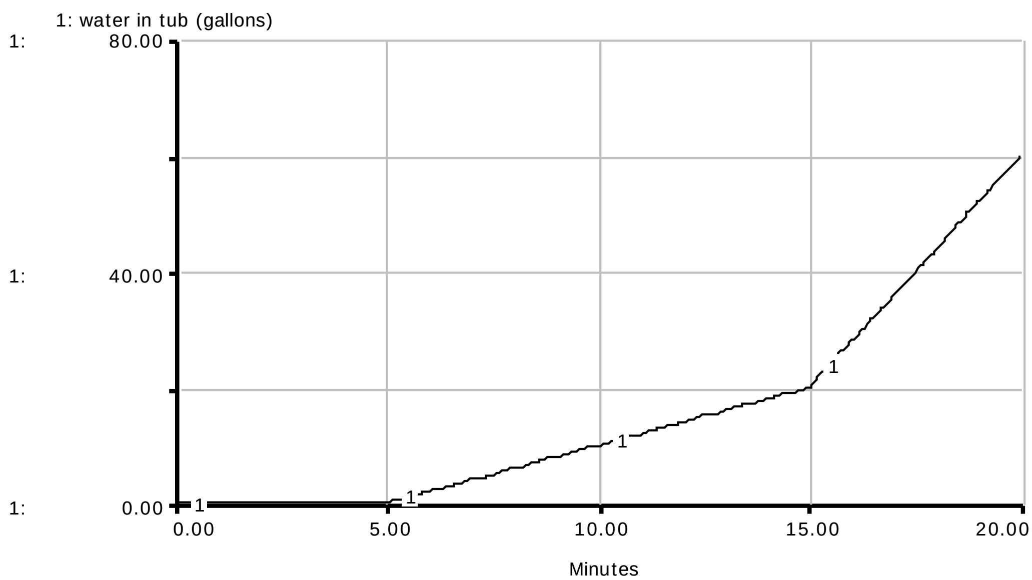


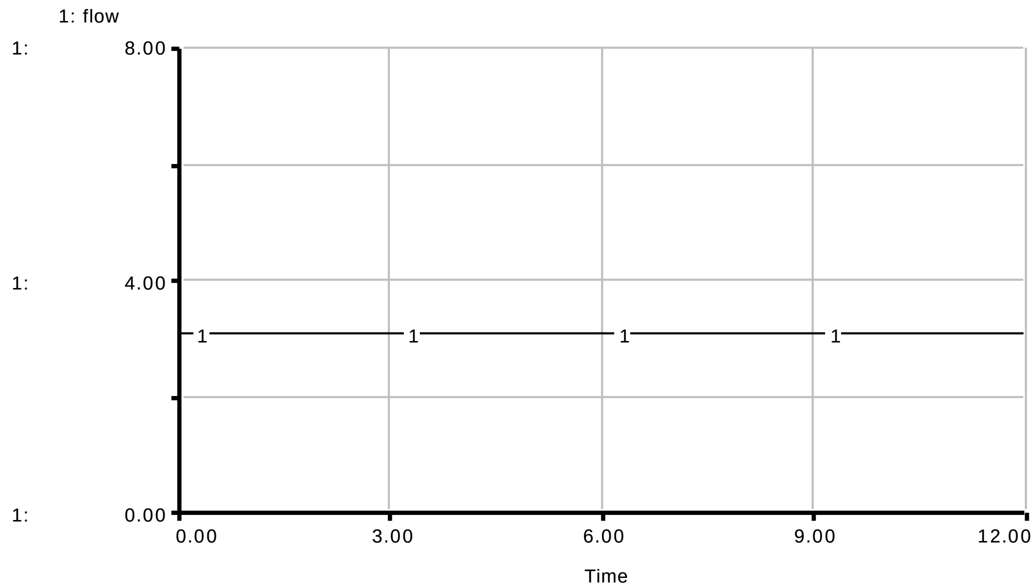
Figure 6: Value of Stock Changing as the Flow Changes in Figure 5
From *Time* = 0 to *Time* = 3, the value of the flow is 0, so the slope of the stock is also 0. From *Time* = 5 to *Time* = 15, the flow is 2 gallons/min, so the slope of the stock is also 2 gallons/min. From *Time* = 15 to *Time* = 20, the flow is 8 gallons/min, so the slope of the stock is 8 gallons/min.

