Power Series and Euler's Formula

At $x=0$, the nth derivative of x^n is the number $n!$ Other derivatives are 0.

Multiply the nth derivatives of $f(x)$ by $x^n/n!$ to match function with series

TAYLOR SERIES $$f(x)=f(0)+f'(0)\frac{x}{1}+f''(0)\frac{x^2}{2}+\cdots+f^{(n)}(0)\frac{x^n}{n!}+\cdots$$

EXAMPLE 1 $f(x)=e^x$ All derivatives $=1$ at $x=0$ Match with $x^n/n!$

Taylor Series
Exponential Series $$= \quad e^x=1+1\frac{x}{1}+1\frac{x^2}{2}+\cdots+1\frac{x^n}{n!}+\cdots$$

EXAMPLE 2 $f=\sin x$ $f'=\cos x$ $f''=-\sin x$ $f'''=-\cos x$

At $x=0$ this is 0 1 0 −1 0 1 0 −1 REPEAT

$$\sin x=1\cdot\frac{x}{1}-1\frac{x^3}{3!}+1\frac{x^5}{5!}-\cdots$$ ODD POWERS $\sin(-x)=-\sin x$

EXAMPLE 3 $f=\cos x$ produces 1 0 −1 0 1 0 −1 0 REPEAT

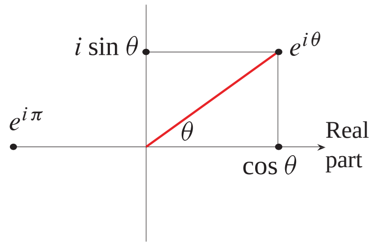
$$\cos x=1-1\frac{x^2}{2!}+1\frac{x^4}{4!}-\cdots$$ EVEN POWERS $\frac{d}{dx}(\cos x)=-\sin x$

Imaginary $i^2=-1$ and then $i^3=-i$ **Find the exponential e^{ix}**

$$e^{ix}=1+ix+\frac{1}{2!}(ix)^2+\frac{1}{3!}(ix)^3+\cdots$$
$$=\left(1-\frac{x^2}{2!}+\cdots\right)+i\left(x-\frac{x^3}{3!}+\cdots\right)$$

Those are **$\cos x + i \sin x$**

EULER'S GREAT FORMULA $e^{ix}=\cos x+i\sin x$



$e^{i\theta}=\cos\theta+i\sin\theta$

$e^{i\theta}+e^{-i\theta}=2\cos\theta$

$e^{i\pi}=-1$ combines 4 great numbers

Two more examples of Power Series (Taylor Series for $f(x)$)

$$f(x)=\frac{1}{1-x}=1+x+x^2+x^3+\cdots$$ "Geometric series"

$$f(x)=-\ln(1-x)=\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots$$ "Integral of geometric series"

Resource: Highlights of Calculus
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