15.997 Practice of Finance: Advanced Corporate Risk Management
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Chapter 9: Pricing Risk

Corporate managers price risk all the time when they discount a cash flow to its present value using a risk-adjusted discount rate. Future cash flows are discounted not only to account for the time value of money, but also to account for risk. Starting from the risk-free discount rate, extra risk is accounted for by adding a risk premium to arrive at a risk-adjusted discount rate. The discounted cash flow methodology using a risk-adjusted discount rate is the workhorse of corporate finance. And for many problems it is a perfectly adequate tool. But it has its limits. Some projects and securities have complicated risk dynamics for which the traditional risk-adjusted discount rate methodology is not flexible enough to do the trick with the ease and power we demand. Modern business and modern financial management is able to slice-up and repackage risk to create novel contingent cash flow patterns for which the usual application of the risk-adjusted discount rate methodology is error prone.

As it is usually implemented, the risk-adjusted discount rate methodology rests on two key assumptions that often do not apply. First, the risky cash flows being discounted are perfectly proportional to the underlying risk variable. One way to think about this is that the risky cash flows are symmetrically distributed, with an upside and a downside in equal proportion. This assumption is violated, for example, in the case of a call option, where the downside is truncated. Second, the risk in the cash flows grows linearly through time. This assumption is violated, for example, when much of the uncertainty in a project is resolved up front in early stages, although the results, whether good or bad, are payed out over a long period of time. This assumption is also violated whenever a key risk factor is not a random walk, but reverts to a mean or has other features that create a changing risk distribution at various horizons.

Many introductory corporate finance textbooks mention these exceptions to the usual discounting rules and construct a couple of simple examples to demonstrate how the usual application of the methodology produces a mistake. But for reasons of space and focus, they do not provide an adequate treatment of the solution. Many corporate managers, in the course of their careers, stumble upon actual projects or problems where the traditional risk-adjusted discount methodology seems inappropriate for these very reasons, whether or not they can articulate it this way. It is the task of this chapter to introduce and develop an alternative, robust risk pricing methodology capable of handling the wide array of complicated and dynamic patterns of risk that corporate managers often face. Ironically, this alternative methodology is widely known as the risk-neutral pricing methodology. Despite the unfortunate moniker, this alternative methodollogy has gained prominence among investment bankers, commodity traders and risk managers because of its power and adaptability. So corporate managers can expect to see a good bit more of it in the coming years. Corporate managers who wish to develop a facility to expertly price and manage risk need to be familiar with this second valuation methodology.

This chapter is your introduction. Before presenting the risk-neutral methodology proper, we first make a brief digression to remind the student that the risk-adjusted discount rate methodology has never been the only methodology available. We do this with an example sketched out in the next section.

9.1 Two Alternative Methods for Discounting Cash Flows

The Hejira oil company is estimating the present value of the next five years of oil revenue. The top panel of Table 9.1 shows the forecasted production, forecasted spot oil price, the risk-free discount rate, and the oil price risk premium for these five years.

Two different methods can be employed to calculate the present value:

- the risk-adjusted discount rate method, and
- the certainty-equivalent method.

The top panel of Table 9.1 also shows the calculation of the present value using the risk-adjusted discount rate method. This is the traditional and familiar method for calculating a present value. In this method, the two tasks of (i) accounting for the time value of money and (ii) accounting for the price of risk are accomplished in a single step by applying a single risk-adjusted discount rate to the expected cash flow. This single risk-adjusted discount rate reflects both the time value of money and the price of risk. The risk-adjusted discount rate is the sum of the risk-free rate and a risk premium appropriate to a stream of revenue tied to the price of oil:

$$r_a = r_f + \lambda. \tag{9.1}$$

We call this risk premium, λ, the market price of risk for cash flows tied to the oil price, or the oil factor risk premium.

The bottom panel of Table 9.1 shows the calculation of the present value using the certainty equivalent method. This is a less familiar, but no less correct method. In this method, the accounting for the price of risk and for the time value of money are performed in two separate steps. First, the expected cash flow is adjusted to account for the price of risk, λ. Second, the certainty-equivalent cash flow is then discounted to account for the time value of money. Because the cash flow being discounted has already been adjusted to account for risk, this step uses the risk-free discount rate.

The present values of each year's cash flow are identical under the two methods. Figure 9.1 captures the point that these are two different routes to the exact same end. In the numerical example shown in Table 9.1 it is obvious that the result must be the same. The risk-adjusted discount rate is just equal to the sum of the risk-free rate and the risk premium, so using the risk-adjusted discount methodology is just literally putting together the two separate steps of the certainty-equivalent method. The result has to be the same whichever method is used. The two methods accomplish the same thing, using a slightly different sequence of steps. Figure 9.1 captures the relationship between these two methodologies.

If the two methods are trivially the same, as this numerical example suggests, then what is the point of making a distinction between them? Separating the risk-premium from the risk-free rate allows us to handle the evolution of risk through time separately from the discounting of time. This is what turns out to be critical. We have already seen that the risk of various factors may vary over time, and that the risk on an asset may vary as the factor grows or falls. It is the dynamic and contingent nature of risk that forces us to resort to the second methodology. When factor risk falls through time, we will need to adjust our discount for risk downward. But the discount for time remains as before. Having separated the risk-premium from the risk-free rate, we can adjust the one without adjusting the other. When the project risk varies as the factor grows, we need to adjust our discount for risk accordingly. But the discount for time remains as before.

It is this greater flexibility in the face of dynamically changing risk structures that gives the risk-neutral methodology its advantage. The risk-adjusted discount rate method remains by far the most familiar method to corporate managers and to investment bankers involved in the purchase and sale of corporate assets and the evaluation of major capital investments. However, the risk-neutral methodology has become very important to specialists involved in valuing and trading derivatives and in evaluating risk management strategies. In addition to the flexibility this method offers, understanding the risk-neutral method is important for reading the information coded into many market prices.

9.2 Risk Neutral Pricing – an Introduction

In Chapter 6 we learned that the uncertain evolution of a variable could be modeled several different ways—as a discrete stochastic process, as a continuous stochastic process, using binomial or other trees, etc. In this section, we start our introduction to the price of risk using the binomial model. Later, based on an analysis developed on branches of the binomial tree, we will explain how this ties back to an analysis using the other representations, such as the discrete or the continuous stochastic process.