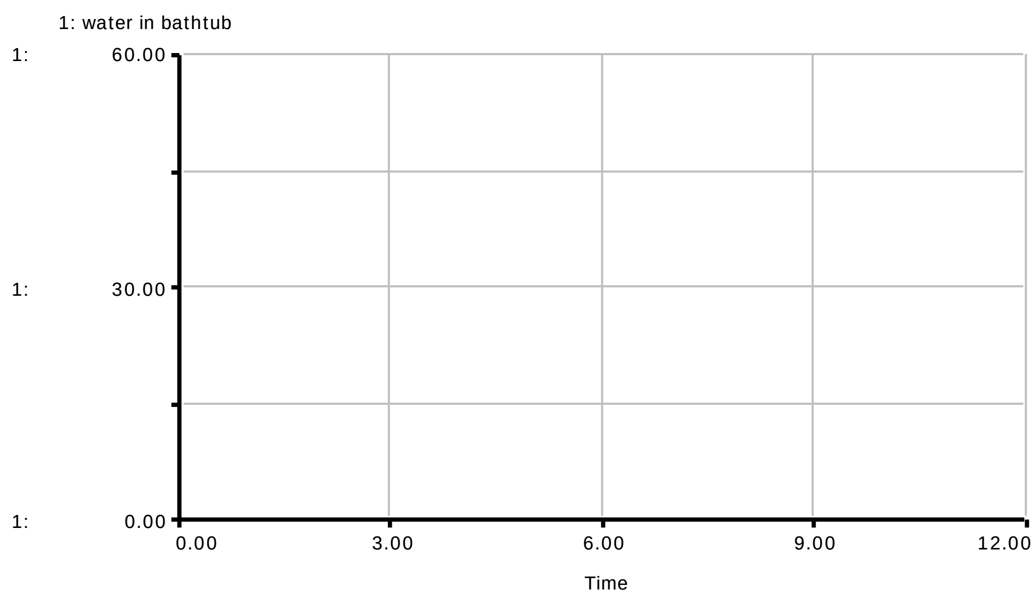
You may be wondering how one can accurately measure the slope of the stock in the above example. The values on the y-axis are not the simple numbers used in the previous examples, and the values of the flow are somewhat difficult to accurately guess. For instance, the first step in Figure 5 could have gone up to 1.9 or 2.1 instead of 2, and the second step could have been up to 7.8 or 7.9. Sometimes you do not know the exact values of flows and stocks. What is important in graphical integration is that you understand the general idea. In Figure 5, you can clearly see that the flow starts out at 0, goes up to some value, and then goes up to a bigger value. From that, you should be able to predict that the stock remains unchanged until the first step, then goes up with a constant slope, and then goes up with a steeper slope.

3.3 Exercises

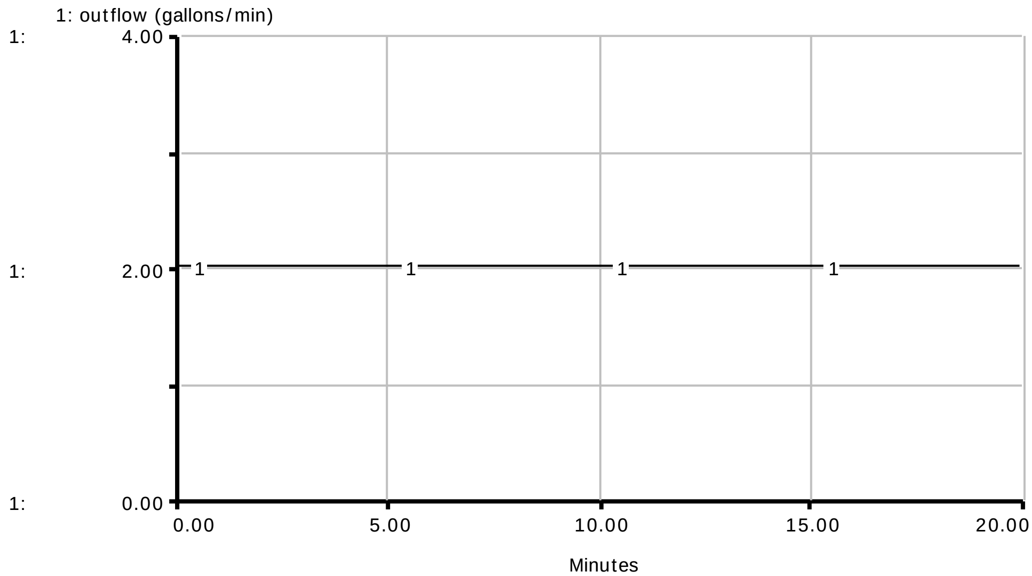
Now, let's try some exercises.

