NOTES ON CALCULUS AND UTILITY FUNCTIONS

These notes have three purposes:

1) To explain why some simple calculus formulae are useful in understanding utility maximization, profit maximization and other problems involving marginal analysis.

2) To give you the three or four calculus formulae you will need for 11.203 so that you can have them handy and use them when needed.

3) To give you a few practice problems so you can determine whether you understand the material.

In terms of notation, we will use letters like x, y and z to denote variables in an equation and a,b, and c to denote constants.

I. Why do we need calculus at all?

The part of calculus we need is first derivatives. Begin with a mathematical function describing a relationship in which a variable, y, which depends on another variable x:

$$y = f(x)$$

Then the symbol for the first derivative of this relationship is: dy/dx

The derivative expresses the number of units of change in y caused by a one unit change in x. For example, suppose that:

y was the number of gallons of gasoline sold per day in Cambridge gas stations.

x was the average price per gallon of gasoline expressed in cents.

and   f( ) was the demand function which expressed gasoline sales as a function of the price per gallon.

If dy/dx equaled -500, it would mean that each penny increase in the average price of gasoline would lower the quantity demanded by 500 gallons per day. (Note that this is related to elasticity but it is not identical since this derivative is a rate expressed in raw numbers while elasticity is a rate expressed in percentage changes.)

A derivative, then, is a rate of change of one variable with respect to another – i.e. the

slope. As long as the relationship, f( ) is a   straight line, this is old news. What will be new here is when we deal with curves. Along a straight line, the slope is always the same. Along a curve, a slope is constantly changing – and so rather than write down the slope as a single number, we have to write down the slope as a new function of where we are on the curve. This function is the derivative. .

Even if you have not seen this concept before, it is not as foreign as it seems. To see this, consider the first formula you will need, the equation for a straight line that runs through the origin:

  1) If y = bx, dy/dx = b (where b is a constant)

  so that if y = 17x, dy/dx = 17.

We can verify this by first looking at the equation itself: if x = 1, then y = 17; if x = 2, then y = 34; if x = 3, then y = 51; and so on. This means that when we increase x by 1, we increase y by 17 or, in different words, the change in y brought about per one unit change in x (or dy/dx) is 17.

Further, if you graph the equation y = 17x, it will be a straight line with a slope of 17. These are different ways of saying the same thing.

A second formula:

  2) If y = a (where a is a constant), dy/dx = 0.

This derivative of a constant (no x involved) is a special case of equation (1). Here is the logic. The line y = 35 is a flat line, parallel to the x axis at height 35. Geometrically, we know that if the line is parallel to the x axis it has a slope of zero. Logically, we know that whether we change x (i.e. move out the x axis) by one unit, two units or a million units, y will still be 35 - again, a zero slope or zero change in y no matter how much x changes.

 A third formula – this time in words:

  3) If y is equal to the sum of two terms, then dy/dx is the sum of:
  (the derivative of the first term) plus (the derivative of the second term).

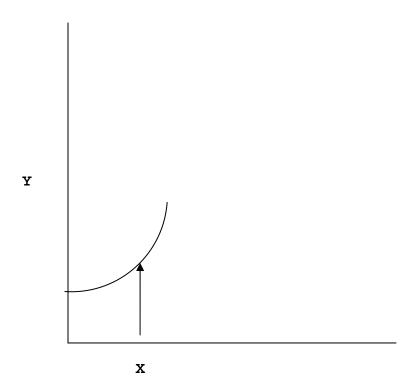 This means that if y = a + bx, dy/dx = 0 + b = b.

So the derivative of y = 35 + 17x is just 17 - i.e. your old friend, the slope of a straight line.

We can think of these straight lines (i.e. linear functions) as a special case of curves (non-linear

functions. With respect to slope, a curve like:

$$y = 35 + 17x^2$$

raises a new problem.. A graph of the curve looks something like this:

At any particular point on the curve (or, said differently, for any particular value of x – e,g, the point identified by the arrow), this curve still has a slope – a tangent to the curve at that particular point. But unlike a straight line, the slope of this curve changes from one value of x to another. Prove this to yourself by calculating the change in y when x goes from 1 to 2 and again when x goes from 50 to 51. In each case, x will have increased by 1 but the change in y will be much different in the two cases. It is for these situations that we need to deal with slope (the rate of change of y for each value of x) using calculus.

II. Where is the derivative useful in economics?

One example of its usefulness is the case of the consumer maximizing her or his utility.

