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# Calculus Revisited Part 1

A Self-Study Course



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**Lecture Notes**

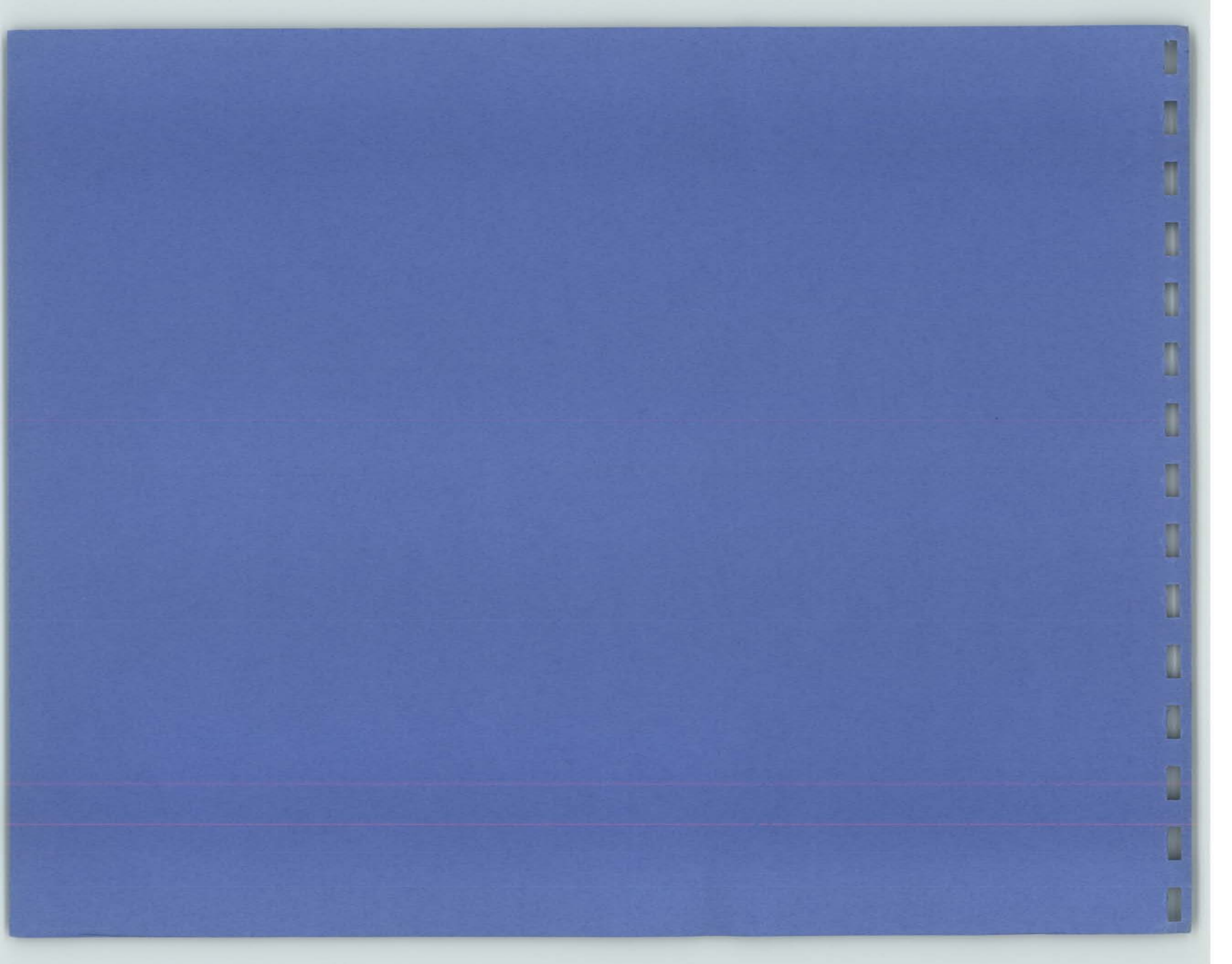
Center for Advanced  
Engineering Study

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CALCULUS REVISITED  
PART 1  
A Self-Study Course