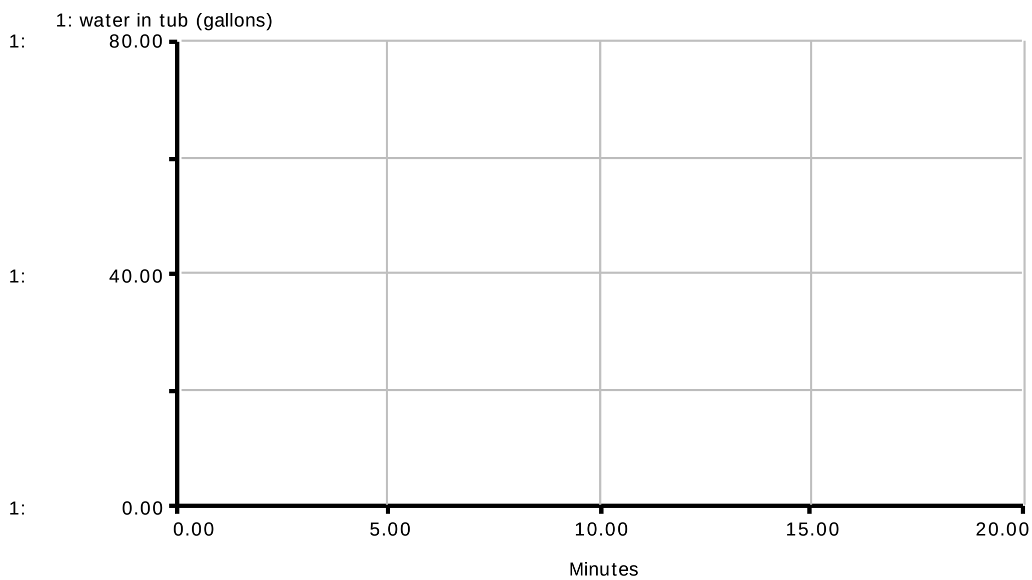
1. In this first exercise, the rate is +3. Using the blank graph pad below, graph the behavior of the system, assuming the initial value of the level is 0.