State Prices

Figure 9.2 shows a single branching of our underlying risk factor, *S*. Also shown in Figure 9.2 is an associated project with cash flows contingent on the evolution of *S*. The variable CF_U is the cash flow earned if the underlying risk factor moves up to S_U. The variable CF_D is the cash flow earned if the underlying risk factor moves down to S_D. The variable V_P is the market value of the project with a claim to these two contingent cash flows, and no other claim. Let us assume the following information:

$$CF_U = \$13.04$$

$$CF_D = \$8.40$$

$$V_P = \$10.00$$

For the moment, we also assume

$$\pi_U = 50\%$$

$$\pi_D = 50\%,$$

although later we will revisit this. Finally, we assume that:

$$r_f = 4\%.$$

Given the market price of the project, we can back out an appropriate risk-adjusted discount rate using the classic discounted cash flow formula:

$$V_P = E\left[CF\right]e^{-r_a} = \left(CF_U \times \pi_U + CF_D \times \pi_D\right)e^{-\left(r_f + \lambda_P\right)}. \tag{9.2}$$

Inputting the relevant values gives us:

$$r_a = 7\%,$$

which also implies:

$$\lambda_P = 3\%.$$

We use the subscript P to denote that this is the risk premium appropriate for the project shown in Figure 9.2.

Figure 9.3 shows two derivatives of the project. One of the derivatives earns the cash flow CF_U if the underlying risk factor moves up to S_U, but earns nothing if the underlying risk factor moves down to S_D. The other derivative earns the cash flow CF_D if the underlying risk factor moves down to S_D, but earns nothing if the underlying risk factor moves up to S_D. Also shown in Figure 9.3 is the payoff structure for a riskless bond earning $1 regardless of whether the underlying risk factor moves up to S_U or down to S_D. Valuing the bond is easy, assuming that we know the risk-free rate of interest, r_f. Can we value the two derivatives?

It is tempting to value the derivatives using the risk-adjusted discount rate of the project, but, as we will see shortly, this would be wrong:

$$E[CF]\,e^{-r_a} = \left(CF_U \times \pi_U\right) e^{-r_a} \neq V_U,$$

and,

$$E[CF]\,e^{-r_a} = \left(CF_D \times \pi_D\right) e^{-r_a} \neq V_D.$$

The risk-premium that should be applied to the derivative V_U is larger than the project risk-premium, so that the correct value is less than what one would get applying the project risk-premium. The risk-premium that should be applied to the derivative V_D is smaller than the project risk-premium, so that the correct value is greater than what one would get applying the project risk-premium. How do we know this?

When we analyzed the market price of the original project, we backed out a risk-premium, λ_P, that we applied to the expected cash flow of the project. In fact, the market price of the original project gives us more information than simply the risk premium to apply to the expected cash flow. It actually gives us two risk adjustment factors, one to be applied to the up cash flow, CF_U, and one to be applied to the down cash flow, CF_D. Applying the project risk-premium to both cash flows makes these two risk adjustment factors equal. But it makes no sense that the two risk adjustment factors should be equal. Under most sensible models of risk and value, cash earned in the down state should capture a premium, while cash in the up state is what should be discounted. Cash in the down state is like insurance, and it is prized. The project risk premium, λ_P, is an average of the discount applied to the up state and the premium paid for the down state cash flow. It works when we are receiving the complete package. But when we are valuing just one cash flow or the other, we need to apply the right risk adjustment factor for that cash flow, not the average. Here's how to extract that information from the value of the project.

We use the terminology "forward state prices" to describe the two variables φ_U and φ_D.[1] We define the forward state prices as the solution to the following two equations:

$$B = \left(\$1\times\pi_U\times\phi_U + \$1\times\pi_D\times\phi_D\right)e^{-r_f} \tag{9.3}$$

$$V_P = \left(CF_U\times\pi_U\times\phi_U + CF_D\times\pi_D\times\phi_D\right)e^{-r_f}\ . \tag{9.4}$$

Solving these two equations gives us:

$$\varphi_U = \left(\frac{V\,e^{r_f} - CF_D}{CF_U - CF_D}\right)\frac{1}{\pi_U} = 0.866 \tag{9.5}$$

$$\varphi_D = \frac{1-\pi_U\ \varphi_U}{1-\pi_U} = 1.134. \tag{9.6}$$

Having derived these forward state prices, we can now go back to value the two derivative claims, V_U and V_D:

$$V_U = \left(CF_U\times\pi_U\times\phi_U\right)e^{-r_f} = \$5.42$$

$$V_D = \left(CF_U\times\pi_U\times\phi_U + CF_D\times\pi_D\times\phi_D\right)e^{-r_f} = \$4.58.$$

Now we can go back and confirm what we had said earlier about the risk-premium that should be applied to the derivative V_U being larger than the project risk-premium, and the risk-premium that should be applied to the derivative V_D being smaller than the project risk-premium. The risk premiums implied by our valuation are:

[1] Some students may be familiar with the term "state price", which is a fundamental concept in the economics of uncertainty originally developed by Kenneth Arrow and Gerard Debreu. An Arrow-Debreu state price is the present value of $1 in a future state, which is a discounted value. Our forward state price is the undiscounted value. The state price and the forward state price differ only by the riskless rate of interest.

$$\lambda_U \text{ solves } V_U = \left(CF_U \times \pi_U\right) e^{-\lambda_U}\ e^{-r_f}\ , \tag{9.7}$$

which gives us:

$$\lambda_U = -\ln\left(\frac{\left(V_U\ e^{r_f}\right)}{CF_U\ \pi_U}\right) = -\ln\left(\frac{\$5.42\,e^{4\%}}{\$13.04\,\frac{1}{2}}\right) = 14.4\%,$$

and,

$$\lambda_D \text{ solves } V_D = \left(CF_D \times \pi_D\right) e^{-\lambda_D}\ e^{-r_f}\ , \tag{9.8}$$

which gives us:

$$\lambda_D = -\ln\left(\frac{\left(V_D\ e^{r_f}\right)}{CF_D\ \pi_D}\right) = -\ln\left(\frac{\$4.58\,e^{4\%}}{\$8.40\,\frac{1}{2}}\right) = -12.6\%.$$

The risk premium λ_U is, as expected, significantly higher than the project risk premium of 3%. The risk premium λ_D is negative! This fits with our earlier discussion in which we pointed out that receiving cash flows in the down state is a type of insurance policy for which a person is willing to pay a premium instead of charging a discount. Although the expected cash flow is $4.20, the value of the claim to the cash flow is $4.58.