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LECTURE NOTES

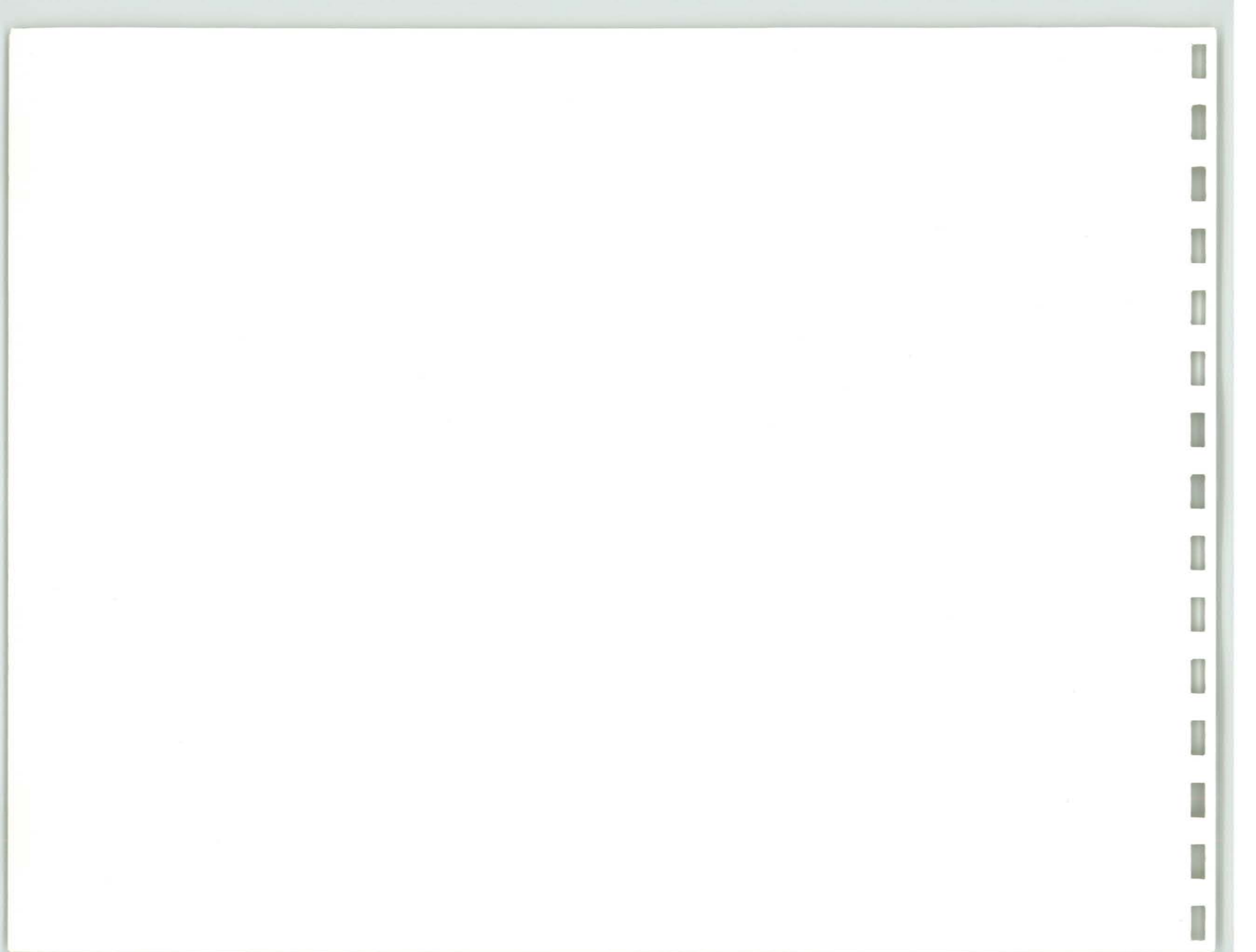
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Block I: Sets, Functions, and Limits

0.000 Preface

32 min.

Preface:  
Lectures  
Text  
Notes  
Study Guide  
Learning Exercises

"Instantaneous Speed"

$\frac{\text{dist } o_1 \rightarrow o_2}{\text{time } o_1 \rightarrow o_2} = v_{ar}$

$\frac{0}{0} = \text{undefined}$   
indeterminate  
 $\frac{10^{-6}}{10^{-12}} = 10^6$ ;  $\frac{10^{-12}}{10^{-6}} = 10^{-6}$

$b \times \left(\frac{a}{b}\right) = a$   
 $0 \times \left(\frac{0}{0}\right) = 0$

$\frac{P}{o_1 \quad o_2}$

How many pairs of observers do we need?

Functions  
 $\Delta = 16t^2$   
We can find  $v$  for each  $t$

$v_{av} = \frac{\Delta \Delta}{\Delta t}$ ;  $v = \lim_{\Delta t \rightarrow 0} \frac{\Delta \Delta}{\Delta t}$

output  
input

$\Delta = 16t^2$

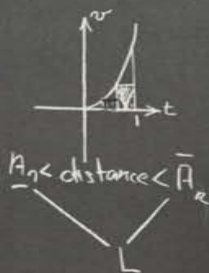
$v = 32t$



### Area under a Curve



Physical Interpretation  
 $v = t^2, 0 \leq t \leq 1$

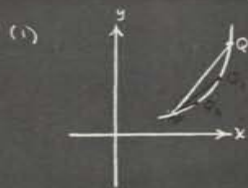


### Areas and Rates of change



Area under the curve is somehow related to the rate of change of height

### Two Limit Concepts



Tangent line passes through "two consecutive points" on the curve

### (a) Discrete Limit

Area is defined as an "endless" sum of areas of rectangles.

How big is an "infinite" sum?