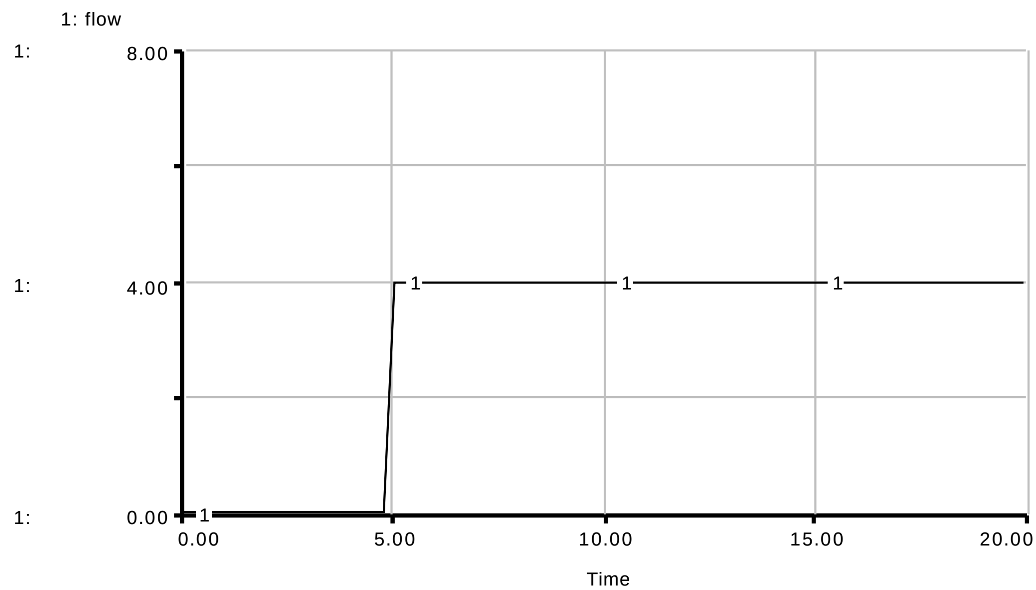


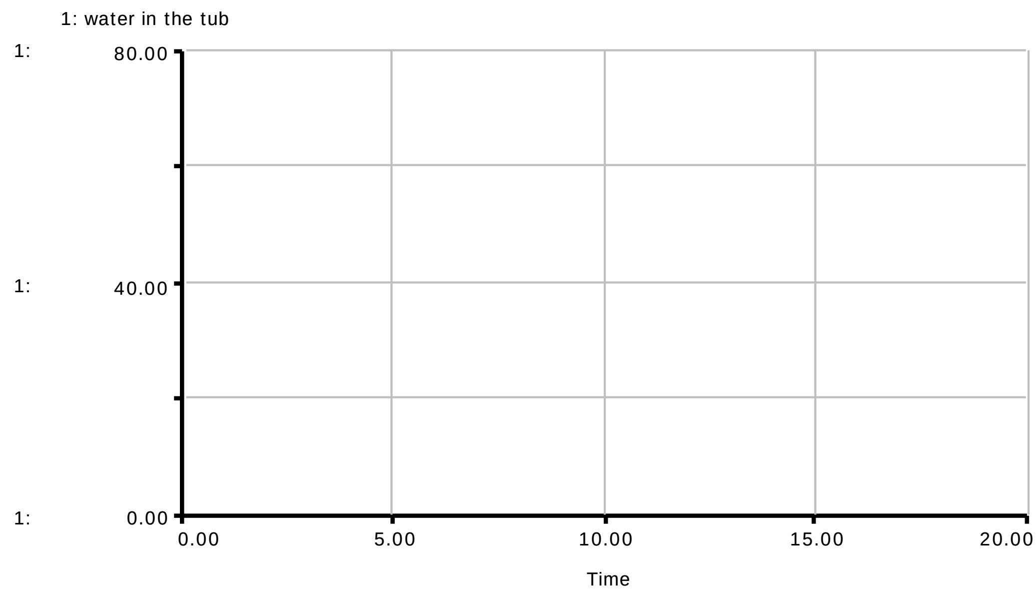
2. For this second exercise, assume there is no inflow and outflow = 2. Graph the behavior of the system below, assuming the initial value of the level is 40.