Paying a premium for insurance against the bad states is a natural feature of all asset pricing models. It is only a surprise to students who are used to estimating risk premia only for typical projects, as opposed to the wide range of risk premia one finds on various derivative projects including derivatives that pay-off primarily in down states. This is a good example of where the traditional risk-adjusted discount rate methodology invites error in the face of specially packaged cash flow patters, while the risk-neutral methodology gets it right as a matter of course.

Indeed, it is the advantage of using state prices that one can readily and easily value packages of cash flows with **any** structure that is derivative of the underlying state variable. This is the key to this methodology's growing acceptance. Once we have backed out the two forward state prices from the market value of a bond and of a project, valuing many other projects with different cash flow patterns is simple.

For example, suppose we borrow $5 to invest in the project. We will call this the levered project and denote it's value as V_{LP}. The net cash flow on the upside is CF_{LU}=$7.84, which is the original cash flow less the debt repayment at the risk-free rate of interest. The net cash flow on the downside is CF_{LD}=\$3.20. The value of the levered project is calculated as:

$$V_{LP} = \left(CF_{LU} \times \pi_U \times \phi_U + CF_{LD} \times \pi_D \times \phi_D\right) e^{-r_a} = \$5.00.$$

The implicit risk premium on the levered project is:

$$\lambda_{LP} \text{ solves } V_{LP} = \left(CF_{LU} \times \pi_U + CF_{LD} \times \pi_D\right) e^{-\lambda_{LP}}\, e^{-r_a},$$

which gives us:

$$\lambda_{LP} = -\ln\left(\frac{\left(V_{LU}\, e^{r_f}\right)}{CF_{LU}\ \pi_U + CF_{LD}\ \pi_D}\right) = -\ln\left(\frac{\$5.00\, e^{4\%}}{\$7.84\,\frac{1}{2} + \$3.20\,\frac{1}{2}}\right) = 5.8\%.$$

The Risk Neutral Distribution

At the top of the previous section, we made an explicit assumption about the probability that the underlying factor, *S*, would move up or down. That assumption was unnecessary. In the series of steps proceeding from the given values of a bond and a project to the derived values of the derivatives, the expressions for the forward state prices are always accompanied by the expression for the probability. Because we assumed a given probability, we could solve for the forward state price. Had we assumed a different pair of probabilities, we would have calculated a different pair of forward state prices. But we would NOT have calculated different values for the two derivative assets! No matter what derivative asset we consider, the calculated value is invariant to changes in the assumed probability.

Let’s confirm that by changing the probabilities and recalculating the results. The first column of Table 9.2 shows the earlier calculations. At the top are the inputs: the cash flows if the factor goes up or down, the market price of the project, the probabilities of the factor going up or down, and the risk-free interest rate. At the bottom of the column are three sets of outputs. The first outputs are the two forward state prices. We also record the product of the probability and the forward state price. The second outputs are the values for the derivative assets. The third outputs are the risk-premium implied for the project by its market value and the risk-premia implied by the calculated market values of the derivative assets.

The second column of Table 9.2 show what happens when we change the assumed probabilities. We keep the cash flows in each state the same as before, and keep the market value of the original project the same. We then recalculate the forward state prices, the values of the derivative assets and the implied risk premia for the two derivatives. Notice that the calculated values for the derivative assets do NOT change!

Take note that the calculated forward state prices do change. So do the implied risk premia for the various derivative assets, including the original project. What matters for valuation is the product of the probability and the forward state price. Looking across our two columns one can see that this product is the same. If we change the assumed probability, we change the calculated forward state price, but just so much as to keep the product constant. That is why our original assumption about the probabilities was an unnecessary one.

This is an important concept to grasp. What is going on here is that we started from a given market value for the project. The market value is already a product of (i) the probability of high or low cash flows, and (ii) the risk-premium applied. If we change the assumed probability, but keep the market value constant, then the risk-premium being applied must change.

The product of the probabilities and the forward state prices are known as the risk-neutral probabilities which we denote by π^*_U and π^*_D. In our numerical example, the risk-neutral probabilities are:

$$\pi^*_U = \pi_U \times \varphi_U = 43.3\% \qquad (9.9)$$

$$\pi^*_D = \pi_D \times \varphi_D = 56.7\%. \qquad (9.10)$$

The rationale behind the terminology is as follows. Suppose the forward state prices were the same for both the up and the down states, $\varphi_U=\varphi_D=1$. This is what they would be if the risk-premium were zero, reflecting the fact that investors were risk-neutral, at least with respect to the risk factor S. Probabilities calculated this way, under this counterfactual assumption, are called risk-neutral probabilities.

We work with the risk neutral probabilities the same way we work with the true probabilities. We refer to the risk-neutral probability distribution and can speak of the risk-neutral expected value of the underlying risk factor, which we denote $E^*[S_1]$. The risk-neutral expected value isn't really an expected value. It is a discounted value. Whenever someone refers to the risk-neutral expectation, what they are really referring to is a properly discounted value. We distinguish the risk-neutral expected value from the expected value calculated using the true probabilities, which we denote $E[S_1]$.

We can illustrate the difference between the risk-neutral expected value and the true expected value by referring back to the numerical example. So far in our numerical example we never specified a value for S, nor for S_U, nor for S_D. Suppose now that we specify

$$S = 2.667$$

$$S_U = 4.347$$

and,

$$S_D = 4.347$$

This implies that:

$$\mathrm{E}[S_1] = \left(\pi_U \times S_U + \pi_D \times S_D\right) = 3.573,$$

and,

$$\mathrm{E}^*[S_1] = \left(\pi_U^* \times S_U^* + \pi_D^* \times S_D^*\right) = 3.469.$$

Note that $S \neq \mathrm{E}[S_1]$ and $S \neq \mathrm{E}^*[S_1]$. Just as there is no general automatic relationship between the current underlying factor price, S, and its expectation at a future date, nor is there such an automatic relationship with its risk-neutral expectation at a future date.

A relationship between the current underlying factor price and its expectation at a future date does arise for specific types of factors. In particular, the current price of a stock must equal the risk neutral expectation of the future stock price, inclusive of all dividends. This can be generalized to any financial asset, where the concept of dividends must be generalized accordingly to incorporate all forms of realized return. It can also be generalized to commodity prices where the concept of convenience yield must be developed to play a role as one form of realized return and costs of storage must be netted out from the realized return. While these are terribly important special cases, they remain special cases. Students who first see the risk neutral distribution developed for stock prices may err when they attempt to understand the risk neutral distribution for other types of factors unless they grasp what is special about stocks.