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \rightarrow 2$$



→ Zeno's Paradox  
 Tortoise and the Hare

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = 2$$

$$\begin{cases} \frac{x}{1} = \frac{x-1}{\frac{1}{2}} \\ \frac{1}{2}x = x-1 \\ x=2 \end{cases}$$

- Functions (sets)
- Limits
- Derivatives (Rate of change)
- Integrals (Area under curves)


Applications  
 More "elaborate" functions  
 More sophisticated techniques  
 Infinite Series

1.010 Analytic Geometry

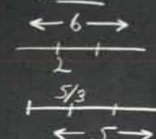
37 min.

ANALYTIC GEOMETRY

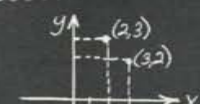
PROFITS



$5 < 3, 6 > 3$



POINTS AND ORDERED PAIRS



greater than  $\left\{ \begin{array}{l} \text{higher than} \\ \text{to the right of} \end{array} \right.$

increasing rising

$s = 16t^2$

$t \xrightarrow{\text{input}} \boxed{\text{distance}} \xrightarrow{\text{output}} 16t^2$   
machine

$V = \pi r^2 h$

$(r, h) \xrightarrow{\text{machine}} \pi r^2 h$

$(2, 3) \rightarrow 12\pi$   
 $(3, 2) \rightarrow 18\pi$

$(a+b)^2$   
 $a^2 + 2ab + b^2$

$(a+b)^3$   
 $a^3 + 3a^2b + 3ab^2 + b^3$

$(a+b)^4 =$   
 $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

	a	b
a	$a^2$	ab
b	ab	$b^2$

$$S = \{(x, y) : x^2 + y^2 = 25\}$$

$$(3, 4) \in S, (1, 2) \notin S$$

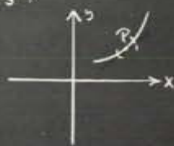


$$x^2 + y^2 = 25$$

$$(1, 2) \notin S$$

↳ "inside" the circle

Why straight lines?



Interpolation:

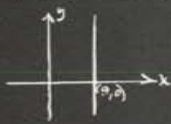
$$\log 2 = 0.301$$

$$\log 4 = 0.602$$

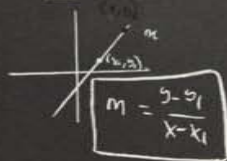
$$0.477$$



Equations of lines



line || to y-axis  
 $\{(x, y) : x = a\} \rightarrow x = a$



Example:

$$m = 3 \quad (2, 5) \in l$$

$$\frac{y-5}{x-2} = 3$$

$$y = 3x - 1$$



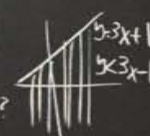
Is  $(8, 23) \in l$ ?

$$y = 3x - 1$$

$$23 = 3(8) - 1$$

$$(8, 23)$$

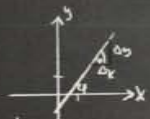
$$(8, 12)$$



slope:



$$\frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$



$$m = \tan \alpha$$



$$m_1 = \tan \alpha_1; m_2 = \tan \alpha_2$$

$$\tan(\alpha_1 - \alpha_2) = \frac{\tan \alpha_1 - \tan \alpha_2}{1 + \tan \alpha_1 \tan \alpha_2}$$

$$l_1 \perp l_2 \Rightarrow m_1 m_2 = -1$$

Simultaneous equations

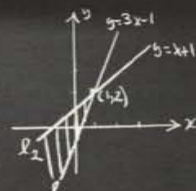
$$\begin{cases} y = 3x - 1 \\ y = x + 1 \end{cases}$$

$$3x - 1 = x + 1$$

$$2x = 2$$

$$x = 1$$

$$(1, 2)$$



$$(1, 2) \in l_1 \cap l_2$$

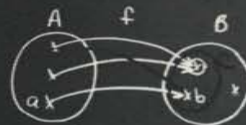
$$\begin{cases} y < x + 1 \\ y > 3x - 1 \end{cases}$$

1.020 Functions

39 min.

Functions:

$f: A \rightarrow B$  means that  $f$  is a rule which assigns to each  $a \in A$  an element  $b \in B$



$A = \text{domain of } f$   
 $B = \text{range of } f$   
 $C = \text{Image of } f = f(A)$

$a \xrightarrow{f} b$   
 $f(a) = b$

Onto Functions

Range = Image

Example:

$\rightarrow A = \{1, 2, 3\}$

$f(a) = 4a, a \in A$

$f(1) = 4, f(2) = 8$

$f(3) = 12$

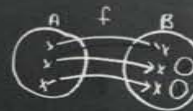
$B = \{4, 8, 12\}$

$f: A \rightarrow B$



One-to-One Functions

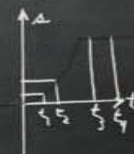
$f(a_1) = f(a_2) \rightarrow a_1 = a_2$



Functions of a Real Variable

$s = 16t^2$

$s = \begin{cases} 0, & t < 0 \\ 16t^2, & 0 \leq t \leq t_0 \\ 16t_0^2, & t > t_0 \end{cases}$





