Combinatorics: The Fine Art of Counting

Week Nine Menu
Graphic Grub

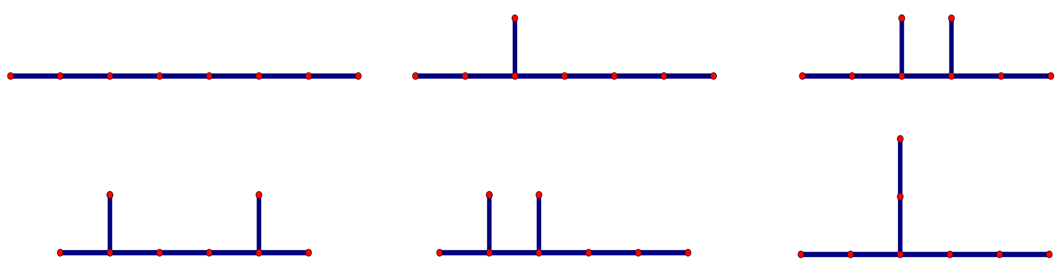
This week's colorful menu includes a number of graphic staples. Feel free to taste a small portion of each item or pick your favorite dish and chow down.

1. Prove that at a party where everyone has at least two friends there must be a circle of friends. Prove that this is still true even if one person has only one friend and everyone else has at least two friends.

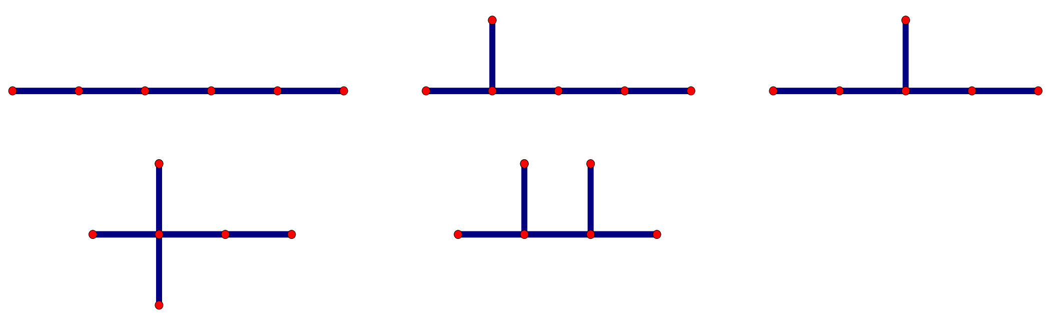
The graph with a vertex for each person and edges between friends must have vertices whose degrees sum to at least 2n or 2n-1. The number of edges is therefore at least n in both cases, hence the graph must contain a cycle.

2. Find all the non-isomorphic spanning trees in the graphs of the cube and the octahedron.

Cube:



Octahedron:



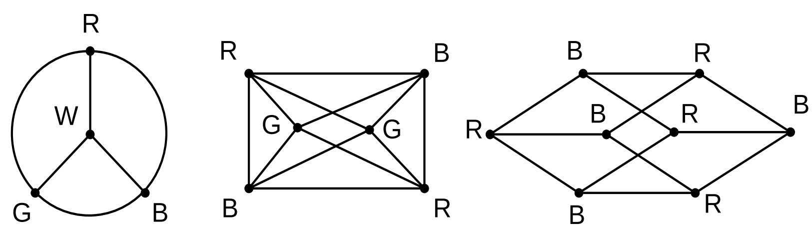
3. K_5 can be decomposed into cycles in two non-isomorphic ways, as two 5-cycles or as a 4-cycle and two 3-cycles. Find all non-isomorphic decompositions into cycles of the following graphs:

- Octahedron: *6-6, 6-3-3, 5-4-3, 4-4-4, 3-3-3-3*
- $K_{4,4}$: *8-8, 6-6-4, 4-4-4-4*
- K_7: *7-7-7, 7-6-5-3, 7-6-4-4, 7-5-5-4, 7-5-3-3-3, 6-6-6-3, 6-6-5-4, 6-6-3-3-3, 6-5-5-5, 6-5-4-3-3, 6-4-4-4-3, 6-3-3-3-3-3, 5-5-5-3-3, 5-5-4-4-3, 5-4-4-4-4, 5-4-3-3-3-3, 4-4-4-3-3-3, 3-3-3-3-3-3-3*

Note that for K_7 *all cycle decomposition which satisfy the obvious constraints (i.e. the # of edges add up, the cycles have size 3-n, and the cycles in a bipartite graph must be even length) are possible. This is conjectured by not known to be true for all complete graphs – it has been verified up through* K_{10}.