The Forward Price as a Certainty-Equivalent Price

Forward prices play an important role in the practical implementation of the risk-neutral method and so it is worth examining forward prices in a little more detail. In the coming analysis, we assume that our underlying risk factor, S, is a price. It may be a price for a commodity, such as oil, or it may be the price of an asset, such as a share of stock or a bond. Denote by F_t the forward price quoted today for delivery of S one-period from now. A forward contract is a linear gamble on the price of the factor S at date t+1. If it turns out that the price S_{t+1} equals F, then the realized profit on the forward contract is zero. For every $1 that the price S_{t+1} is above F, the realized profit is greater by $1, and for every $1 that the price S_{t+1} is below F, the realized profit is less by $1. By definition, the forward price is the certainty-equivalent for this linear gamble:

$$F_t = CEQ_t\left[S_{t+1}\right].$$

We can translate this into the risk-adjusted discounting framework by writing:

$$F_t = CEQ_t\left[S_{t+1}\right] = E\left[S_{t+1}\right]e^{-\lambda}, \tag{9.11}$$

where λ is the one-period risk-premium on a payoff linear in S. In a binomial tree we can expand this further as:

$$F_t = CEQ_t\left[S_{t+1}\right] = E\left[S_{t+1}\right]e^{-\lambda} = \left(\pi_U\, S_{t+1,U} + \pi_D\, S_{t+1,D}\right)e^{-\lambda}. \tag{9.12}$$

In the risk-neutral valuation framework we can write this as:

$$F_t = CEQ_t\left[S_{t+1}\right] = E_t^*\left[S_{t+1}\right] = \left(\pi_U^*\, S_{t+1,U} + \pi_D^*\, S_{t+1,D}\right). \tag{9.13}$$

From equation 9.12 we see that if we know the true probabilities and the risk-premium, then we can calculate the forward price. If we know the forward price, we cannot back out the risk-premium without making an assumption on the true probabilities, and we cannot back out the true probabilities without making an assumption on the risk-premium. Equation 9.12 puts a restriction on the combination of the true probabilities and the risk-premium. Rewriting the equation as 9.13 we see that if we avoid taking a stand on the specific combination of the true probabilities and risk-premium. If we know the forward price we can back out the risk-neutral probabilities without making any additional assumptions.

9.3 Implementing Risk-Neutral Valuation

Solving for the Risk Neutral Probability in Binomial Trees

In Chapter 6 we showed how a binomial tree could be constructed to model the risk of a factor S. Here we show how to add pricing for risk on top of that model.

Viewed period-by-period, the general mechanics apply to any binomial tree, whether it is used to model a random walk or a mean reverting process or any other risk structure. The resulting valuations of assets with payoffs in distant states will vary according to the assumed factor risk structure since, as we saw in Chapter 6, the risk at distant horizons varies under the different models. We begin by showing the period-by-period mechanics, and then later we discuss the implication over longer horizons.

Random Walk Example

Recall from Chapter 6 the binomial tree shown in Figure 6.7. The example began with a factor price of S_0=\$10 so that ln(S_0)=2.303. We assumed that the factor evolved as a random walk with a drift parameter μ=7% and a volatility σ=22%. We constructed the next period outcomes to be equally distributed around the expected value E[ln(S_1)]=ln(S_0)+(μ –½σ^2)=2.348, so that ln(S_{1U})= E[ln(S_1)]+σ=2.568 and ln(S_{1D})= E[ln(S_1)]–σ=2.128, giving S_{1U}=\$13.04 and S_{1D}=\$8.40. By specifying that the two outcomes be centered around the expected value, we have effectively set π_U=π_D=½.

On top of this structure we now want to add the risk pricing. We can do that either by (i) specifying the one-period forward price on S, or (ii) specifying λ, the one-period risk-premium on a linear gamble on S. If we do (i), it implies (ii), and if we do (ii) it implies (i). In this calculation we specify λ=5% and back out F_t as follows:

$$F_t = E\left[S_{t+1}\right]e^{-\lambda} = \left(\pi_U \; S_{t+1,U} + \pi_D \; S_{t+1,D}\right)e^{-\lambda} = \$10.20. \tag{9.14}$$

Having defined the forward price, we can back out the risk-neutral probabilities using the equation:

$$\left(\pi_U^* S_U + \pi_D^* S_D\right) = F_t = \$10.20 \tag{9.15}$$

which can be rewritten and solved for π_U^*

$$\pi_U^* = \frac{\left(\pi_U S_U + \pi_D S_D\right)e^{-\lambda} - S_D}{\left(S_U - S_D\right)} = \frac{CEQ[\tilde{S}_1] - S_D}{\left(S_U - S_D\right)} = 39\% \,. \tag{9.16}$$

We assume that the risk pricing structure of the tree is constant throughout, meaning that the one-period risk premium is fixed at λ=5%. This means that the risk-neutral probability will be constant throughout the tree at π^*_U=39%.

Using this risk-neutral probability instead of the original probability essentially defines a new binomial model as shown in Figure 9.4. All of the nodes in this tree are the same as the nodes of the original tree shown in Figure 6.7: S_0=\$10, S_{1U}=\$13.04 and S_{1D}=\$8.40. However, with the altered probabilities, we have changed the expected price at each node: $E^*[\ln(S_1)]=2.299<2.348=E[\ln(S_1)]$. We say that the expected price under the risk neutral probabilities is less than the expected price under the true probabilities. The difference represents the discounting for risk implicit in the risk neutral probabilities.

Having constructed a new binomial tree with the correct risk-neutral probability at every branching, we are now capable of valuing any asset with cash flows contingent on the underlying risk factor.

Before moving forward to do the valuations, let's repeat the exercise of finding the risk-neutral distribution, but this time on one step of a binomial tree constructed to describe a mean reverting process. This will make clear that the mechanics of solving for the risk-neutral distribution are the same regardless of the process being modeled.

Mean Reversion Example

We take the example from Chapter 6, Part B on Mean Reverting Processes. The example began at the same starting point as the previous example, with a factor price S_0=\$10 so that $\ln(S_0)=2.303$. We assumed that the mean log price to which the process reverts is $\ln(\bar{S})=2.079$ so that $\bar{S}$=\$8. The rate of mean reversion is $\kappa=0.75$ and the volatility is $\sigma=22\%$. In our default methodology for constructing binomial trees, we constructed the next period outcomes to be equally distributed around the expected value $E[\ln(S_1)]=\ln(S_0)+e^{-\kappa}(\ln(S_0)-\bar{S})=2.185$, so that $\ln(S_{1U})=E[\ln(S_1)]+\sigma=2.405$ and $\ln(S_{1D})=E[\ln(S_1)]-\sigma=1.965$, giving S_{1U}=\$11.08 and S_{1D}=\$7.13. This methodology necessarily sets $\pi_U=\pi_D=½$ for the mean reverting tree, too.