### Intervals

$$\{x: a < x < b\} = (a, b)$$

$$\{x: a \leq x \leq b\} = [a, b]$$



$$\{x: a \leq x < b\} = [a, b)$$

$$\frac{1}{2} \in (0, 1)$$

$$0 < \frac{1}{2} < 1$$

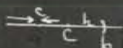


$$0 \notin (0, 1)$$

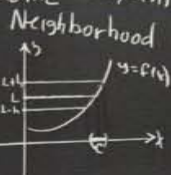
$$0 \in [0, 1]$$

### Neighborhood of $x=c$

(open) Interval which contains  $c$  "inside"



Symmetric  $(c-h, c+h)$



### Arithmetic of (Real) Functions

$f \pm g$  means

$$1. \text{dom } f = \text{dom } g (=A)$$

$$2. f(x) \pm g(x), \text{ all } x \in A$$

Example:

$$A = \{0, 1\}, B = \{0, 1\}$$



### Sums, Products, etc of functions

$$A = \{1, 2, 3\}$$

$$f(x) = 2x, x \in A$$

$$g(x) = x+1, x \in A$$

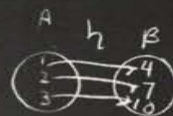
$$h(x) = f(x) + g(x)$$

$$= 2x + (x+1)$$

$$= 3x+1$$

$$k(x) = f(x)g(x)$$

$$= 2x^2 + 2x$$



### Deleted Neighborhoods

$$f(x) = \frac{x^2 - 9}{x - 3}$$

$$f(3) = \frac{0}{0}$$

$$\frac{(+/-)}{(+/-)} = 37$$

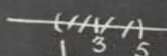
$$f(x) = \frac{(x+3)(x-3)}{(x-3)}$$

"distance"  $(x_1, x_2) = \begin{cases} x_2 - x_1 \\ x_1 - x_2 \end{cases}$

### Absolute Value

$$|x_1 - x_2| = \sqrt{(x_1 - x_2)^2}$$

$$|x - 3| < 2$$



$$1 < x < 5$$

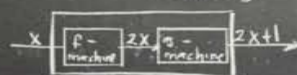
$$+\sqrt{(x-3)^2} < 2$$

$$(x-3)^2 < 4$$

$$\left. \begin{aligned} x^2 - 6x + 9 < 4 \\ (x-1)(x-5) < 0 \end{aligned} \right\} 1 < x < 5$$

$$\begin{matrix} > 0 < 0 \end{matrix}$$

### Composition of functions



$$f \circ g = f \circ g(x) = g(f(x))$$

$$f(x) = 2x, g(x) = x+1$$

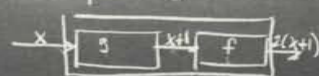
$$f(x) = 2x+1$$

$$\text{dom } f = \text{dom } g$$

$$a+b = b+a$$

$$a \div b \neq b \div a$$

$$p = f \circ g$$



$$p(x) = 2(x+1) = 2x+2$$

$$q(x) = 2x+1$$

$$\begin{cases} f(x)g(x) = 2x(x+1) \\ f \circ g(x) = 2x+2 \\ g \circ f(x) = 2x+1 \end{cases}$$

1.025 Inverse Functions

40 min.

Inverse Functions

"Switch in Emphasis"

$$5 - 3 = 2 \quad 2 + 3 = 5$$

$$3 + \frac{(5-3)}{2} = 5$$

$$y = \log_b x ; b = x$$

$$y = \sin^{-1} x ; x = \sin y$$

$y = f(x) ; x = f^{-1}(y)$

Example  
 $\rightarrow y = 2x - 7 ; y = f(x) = 2x - 7$   
 $x = \frac{y+7}{2} ; x = f^{-1}(y) = \frac{y+7}{2}$

input  $\xrightarrow{x}$  f-machine  $\xrightarrow{\text{output}}$   $y$

$f^{-1} \circ f = \text{Id}_A$   
 $f^{-1}(f(c)) = c$   
 $f(f^{-1}(d)) = d$

$c \xrightarrow{f} 2c-7 \xrightarrow{f^{-1}} \frac{(2c-7)+7}{2} = c$   
 $d \xrightarrow{f^{-1}} \frac{d+7}{2} \xrightarrow{f} 2\left(\frac{d+7}{2}\right) - 7 = d$