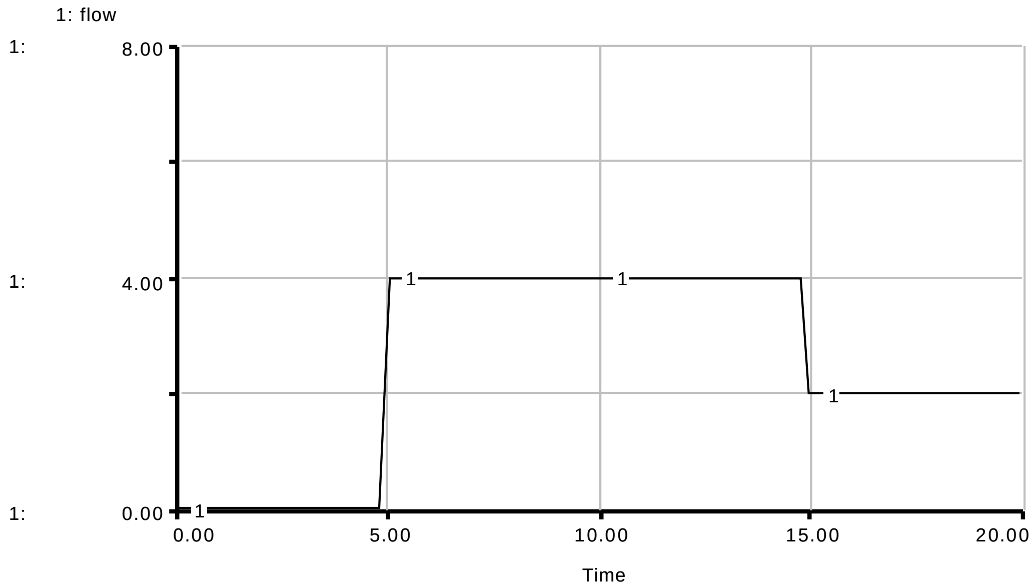


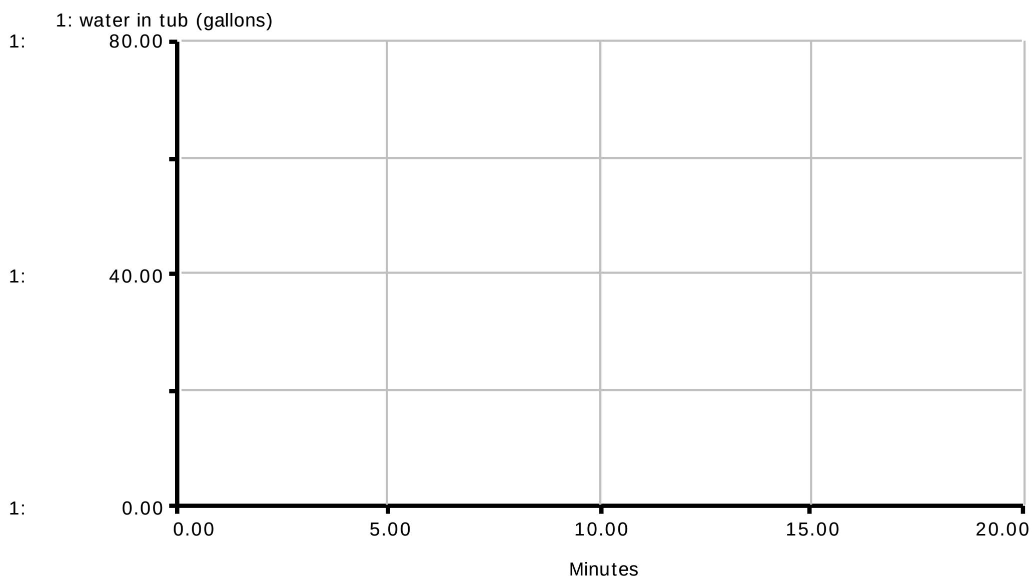
3. Now, let's try a step function. The flow in this exercise remains 0 until a step up to 4 at *Time* = 5. Assume the initial value of the stock is 0, and graph the predicted behavior below.