On top of this structure we now add the risk pricing by specifying $\lambda=5\%$ and backing out F_t:

$$F_t = E[S_{t+1}]e^{-\lambda} = (\pi_U\, S_{t+1,U} + \pi_D\, S_{t+1,D})e^{-\lambda} = \$8.69.$$

The forward price in the mean reverting process is less than the forward price in the random walk. This is because we started at an initial price, S_0=\$10.00 that was above the price to which we assumed the price reverted, $\bar{S}$=\$8.00. Therefore the mean reverting process adds a negative drift within this period. Had we started at an initial price below the mean level, there would have been a positive drift.

Having defined the forward price, we can back out the risk-neutral probabilities using the equation:

$$\left(\pi_U^* S_U + \pi_D^* S_D\right) = F_t = \$8.69$$

which can be rewritten and solved for π_U^*

$$S_D + \pi_U^*\left(S_U - S_D\right) = \left(\pi_U S_U + \pi_D S_D\right)e^{-\lambda}$$

$$\pi_U^* = \frac{\left(\pi_U S_U + \pi_D S_D\right)e^{-\lambda} - S_D}{\left(S_U - S_D\right)} = \frac{CEQ[\tilde{S}_1] - S_D}{\left(S_U - S_D\right)} = 39\% .$$

Although the forward price in the mean reverting binomial step differs from the forward price in the random walk step, the risk-neutral probabilities are the same.

If we assume that the structure of the tree is essentially constant throughout, then this is the risk-neutral probability at each branching.

Using this risk-neutral probability instead of the original probability essentially defines a new binomial model as shown in Figure 9.5. All of the nodes in this tree are the same as the nodes of the original tree shown in Figure 6.X: S_0=\$10, S_{1U}=\$11.08 and S_{1D}=\$7.13. However, with the altered probabilities, we have changed the expected price at each node: E*[ln(S_1)]=2.139<2.185= E[ln(S_1)].

Valuation in Binomial Trees Using The Risk Neutral Distribution

Now that we know how to derive the risk-neutral probability for a binomial tree, we are ready to value any series of cash flows through time that are contingent on the underlying risk factors, *S*. This is where we first see the advantage that the risk-neutral method has over the risk-adjusted method. Once we have defined the structure of the tree and derived the risk-neutral probability, which we need only do once, the algorithm for valuing a contingent cash flow is the same for all cash flows. There is no need to devise the algorithm for the cash flow.

Let $C_t(S)$ be a cash flow received at time *t* and contingent on the realization of the underlying risk factor, *S*. In the traditional risk-adjusted discount rate method, the present value of

C is given by calculating the expected value of a cash flow and applying a risk-adjusted discount rate:

$$PV_0\left[\tilde{C}_t\right] = E_0\left[\tilde{C}_t\right] e^{-r_a t}, \qquad (9.17)$$

where $\tilde{C}_t$ is the risky contingent cash flow at time, t, determined by the function $C_t(S)$. The expectation uses the true distribution. The problem is that the correct risk-adjusted discount rate depends upon the contingent structure of the cash flows, as we have already seen. There is no universal algorithm that determines the correct risk-adjusted rate.

In the risk-neutral methodology we mimic the traditional valuation formula, but with minor differences. We take the expected value of the cash flow, but we use the risk neutral probability distribution. This expected value will be lower than had we used the true distribution. Then we apply the risk-free discount rate:

$$PV_0\left[\tilde{C}_t\right] = E_0^*\left[\tilde{C}_t\right] e^{-r_f t}. \qquad (9.18)$$

All of the discounting for risk occurs through the reduction in the expected value produced by the use of the risk neutral distribution. The risk-neutral distribution, remember, is really a discounted valuation.

Example 1. Symmetric Risk, Single-Period

A simple case is a cash flow earned at t=1 and proportional to the underlying risk factor: $C_U = qS_U$ and $C_D = qS_D$, where q is the proportionality constant. The present value is

$$PV_0\left[\tilde{C}_1\right] = E_0^*\left[\tilde{C}_1\right] e^{-r_f} = q\left(\pi_U^* S_U + \pi_D^* S_D\right) e^{-r_f} .$$

Assuming the same values for the inputs as used in our earlier example of the random walk, we have S_{1U}=\$13.04 and S_{1D}=\$8.40, π_U=50%, and π_U^*=39%. We also assume a risk-free rate, r_f=4%. Then the present value is

$$PV_0\left[\tilde{C}_1\right] = q\left(39\%(13.04) + 61\%(8.40)\right)0.961 = q\ 9.80 .$$

Note that this could just as easily have been calculated using the risk adjusted discount rate, $r_a = r_f + \lambda$=4%+5%=9%:

$$PV_0\left[\tilde{C}_1\right] = E_0\left[\tilde{C}_1\right] e^{-r_a} = q\left(\pi_U S_U + \pi_D S_D\right) e^{-r_a} .$$

$$=q\left(50\%(13.04)+50\%(8.40)\right)0.914 = q\ 9.80\,.$$

In this first example, the risk-adjusted method and the risk neutral method give the same answer. Indeed, all this simple one-period case really does is repeat the definition of the risk neutral probability. But in the following examples we shall see that the risk-neutral methodology proves both easy to use and offers telling insights about the risk embedded in various cash flow patterns.

Example 2. Skewed Risk, Single-Period

Consider a payoff like a call option on *S*: $C_U = X$ and $C_D = 0$. We can think of $X=q(S_U-K)$, where *q* is the number of options and *K* is an exercise price. Obviously, by adjusting *q* and *K* we can arrive at any value of *X*. But there is no need to think of the payoff *X* as coming from a call option. This is just a risky payoff with a distribution that is skewed relative to the underlying risk factor, *S*.