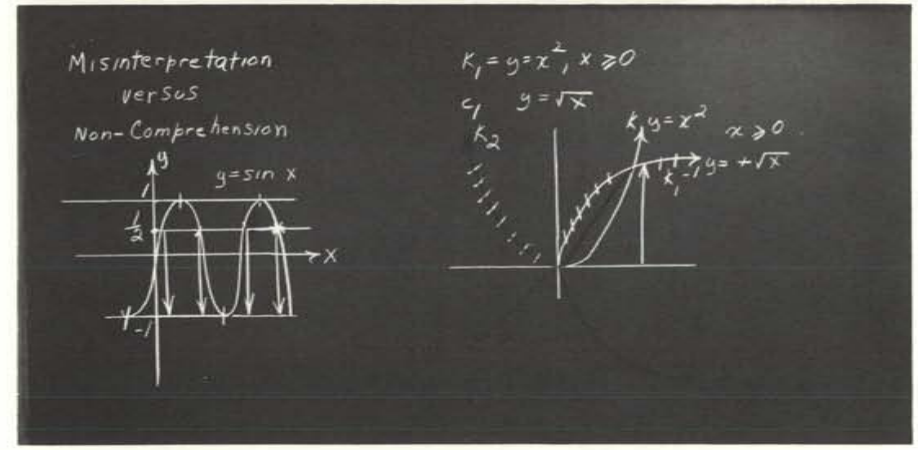
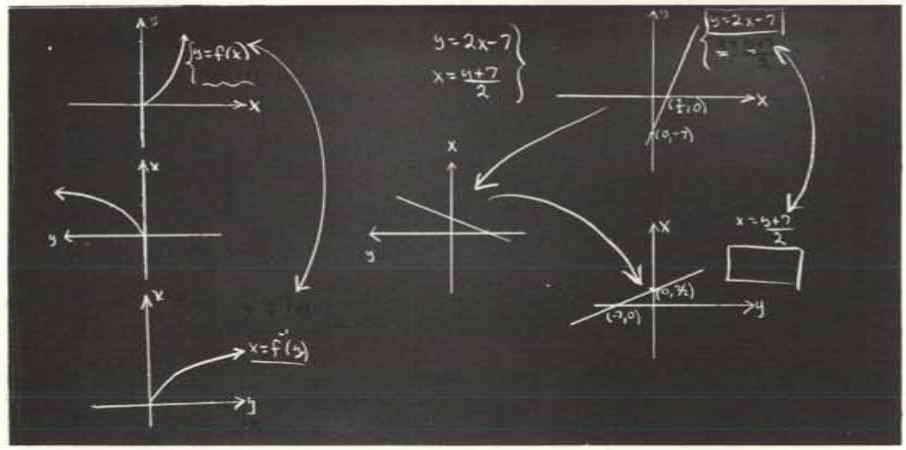
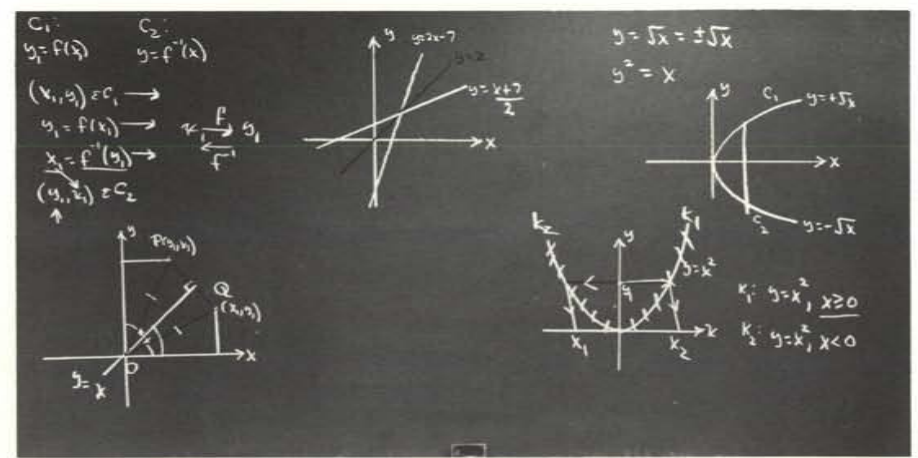
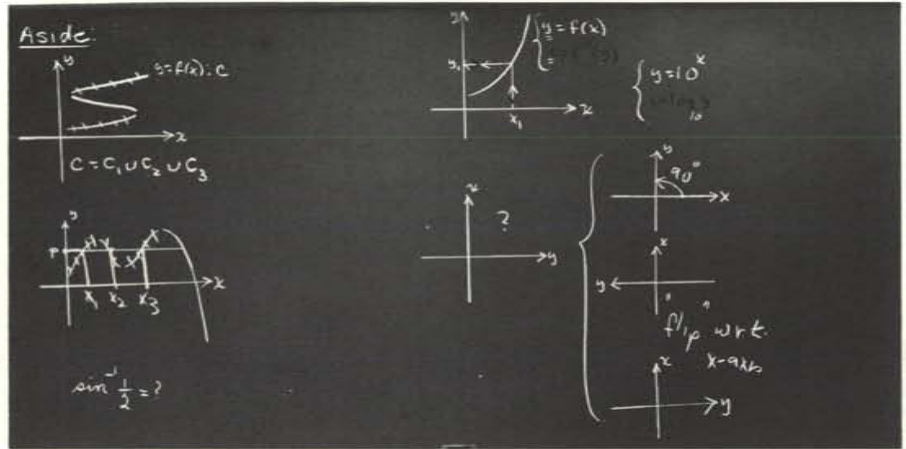
$\text{dom } f = [a, b]$   
 $\text{Range } f = [c, d]$