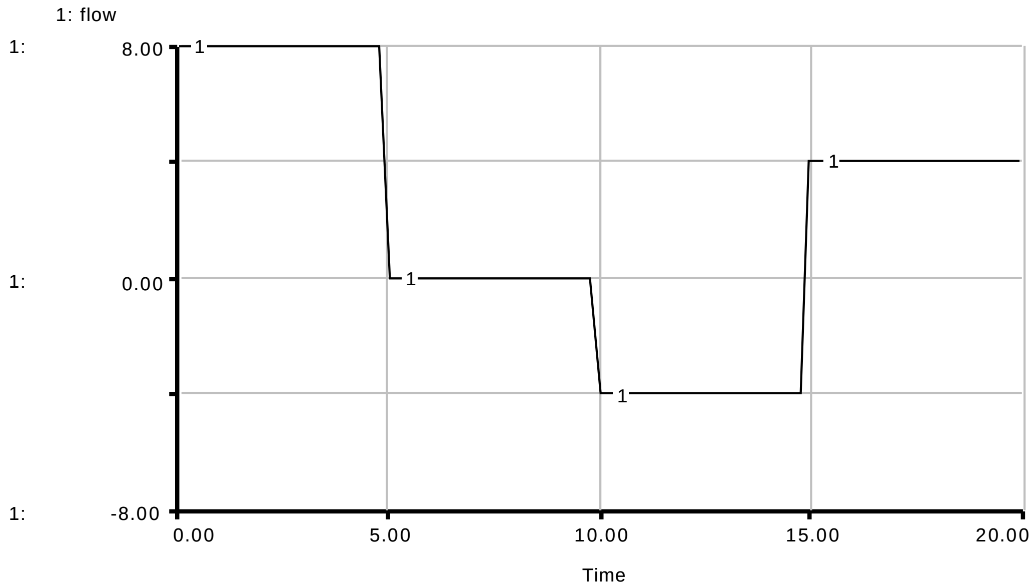


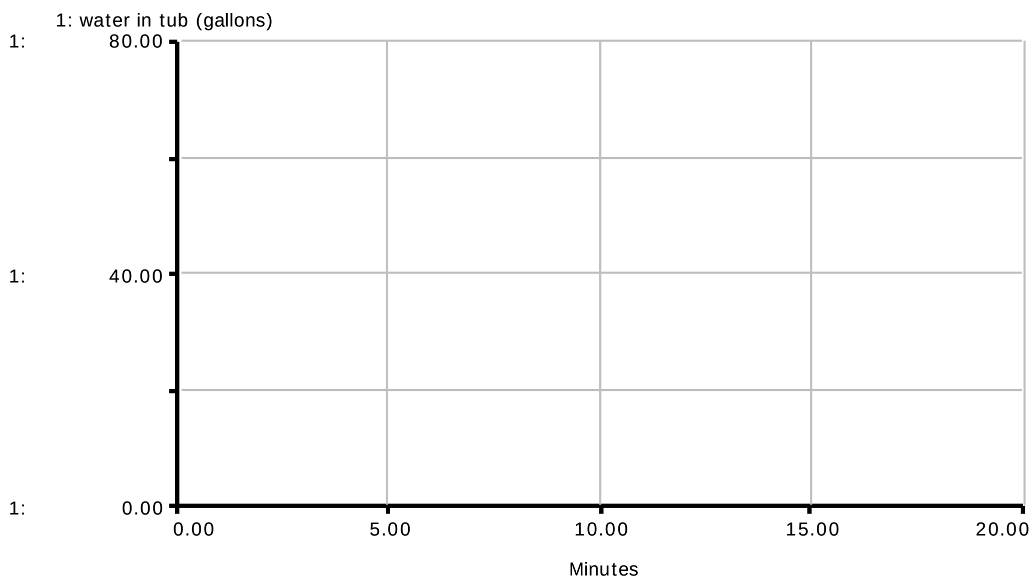
4. Let us try a more complex problem. We start with *flow* = 0, then step up to 4 at *Time*=5, then step down to 2 at *Time*=15. Assuming the initial value of the stock is 0, graph the result below.





5. Here is another one. It might look complicated, but if you break it up into small pieces, it is not difficult. The initial value of the net flow is +8, then it steps down to 0 at *Time* = 5, then down again to -4 at *Time* = 10, then up to +4 at *Time* = 15. Don't forget that the value of the net flow at a given time is equal to the slope of the stock.





The answers to the above exercises are in the Appendix.

4. Conclusion

So far we've looked at exogenous rates and how they affect the behavior of the stock. You should now feel comfortable using graphical integration method to predict the results of constant positive and negative rates as well as step function rates.