Using the risk neutral valuation methodology, we have,

$$PV_0\left[\tilde{C}_1\right] = E_0^*\left[\tilde{C}_1\right]e^{-r_f} = \pi_U^*\ X\ e^{-r_f}\,,$$

Assuming the numbers from our random walk example, so that π_U^*=39% and r_f=4%, this reduces to,

$$PV_0\left[\tilde{C}_1\right] = 39\%(X)0.961 = 0.373\,X\ .$$

In this case, we cannot replicate the result using the standard risk-adjusted discount rate, where $r_a=r_f+\lambda=4\%+5\%=9\%$:

$$PV_0\left[\tilde{C}_1\right] \neq E_0\left[\tilde{C}_1\right]e^{-r_a} = \pi_U\ X\ e^{-r_a}$$

$$=50\%\,(X)\,0.914 = 0.454\,X\ > 0.373\,X\ .$$

We could, of course, find some other risk-adjusted discount rate which would give the correct answer. After all, we can just solve the one equation for the one unknown variable, telling us that the risk-adjusted rate that produces the correct value is 29.4%. But that assumes the correct answer to the valuation problem, instead of producing it!

Note that the implicit risk-adjusted discount rate on the skewed payoff is higher than the rate used for the underlying risk factor and for any linear payoff structure—29.4% as compared to

9%. This is because the cash flow in skewed payoff structure is “riskier” than the underlying risk factor. We saw this in Chapter 6 when we showed that the risk of a call option was always greater than the risk of the underlying stock, with the amount of extra risk depending upon how in or out-of-the-money is the option.[2]

Example 3. Two-Period, Compounded Risk

We now consider a case like Example #1 above where the cash flow earned is proportional to the underlying risk factor, but we assume that the cash flow is earned only in the second period, t=2. The figure below shows the payoffs: $C_{UU}=qS_{UU}$, $C_{UD}=C_{DU}=qS_{UD}$, and $C_{DD}=qS_{DD}$. Assuming the same values for the inputs as used in our earlier example of the random walk, and extending it to the second period, we have S_{UU}=\$17.02, $S_{UD}=S_{DU}$=\$10.96, S_{DD}=\$7.06. Of course, π_U=50%, and π_U^*=39%, and r_f=4%. The present value is

$$PV_0\left[\tilde{C}_2\right]=E_0^*\left[\tilde{C}_2\right]e^{-2r_f}=q\left[\pi_U^*\left(\pi_U^* S_{UU}+\pi_D^* S_{UD}\right)+\pi_D^*\left(\pi_U^* S_{DU}+\pi_D^* S_{DD}\right)\right]e^{-2r_f}$$

$$= q\ 9.61.$$

Once again we have a result that could just as easily have been calculated using the traditional risk-adjusted discount rate method with $r_a=r_f+\lambda$=4%+5%=9% as follows:

$$PV_0\left[\tilde{C}_2\right]=E_0\left[\tilde{C}_2\right]e^{-2r_a}=q\left[\pi_U\left(\pi_U S_{UU}+\pi_D S_{UD}\right)+\pi_D\left(\pi_U S_{DU}+\pi_D S_{DD}\right)\right]e^{-2r_a}$$

$$= q\ 9.60.$$

So in this particular case, for this symmetric distribution with risk that grows linearly with time, the risk neutral valuation is equivalent to discounting by the compounded risk-adjusted discount rate.

Example 4. Two-Period, Non-compounded Risk

We now consider another example in which the cash flow earned at t=2 is proportional to the underlying risk factor: $C_{UU}=qS_{UU}$, $C_{UD}=qS_{UD}$, $C_{DU}=qS_{DU}$, and $C_{DD}=qS_{DD}$. However, in this case

[2] For the special case of payoff structures that are pure European options on the underlying stock there is a simple formula relating the Beta of the option to the Beta of the stock, depending upon how in the money is the option and how long to maturity. This can, of course, then be tied back to the required rate of return on the option as a function of the required rate of return on the stock. See Cox, John and Mark Rubenstein, Option Markets, Englewood Cliffs: Prentice Hall, section 5-5.

the underlying risk factor, S, is mean reverting. Therefore, we have S_{UU}=\$11.62, S_{UD}=\$7.49, S_{DU}=\$9.44, S_{DD}=\$6.08. Recall that our standard methodology for constructing a binomial tree for a mean reverting process does not yield a recombining tree, so the second period has four distinct nodes. As before, π_U=50%, and π_U^*=39%, and r_f=4%. Using the risk neutral method, the present value is

$$PV_0\left[\tilde{C}_2\right] = E_0^*\left[\tilde{C}_2\right]e^{-2r_f} = q\left[\pi_U^*\left(\pi_U^* S_{UU} + \pi_D^* S_{UD}\right) + \pi_D^*\left(\pi_U^* S_{DU} + \pi_D^* S_{DD}\right)\right]e^{-2r_f}$$

$$= q\ 7.46.$$

What happens if we try to apply the risk-adjusted discount rate, compounding for the two year horizon:

$$PV_0\left[\tilde{C}_2\right] \neq E_0\left[\tilde{C}_2\right]e^{-2r_a} = q\left[\pi_U\left(\pi_U S_{UU} + \pi_D S_{UD}\right) + \pi_D\left(\pi_U S_{DU} + \pi_D S_{DD}\right)\right]e^{-2r_a}$$

$$= q\ 7.23 < q\ 7.46.$$

In this case, compounding the discount rate produces too low of a value. Because of mean reversion, the total risk in the second period cash flows is not equal to twice the risk in the first period cash flows.

Examples #1 and #3 illustrate that where the cash flow pattern fits the strict assumptions behind the traditional risk-adjusted discount rate methodology, the risk-neutral method arrives at the same result. Examples #2 and #4 illustrate cases that do not fit the traditional risk-adjusted discount rate method. Nevertheless, the risk-neutral method can be applied without amendment. It is this robustness that makes the risk-neutral method such a powerful and valuable tool.

