Combinatorics: The Fine Art of Counting

Week One Solutions

1. According to legend the ancient Greeks used to play soccer using a regular icosahedron for a ball, until Archimedes came along and suggested that should shave off the corners of the icosahedron to create a truncated icosahedron. This led to the modern soccer ball shape we use today which is semi-regular polyhedron with vertex degree 3 and two hexagons and one pentagon incident to each vertex.

   Compute the number of vertices, edges, and faces of the soccer ball and verify that they satisfy Euler's formula V + F – E = 2.

   *Slicing off each corner of the icosahedron replaces each vertex with a pengtagon with 5 new vertices of degree 3. An icosahedron has 12 vertices, so there are 5*12 = 60 vertices in the truncated icosahedron. 2E = dV = 3 * 60, so the there are 90 edges. The 12 new faces are all pentagons, and the 20 original triangular faces become 20 hexagons giving a total of 32 faces. 60 + 32 – 90 = 2.*

2. The complete graphs K_1, K_2, K_3, and K_4 are all planar. Prove that K_5 is not planar. What about K_n for n > 5?

   *K_5 has 5 vertices each with degree 4, so it has 5*4/2 = 10 edges. It is a connected graph and by corollary 1 of Euler's formula, every connected planar graph must satisfy $E \leq 3V - 6$, but 10 > 3*4 – 6 so K_5 cannot be connected.*

   For n > 5, note that K_n contains a copy of K_5 (many copies in fact) inside it – pick any 5 vertices and the edges between them. Thus any planar embedding of K_n would contain a planar embedding of K_5 which is not possible since K_5 is not planar.

   Alternatively, note that K_n is a connected graph with n vertices all of degree n-1, so it has n(n-1)/2 edges. For n > 5, n/2 is at least 3 so we have $E \geq 3(n-1) = 3n - 3 > 3n - 6$, which would contradict corollary 1 of Euler's formula if K_n were planar.

3. The hypercube graphs H_1, H_2, and H_3 are all planar. Prove that H_4 is not planar. What about H_n for n > 4?

   *H_4 has 2^4 = 16 vertices, each of which has degree 4 so it has16*4/2 = 32 edges. H_4 is a connected graph without any triangles (see below), so corollary 2 of Euler's formula applies. Thus. $E \leq 2V - 4$, but 32 > 2*16 – 4*

so H_4 cannot be planar. For $n > 4$, H_n contains a copy of H_4 inside it so H_n cannot be planar either.

One very useful way to think about the graph H_n is to construct H_n so that each vertex is labeled with a binary sequence of '0's and '1's which represent its coordinates in n-dimensional space – when we build H_{n+1} out of two copies of H_n we just add a '0' to all the vertex labels in one copy, and add a '1' to all the labels in the other copy. When constructed in this fashion, each edge will lie between vertices whose labels differ by exactly one bit, e.g. in H_4 vertex 0101 would be adjacent to the vertices 1101, 0001, 0111, and 0100. Changing the i^{th} bit means moving along an edge in the direction of the i^{th} dimension.

To see that H_n does not contain any triangles nor any cycles of odd length, notice that as we travel along a path starting from a vertex with some particular label, each vertex along our path has a label which differs from the starting label in some number of bits. Each step along an edge either increases or decreases the number of bits which are different by exactly 1. Only a path with an even number of edges can bring this difference back to 0 (think of a sequence of +1s and -1s that add up to 0, there must be the same number of +1s and -1s). Thus any cycle must have even length.

4. Given a regular polyhedron with V vertices of degree d, F faces of degree c, and E edges, truncating the polyhedron will result in a semi-regular polyhedron. Let V', F', E', and d' be the vertices, faces, edges and vertex degree of the truncated polyhedron and let c_1' and c_2' be the degrees of the faces of the truncated polyhedron.

   Find simple expressions for V', F', E', d', and c_1' and c_2' in terms of V, F, E, c, and d. Use these expressions to compute the result of truncating each of the five regular polyhedra.

   $V' = dV$ $F' = F + V$ $E' = E + dV$ $d' = 3$ $c_1' = 2c$ $c_2' = d$

Polyhedron	***F'***	***V'***	***E'***	***d'***	***c_1', c_2'***
truncated tetrahedron	*4+4=8*	*3*4=12*	*6+12=18*	*3*	*6,3*
truncated cube	*6+8=14*	*3*8=24*	*12+24=36*	*3*	*8,3*
truncated octahedron	*8+6=14*	*4*6=24*	*12+24=36*	*3*	*6,4*
truncated dodecahedron	*12+20=32*	*3*20=60*	*30+60=90*	*3*	*10,3*
truncated icosahedron	*20+12=32*	*5*12=60*	*30+60=90*	*3*	*6,5*

5. An extreme way to truncate a regular polyhedron is to slice off the vertices at a depth which bisects the edges creating a single vertex at the center of each edge, rather than two vertices along each edge as in a normal truncation. The resulting polyhedron will be different from than the one obtained by the normal truncation process, but it will produce either a semi-regular or in one case a regular polyhedron.