$f(a) = b$   
 $f(b) = a$   
 $f^{-1}(b) = a$

$\text{dom } f^{-1} = \text{Im } f = [c, d]$   
 $\text{Im } f^{-1} = \text{dom } f = [a, b]$

$f^{-1} = f^{-1}$





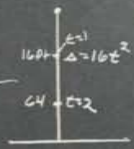
1.030 Derivatives and Limits

45 min.

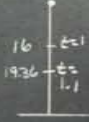
Derivatives and Limits

How fast does ball fall when  $t=1$ ?

Ave speed from  $t=1$  to  $t=2$ :

$$\frac{s(2) - s(1)}{2 - 1} = \frac{64 - 16}{1} = 48 \text{ ft/sec}$$


Ave speed from  $t=1$  to  $t=1.1$ :

$$\frac{s(1.1) - s(1)}{.1} = \frac{19.36 - 16}{.1} = 33.6 \text{ ft/sec}$$


Ave speed from  $t=1$  to  $t=1+h$ :

$$\frac{s(1+h) - s(1)}{h} = \frac{16(1+h)^2 - 16}{h} = \frac{32h + 16h^2}{h} = 32 + 16h$$

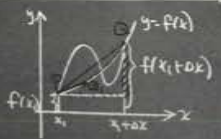
"Appears" that speed is 32 ft/sec when  $t=1$ .  
I.e.,

$$\lim_{h \rightarrow 0} \left[ \frac{s(1+h) - s(1)}{h} \right] = \lim_{h \rightarrow 0} (32 + 16h) = 32 \text{ ft/sec}$$

By this approach, if  $s = 16t^2$  then at time  $t = t_1$ ,  $v = 32t_1$ , or:  $v = 32t_1$  where (instantaneous) speed has been defined as a limit of average speeds

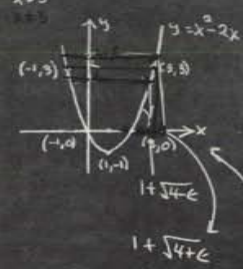
$$\frac{s(t_1+h) - s(t_1)}{h} = \frac{16(t_1+h)^2 - 16t_1^2}{h} = 32t_1 + 16h$$

$y = f(x)$   
 $\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$   
 $f'(x_1) = \lim_{\Delta x \rightarrow 0} \left[ \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \right]$



$\lim_{x \rightarrow a} f(x) = f(a)$   
 $\lim_{x \rightarrow 3} \left[ \frac{x^2 - 9}{x - 3} \right] = \frac{0}{0}$   
 $\frac{x^2 - 9}{x - 3} = \frac{(x+3)(x-3)}{(x-3)} = x+3$

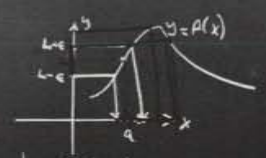
$\lim_{x \rightarrow 3} x^2 - 2x = 3$



Given  $\epsilon > 0$  must find  $\delta > 0$  such that:  
 $0 < |x-3| < \delta \rightarrow |x^2 - 2x - 3| < \epsilon$   
 $-\epsilon < x^2 - 2x - 3 < \epsilon$   
 $-\epsilon < x^2 - 2x + 1 - 4 < \epsilon$   
 $-\epsilon < (x-1)^2 - 4 < \epsilon$   
 $4 - \epsilon < (x-1)^2 < 4 + \epsilon$   
 If  $x$  is "near" 3,  $x-1$  is positive  
 $\therefore \sqrt{4 - \epsilon} < x-1 < \sqrt{4 + \epsilon}$   
 $1 + \sqrt{4 - \epsilon} < x < 1 + \sqrt{4 + \epsilon}$

$\frac{x^2 - 9}{x - 3} = x + 3$   
 If  $x=3$ ,  $\frac{x^2 - 9}{x - 3}$  is undefined  
 $\lim_{x \rightarrow 3} \left[ \frac{x^2 - 9}{x - 3} \right] = \lim_{x \rightarrow 3} (x + 3) = 6$

$\lim_{x \rightarrow a} f(x) = L$  means  
 Given  $\epsilon > 0$  we can find  $\delta > 0$  such that  
 $0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon$



$\lim_{x \rightarrow a} f(x) = L$

$\sqrt{4 - \epsilon} - 2 < x - 3 < \sqrt{4 + \epsilon} - 2$

Let  $\delta = \min \left\{ \frac{\sqrt{4 + \epsilon} - 2}{2}, \frac{2 - \sqrt{4 - \epsilon}}{2} \right\}$