Figure 9.1
Two Methods for Discounting

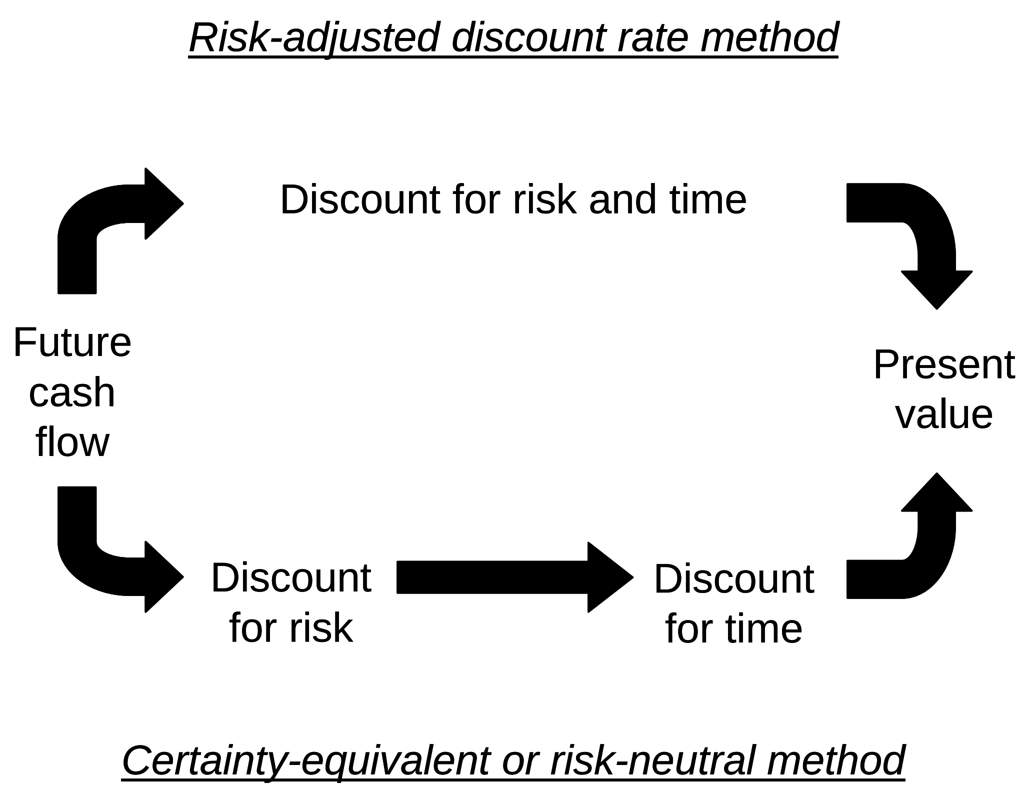

Figure 9.2
Project Cash Flows

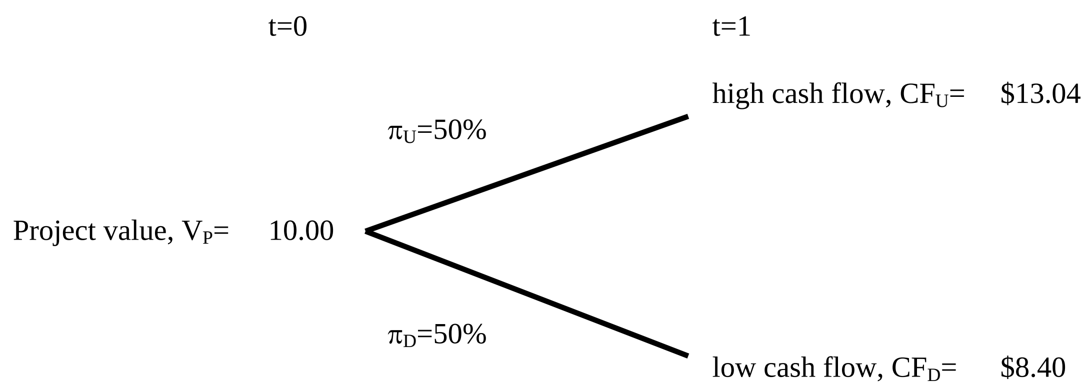

Figure 9.3
Derivative Cash Flows

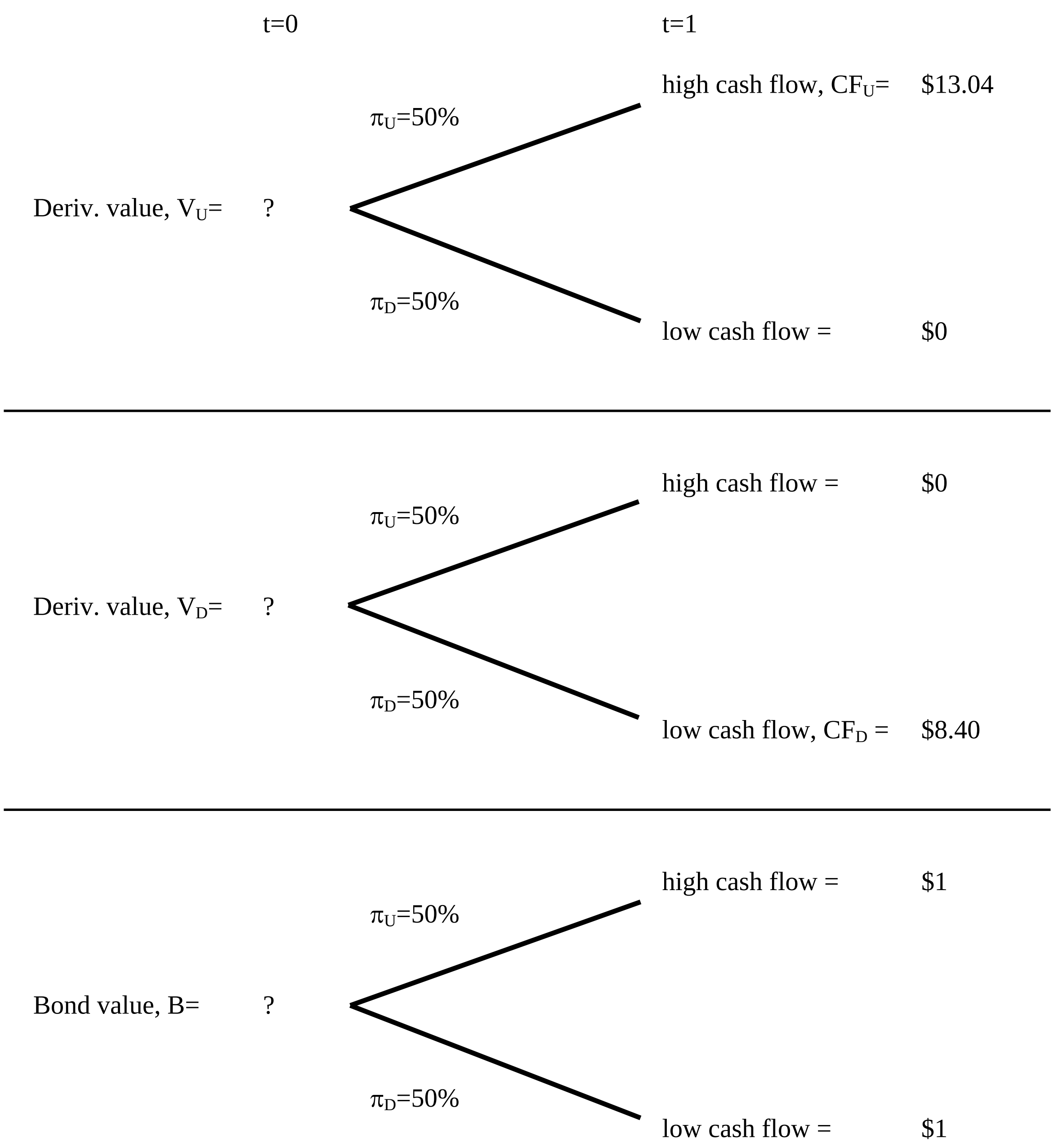

Figure 9.4
Revised Binomial Tree: Risk-Neutral Probabilities for the Random Walk

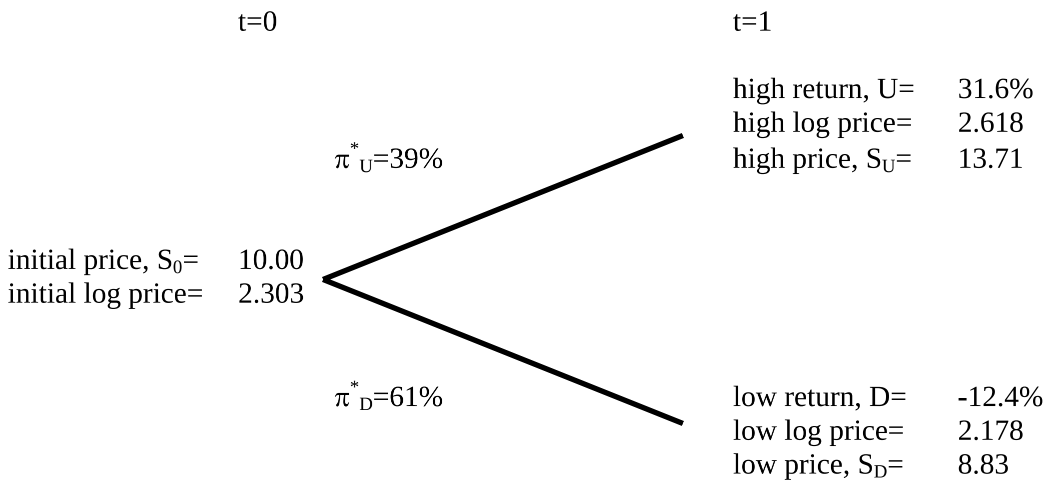

Figure 9.5
Revised Binomial Tree: Risk-Neutral Probabilities for Mean Reversion

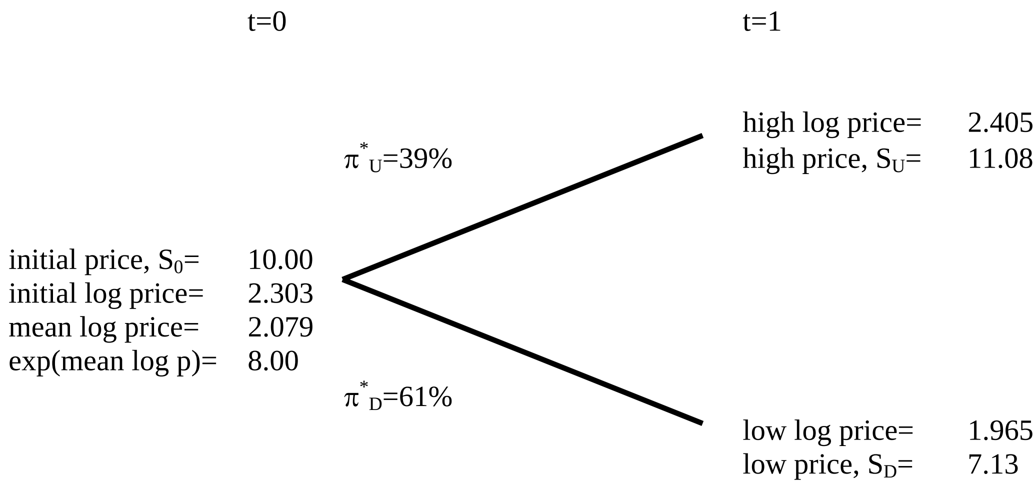

Table 9.1
The Hejira Oil Corp: Two Alternative Methods for Valuing Production

Method #1: Risk Adjusted Discount Rate Method -- simultaneously adjust for risk and time

Year	1	2	3	4	5
Forecasted Production (000 bbls)	10,000	9,000	8,000	7,000	6,000
Forecasted Spot Price ($/bbl) -- current price $38	35.00	33.50	32.75	32.38	32.19
Forecasted Spot Revenue ($ 000)	350,000	301,500	262,000	226,625	193,125
Risk-adjusted Discount Rate, r_a	10.0%	10.0%	10.0%	10.0%	10.0%
Risk-adjusted Discount Factor	0.9048	0.8187	0.7408	0.6703	0.6065
PV ($ 000)	316,693	246,847	194,094	151,911	117,136