   As above, find simple expressions for V', F', E', d', and c_1' and c_2' in terms of V, F, E, c, and d. Use these expressions to compute the result of truncating each of the five regular polyhedra in this fashion. How many new semi-regular polyhedra can be obtained in this way?

$V' = dV/2$ $F' = F + V$ $E' = dV$ $d' = 4$ $c_1' = c$ $c_2' = d$

Each vertex in the resulting polyhedron will have degree 4 with two opposing pairs of congruent faces incident to each vertex. Truncating the tetrahedron in this extreme way simply yields an octahedron since the opposing pairs are both triangles.

The extreme truncations of the cube and octahedron both result in the same polyhedron known as the cuboctahedron. Similarly the extreme truncation of the dodecahedron and icosahedron both result in the same polyhedron, known as the icosidodecahedron. Thus only two new semi-regular polyhedra are obtained.

Polyhedron	***F'***	***V'***	***E'***	***d'***	***c_1', c_2'***
cuboctahedron	*6+8=14*	*3*8/2=12*	*3*8=24*	*4*	*4,3*
icosidodecahedron	*20+12=32*	*5*12/2=30*	*5*12=60*	*4*	*3,5*

Desert

6. All of the proofs of non-planarity we have seen rely on showing that the graph in question has "too many edges" to be planar (either $E > 3V - 6$ or $E > 2V - 4$ for triangle-free graphs). While having a small number of edges is a necessary condition for a graph to be planar, it is not always sufficient.

   Give an example of a non-planar graph where $E \leq 2V - 4$.

   Add a vertex in the middle of one edge of the graph $K_{3,3}$ splitting the edge into two adjacent edges. The resulting graph has 10 edges and 7 vertices and $10 \leq 2*7 - 4$, but it is still non-planar since if we could embed it in the plane we could then remove the vertex we added and simply merge the two adjacent edges to obtain a planar embedding of $K_{3,3}$, but we have already shown that $K_{3,3}$ is not planar.

7. There are seven distinct semi-regular polyhedra which can be obtained by truncating or completely truncating the five regular polyhedra as described in problems 4 and 5 above. There are six other convex semi-regular polyhedra which together with these seven make up the thirteen Archimedean solids. There are also two infinite families of convex semi-regular polyhedra which are usually not classified as Archimedean solids.

   How many of these semi-regular polyhedra can you find? (Hint: one of the infinite families is a very familiar class of solid shapes).

   The two infinite families of convex semi-regular polyhedra are the regular prisms and anti-prisms. A regular prism consists of two parallel congruent regular polygons with edges between the corresponding vertices. A *regular prism has n+2 faces (2 n-gons and n squares), 2n vertices of degree 3, and 3n edges. A square prism is a cube.*

A regular anti-prism consists of two parallel congruent regular polygons with one rotated with respect to the other so that each vertex is positioned opposite the center of an edge on the opposing polygon. Each vertex in the anti-prism is connected to both vertices on the opposing edge, yielding 2n+2 faces (2 n-gons and 2n triangles), 2n vertices of degree 4, and 4n edges. A triangular anti-prism is an octahedron.

The six other semi-regular polyhedra are listed in the table below.

Polyhedron	***Faces***	***Vertices***	***Edges***	***d***	***Faces/Vertex***
rhombicuboctahedron	*8+18=26*	*24*	*48*	*4*	*1 triangle 3 squares*
snub cube	*32+6=38*	*24*	*60*	*5*	*4 triangles 1 square*
snub dodecahedron	*80+12=92*	*60*	*150*	*5*	*4 triangles 1 pentagon*
truncated cuboctahedron	*12+8+6=26*	*48*	*72*	*3*	*1 square 1 hexagon 1 octagon*
truncated icosadodecahedron	*30+20+12=62*	*120*	*180*	*3*	*1 square 1 hexagon 1 decagon*
rhombicosidodecahedron	*20+30+12=62*	*60*	*120*	*4*	*1 triangle 2 squares 1 pentagon*

For more information about Archimedean solids including animated images of each solid, check out the two web-sites below:

http://en.wikipedia.org/wiki/Archimedean_solids

http://mathworld.wolfram.com/ArchimedeanSolid.html

For those who just can't get enough polyhedra, some interesting departure points for further exploration are:

The duals of the 13 Archimedean solids are known as the Catalan solids. The duals of the cuboctahedron and the icosidodecahedron, are particularly interesting in that they are also edge uniform. See the web-site below:

http://en.wikipedia.org/wiki/Catalan_solid

There are analogs of the Platonic Solids in higher dimensions. An n-dimensional polyhedron is called an n-polytope (or polychoron in dimension 4). There are 6 convex regular 4-polytopes. They include the tesseract (4-cube) and its dual the hexadexachoron, the pentachoron (4-d tetrahedron or 4-simplex), and 3 others. For more details check out wikipedia and/or the web-site below:

http://members.aol.com/Polycell/nets.html