Then:  
 $0 < |x - 3| < \delta \rightarrow |x^2 - 2x - 3| < \epsilon$

$\lim_{x \rightarrow 3} (x^2 - 2x) = 3$   
 $(1 + \sqrt{4 + \epsilon})^2 - 2(1 + \sqrt{4 + \epsilon}) = 1 + 2\sqrt{4 + \epsilon} + 4 + \epsilon - 2 - 2\sqrt{4 + \epsilon} = 3 + \epsilon$

$|x^2 - 2x - 3| = |x-3||x+1|$   
 "Near"  $x=3$ ,  $x+1$  is "near" 4  
 I.e.,  $-\epsilon < x-3 < \epsilon$   $|x+1| < 5$   
 $3 - 2 - \epsilon < x+1 < 4 + \epsilon < 5$   
 $\therefore$  If  $\epsilon < 1$  there exists  $\delta > 0$  such that  
 $0 < |x-3| < \delta \rightarrow |x-3||x+1| < \frac{\epsilon}{5} < \frac{\epsilon}{5} < \frac{\epsilon}{5}$

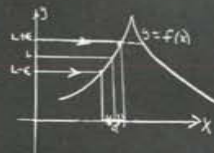
1.040 Limits: A More Rigorous Approach

46 min.

Limits - A More Rigorous Approach

$$\lim_{x \rightarrow a} f(x) = L$$

For each  $\epsilon > 0$ , can find  $\delta > 0$  such that  
 $0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon$



Theorem

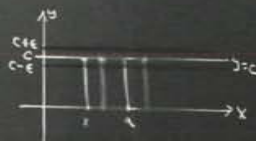
$$\lim_{x \rightarrow a} c = c$$

Let  $f(x) = c$

Then  $\lim_{x \rightarrow a} f(x) = c$

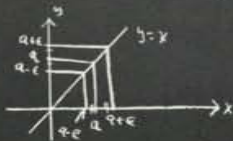
Given  $\epsilon > 0$  we must find  $\delta > 0$  such that

$$0 < |x - a| < \delta \rightarrow |c - c| < \epsilon$$



Thm:  
 $\lim_{x \rightarrow a} x = a$

Given  $\epsilon > 0$  <sup>choose  $\delta = \epsilon$</sup>  must find  $\delta > 0$  such that  $0 < |x - a| < \delta \xrightarrow{\epsilon}$   
 Implies that  $|x - a| < \epsilon$



Thm:

$$\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

$$\lim_{x \rightarrow a} f(x) = L_1 \quad \lim_{x \rightarrow a} g(x) = L_2 \quad \text{dom } f$$

Let  $h(x) = f(x) + g(x)$

To prove:  $\lim_{x \rightarrow a} h(x) = L_1 + L_2$  dom g





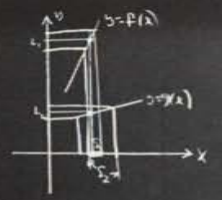
Given  $\epsilon > 0$  must find  $\delta > 0$   
 such that  $0 < |x-a| < \delta \rightarrow$   
 $|f(x) + g(x) - L_1 + L_2| < \epsilon$

$$|f(x) + g(x) - L_1 - L_2| = |(f(x) - L_1) + (g(x) - L_2)| \leq |f(x) - L_1| + |g(x) - L_2|$$

Given  $\epsilon > 0$ , let  
 $\epsilon_1 = \frac{\epsilon}{2}$   
 Can find  $\delta_1 > 0$  such that  
 $0 < |x-a| < \delta_1 \rightarrow |f(x) - L_1| < \epsilon_1$   $\left(\lim_{x \rightarrow a} f(x) = L_1\right)$   
 Can find  $\delta_2 > 0$  such that  
 $0 < |x-a| < \delta_2 \rightarrow |g(x) - L_2| < \epsilon_1$   
 Let  $\delta = \min\{\delta_1, \delta_2\}$   
 $0 < |x-a| < \delta \rightarrow$   
 $|f(x) - L_1| + |g(x) - L_2| < \epsilon_1 + \epsilon_1 = \epsilon$

$$|f(x)g(x) - L_1L_2| = |f(x)g(x) - L_1g(x) + L_1g(x) - L_1L_2| \leq |g(x) - L_2| |f(x) - L_1| + |L_1| |g(x) - L_2|$$

Example  
 $\lim_{x \rightarrow 3} (x^2 + 7x) = 30$   
 $x \rightarrow 3 \quad |x-3| < ?$   
 $x^2 + 7x < 30.023$   
 $\lim_{x \rightarrow 3} x^2 + \lim_{x \rightarrow 3} 7x =$   
 $(\lim_{x \rightarrow 3} x)(\lim_{x \rightarrow 3} x) + (\lim_{x \rightarrow 3} 7)(\lim_{x \rightarrow 3} x) =$   
 $(3)(3) + 7(3) = 30$



Thm:  
 $\lim_{x \rightarrow a} f(x) = L_1, \lim_{x \rightarrow a} g(x) = L_2 \rightarrow$   
 $\lim_{x \rightarrow a} f(x)g(x) = L_1L_2$

$$f(x) = L_1 + [f(x) - L_1]$$

$$g(x) = L_2 + [g(x) - L_2]$$

$$|f(x)g(x) - L_1L_2| \leq |L_1| |g(x) - L_2| + |L_2| |f(x) - L_1|$$

$$|L_1| |g(x) - L_2| + |L_2| |f(x) - L_1|$$

1.060 Mathematical Induction

29 min.