Total PV Spot Sales ($ 000)	1,026,682				

Method #2: Certainty Equivalent or Risk-Neutral Method -- separately adjust for risk then for time

Forecasted Spot Revenue ($ 000)	350,000	301,500	262,000	226,625	193,125
Certainty Equivalence Risk Premium, λ	6.0%	6.0%	6.0%	6.0%	6.0%
Certainty Equivalence Factor	94.2%	88.7%	83.5%	78.7%	74.1%
Certainty Equivalent Revenue	329,618	267,407	218,841	178,270	143,071
Riskless Discount Rate, r_f	4.0%	4.0%	4.0%	4.0%	4.0%
Riskless Discount Factor	0.9608	0.9231	0.8869	0.8521	0.8187
PV ($ 000)	316,693	246,847	194,094	151,911	117,136
Total PV Spot Sales ($ 000)	1,026,682				

Table 9.2
Derivative Valuation for an Alternative Probability Assumption

	Original	Revised
Inputs		
CF_U	\$13.04	
CF_D	\$8.40	
V_P	\$10.00	
π_U	50%	60%
π_D	50%	40%
r_f	4.0%	
Outputs		
φ_U	0.866	0.721
φ_D	1.134	1.418
$\pi^*_U=\pi_U\varphi_U$	0.433	0.433
$\pi^*_D=\pi_D\varphi_D$	0.567	0.567
V_U	\$5.42	\$5.42
V_D	\$4.58	\$4.58
V_{LP}	\$5.00	\$5.00
λ_P	3.0%	7.2%
λ_U	14.4%	32.7%
λ_D	-12.6%	-34.9%
λ_{LP}	5.8%	13.9%