4. Find a proper vertex-coloring of each of the five regular polyhedral graphs using the minimum number of colors possible.

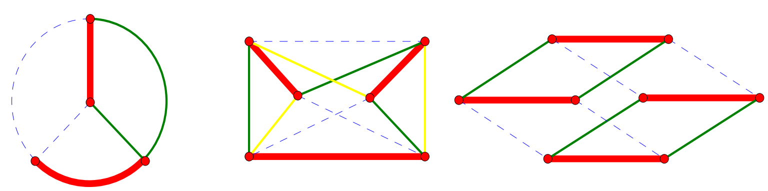
Proper vertex-colorings of the tetrahedron (4 colors), octahedron (3 colors) and cube (2 colors) are shown below.



The dodecahedron can be properly vertex-colored with 3 colors, while the icosahedron requires 4 colors.

5. Find a proper edge-coloring of each of the five regular polyhedral graphs using the minimum number of colors possible.

Proper edge-colorings of the tetrahedron (3 colors), octahedron (4 colors) and cube (3 colors) are shown below.



The dodecahedron and icosahedron can both be properly edge-colored with 3 colors.

6. Construct a round-robin tournament schedule for 9 players {1,2,3,4,5,6,7,8,9} using 9 rounds. For each round list the matches played and any players who have a "bye" (i.e. no opponent in that round).

The following schedule was constructed by using the "turning" technique to obtain a decomposition of K_{10} into perfect matchings. Vertex 0 representing a "bye" and was placed in the center, with vertex 1 at the top and the remaining vertices numbered in increasing order clock-wise.

Round 1	Round 2	Round 3	Round 4	Round 5	Round 6	Round 7	Round 8	Round 9
1 bye	2 bye	3 bye	4 bye	5 bye	6 bye	7 bye	8 bye	9 bye
2 v 9	1 v 3	2 v 4	3 v 5	4 v 6	5 v 7	6 v 8	7 v 9	1 v 8
3 v 8	4 v 9	1 v 5	2 v 6	3 v 7	4 v 8	5 v 9	1 v 6	2 v 7
4 v 7	5 v 8	6 v 9	1 v 7	2 v 8	3 v 9	1 v 4	2 v 5	3 v 6
5 v 6	6 v 7	7 v 8	8 v 9	1 v 9	1 v 2	2 v 3	3 v 4	4 v 5

7. Prove or disprove: every 2-coloring of the edges of K_6 must contain a mono-chromatic 4-cycle.

Any 2-coloring of the edges of K_6 contains a mono-chromatic 4-cycle. The simplest proof of this fact involves proving something even stronger, that any graph with 8 edges and 6 vertices contains a 4-cycle (since any 2-coloring must color at least 8 of the 15 edges the same color, this implies a mono-chromatic 4-cycle).

*There are many ways to prove this, but perhaps the shortest involves noting that a graph with 8 edges must be planar ($K_{3,3}$ is the smallest non-planer graph) and it is not difficult to show that every edge must be contained in a cycle. This graph must have 4 faces (5 edges span the graph plus 3 additional edges each create an additional face). The sum of the degrees of the faces must be twice the number of edges in such a graph (since it is 2-connected it is composed entirely of polygons), so 4 faces with between 3 and 6 sides must add up to a total of 2*8 = 16. At least one of the faces must have 4 sides.*

8. Find the largest graph with 6 vertices which does not contain a sub-graph isomorphic to K_4.

The ***octahedron*** *has 12 edges and does not contain K_4 and this is the largest number of edges possible. Any graph with 13 edges and 6 vertices can be constructed by removing two edges from K_6. These two edges must contain at* *least three distinct vertices. One of the edges between these three or four vertices must still be present. If we take the two vertices contained in this edge along with two vertices not contained in either of the edges removed, all edges within this set of 4 will be present which means the graph contains a sub-graph isomorphic to K_4.*

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Summer 2007