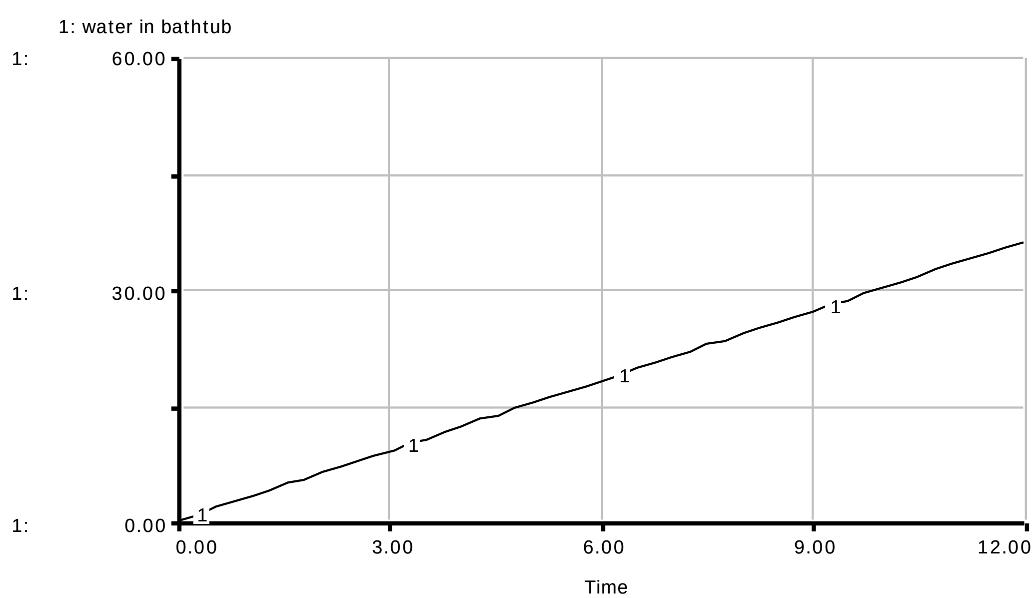
In the next paper in this series, you will go through similar steps, but with ramp functions and feedback systems. If you do not feel you have built a strong foundation of graphically integrating constant and step function rates, please go back and review this paper before going on to the next step.

Good luck!

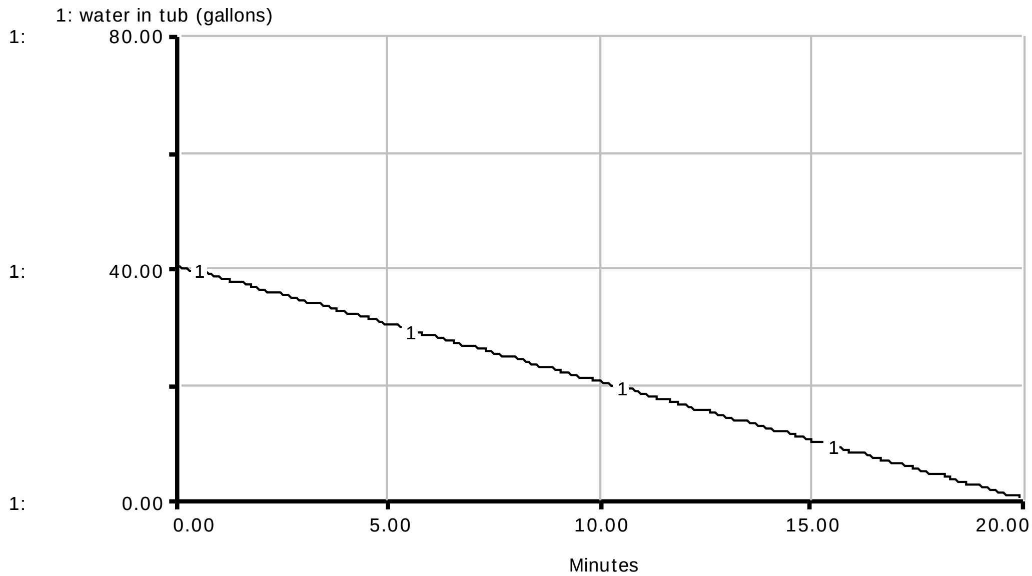
5. Appendix

Here are the answers to exercises.