Begin with a consumer who has a utility function of the form:

U = U(Pizza, Beer)

- i.e. their level of utility or satisfaction depends upon the amount of pizza and of beer they consume.

The need for derivatives in this situation comes from the following reasoning. Suppose you have allocated some, but not all, of your income and you are now considering how to allocate your next dollar. You can analyze the situation in the following way.

Should I spend this next dollar on pizza or beer?

If I spend the dollar on pizza, how much extra pizza can I get? (Let's say l/8 extra pizza.)

And if I consume the extra 1/8 of pizza how much extra utility will I get? (i.e. the change in utility per 1/8 of pizza or, said differently, the marginal utility of an extra 1/8 of pizza)

If I spend the dollar on beer, how much extra beer can I get? (Let's say 1 glass.)

And if I consume the extra glass, how much extra utility will I get from that? (i.e. the marginal utility from an extra glass of beer)

Having made these calculations, I will invest my dollar in the commodity - pizza or beer - that increases my utility the most. One way of thinking of the consumer's choice problem is to allocate the budget, dollar by dollar, in just this fashion.

Because of this logic, it is important to be able to know the amount of extra utility that comes from a small additional amount of a commodity. That amount is known as the marginal utility of the commodity and is identical to the first derivative of the utility function with respect to that commodity.[1]

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[1] More precisely, this is called a first partial derivative because the derivative is taken with respect to one variable - either pizza or beer - while the other variable as a constant. Thus the marginal utility of beer is the first derivative of the utility function with respect to the quantity of

We need calculus for this problem because, as you might suspect, the marginal utility of either commodity is not constant: the marginal utility (i.e. the increase in utility) you get from the first 1/8 of pizza is probably a lot higher than the marginal utility you get from the 97'th 1/8 of pizza (experienced while in an ambulance on the way the ER).

III. The specific examples of derivatives we will need.

1. A constant term: If y = a, then dy/dx = 0   where a is a constant. (We already saw this.)

2. The Linear Function (straight line):
    If y = a + bx, then dy/dx = b   where a, b are constants. (We already saw this too.)

  3. The Quadratic and Cubic functions:

    If $y = bx^2$, then dy/dx = 2bx   where b is a constant.

    If $y = bx^3$, then dy/dx = $3bx^2$ where b is a constant.

    In fact, these are part of a more general formula:

    If $y = bx^a$, then dy/dx = $abx^{(a-1)}$ where the exponent "a" can be positive or negative and an integer or a fraction.

  4. The natural log function or LN( ):

    If y = bLN(x), dy/dx = b/x   where b is a constant.

(We use the natural log function – LN (   ) in part because it's first derivative is so simple to work with/)

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beer, treating the quantity of pizza as a constant.

IV. Some Practice Problems

  1) In most cases, we represent a utility function as having positive, but diminishing, marginal utility. That is, one more unit of a good increases your total utility but this increase is smaller than the increase in utility of the previous unit.

   Below are four functions. Explain why each function meets or fails to meet the two conditions.

a) $Y = 17X$

 b) $Y = 17X^2$

 c) $Y = 17X - 5X^2$

 d) $Y = 17X^{(1/2)}$ (i.e. 17 x the square root of X)

e) $Y = 17LN(X)$

In each case, it is useful to do the problem in two ways. The first is to actually take the derivative and evaluate its properties – is the derivative always positive for any value of X? Does the derivative get smaller as X gets larger? Then, using Excell, graph the function and visually inspect it to see if it satisfies the two properties.

2) Suppose your utility function for hamburgers and apples can be written:

$$U(H, A) = 17H^{.35}A^{.88}$$

and suppose you have 10 hamburgers 6 apples.

a) Set up an Excel formula using exponents to calculate your utility at U(10,6).

b) Copy the formula to an adjacent cell. In that cell, keep hamburgers fixed at ten and increase apples to 7. By how much does your utility increase over your answer in (a)?

c) Then, on a piece of paper, take the partial derivative of the utility function with respect to apples - (dU/dA) - and evaluate the partial derivative at (H = 10 and A = 6). That is, plug the values of 10 for H and 6 for A into the formula you get when you take the partial derivative. Is your answer from the derivative reasonably close to the change between the two cells in the spreadsheet? We certainly hope so.

d) Back to the spread sheet, recopy the formula to a third cell, this time holding hamburgers at 10 and increasing apples from 7 to 8. How does the increase in utility compare to the increase you obtained in the move from 6 apples to 7 apples? Reevaluate your partial derivative at (H = 10 and A = 7) to check the fit.

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