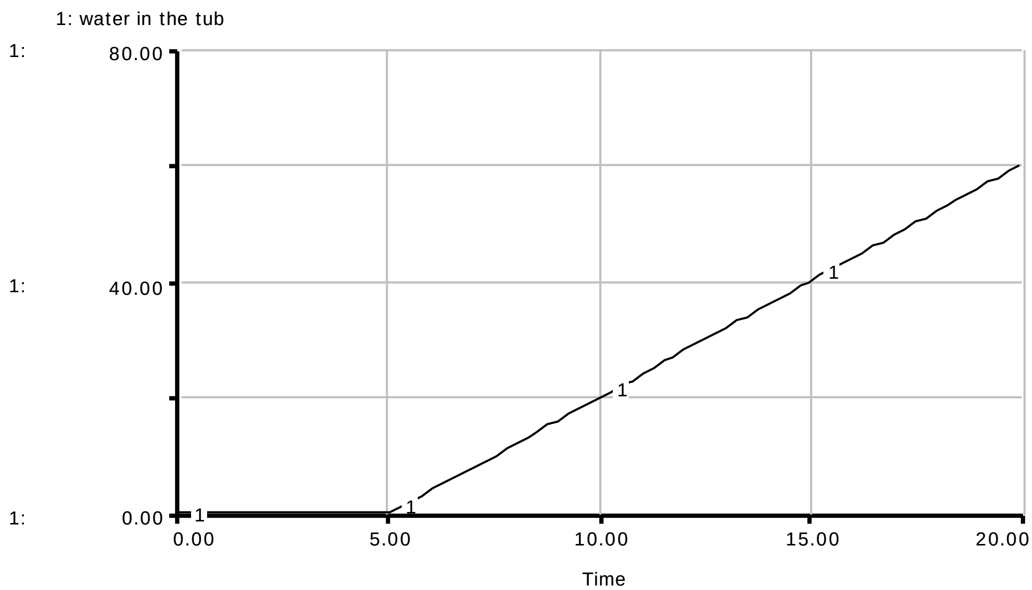
1. This first one is just a straight line with a slope of +3. Notice that the value of the constant rate was also +3.



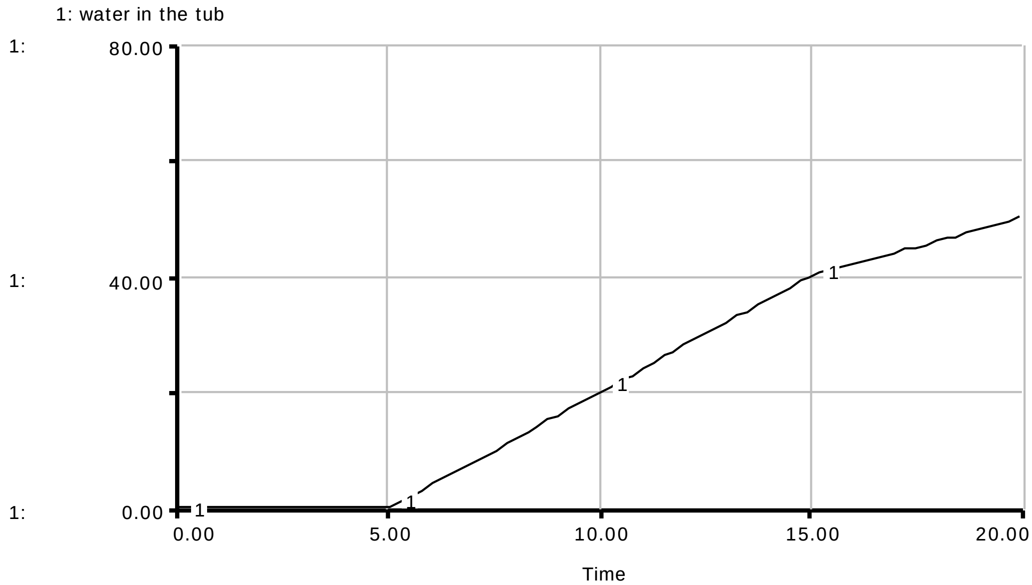
2. This is similar to exercise 1, but the slope is -2. We see that the tub is being drained.



3. There is a delay in turning on the faucet. The water level remains at 0 until the value of the rate steps up to 4 at *Time* = 5.



4. This is the same as exercise 3 until *Time* = 15, at which point the slope changes to 2.



5. The initial rate is +8, so the value of the stock at *Time* = 5 is 8 units of water / unit time x 5 units of time = 40 units of water. The value of the net flow from *Time* = 5 to *Time* = 10 is 0, so the stock remains at 40. Then the slope drops down to -4, and then back up to +4.