Mathematical Induction

$$\lim_{x \rightarrow a} [f_1(x) + f_2(x)] = \lim_{x \rightarrow a} f_1(x) + \lim_{x \rightarrow a} f_2(x)$$

How about

$$\lim_{x \rightarrow a} [f_1(x) + f_2(x) + f_3(x)] ?$$

$$\lim_{x \rightarrow a} [f_1(x) + f_2(x)] + \lim_{x \rightarrow a} f_3(x)$$

$$[\lim_{x \rightarrow a} f_1(x) + \lim_{x \rightarrow a} f_2(x)] + \lim_{x \rightarrow a} f_3(x)$$

$$\lim_{x \rightarrow a} [f_1(x) + f_2(x) + f_3(x) + f_4(x)] = \lim_{x \rightarrow a} [f_1(x) + f_2(x) + f_3(x)] + \lim_{x \rightarrow a} f_4(x) = \lim_{x \rightarrow a} f_1(x) + \lim_{x \rightarrow a} f_2(x) + \lim_{x \rightarrow a} f_3(x) + \lim_{x \rightarrow a} f_4(x)$$

$$1 + 2 + 3 + 4 = 10$$

1, 2, 3, 4

$$12 \div 6 \div 2 = 1$$

$$12 \div 6 \div 2 = 4$$

odd + odd = ? even  
 positive - positive = ?  
 3 - 5



Suppose

$$\lim_{x \rightarrow a} [f_1(x) + \dots + f_n(x)] =$$

$$\lim_{x \rightarrow a} f_1(x) + \dots + \lim_{x \rightarrow a} f_n(x)$$

$$\lim_{x \rightarrow a} [f_1(x) + \dots + f_n(x) + f_{n+1}(x)] =$$

MATHEMATICAL INDUCTION:

(1) SHOW CONJECTURE TRUE FOR  $n=1$  ✓

(2) Prove that truth for  $n=k$  implies the truth for  $n=k+1$  ✓

$$1 + 2 + \dots + n = \frac{n(n+1)}{2}$$

Assume:

$$1 + 2 + \dots + k = \frac{k(k+1)}{2}$$

$$1 + 2 + \dots + k + k + 1 =$$

$$\frac{k(k+1)}{2} + k + 1 =$$

$$(k+1) \left( \frac{k}{2} + 1 \right) = \frac{(k+1)(k+2)}{2}$$

$$P(n) = n^2 - n + 41$$

$$P(1) = 1^2 - 1 + 41 = 41$$

$$P(2) = 2^2 - 2 + 41 = 43$$

$$P(3) = 9 - 3 + 41 = 47$$

$P(40)$  = prime

$$P(41) = 41^2 - 41 + 41 =$$

$$= 41^2 = 41 \times 41$$

UNIQUE

FACTORIZATION

THEOREM

$$2 = 2$$

$$6 = 2 \times 3$$

$$3 = 3$$

$$7 = 7$$

$$4 = 2 \times 2$$

$$8 = 2 \times 2 \times 2$$

$$5 = 5$$

$$9 = 3 \times 3$$

$$10 = 2 \times 5$$

$$11 = 11$$

$n+1$  factors "considerably different" than  $n$

$$59 \leftarrow 60$$

$$61 \leftarrow 60$$

$$1, 2, 3, 4, 5, 6, \dots$$

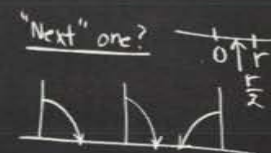
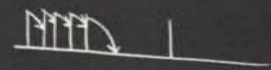
True for  $n=7$  →

$x \rightarrow k+1$

True for  $n \geq 7$

$$+2+3+\dots+98+99+100 = \frac{(100)(101)}{2}$$

$$1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$



Block II: Differentiation

2.010 Derivatives of Some Simple Functions

28 min.

Derivatives of Some Simple Functions

$$f'(x_1) = \lim_{\Delta x \rightarrow 0} \left[ \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \right]$$

Example 1:  $f(x) = x^3$

$$f(x_1 + \Delta x) - f(x_1) = (x_1 + \Delta x)^3 - x_1^3$$

$$= 3x_1^2 \Delta x + 3x_1 \Delta x^2 + \Delta x^3$$

$$= \Delta x (3x_1^2 + 3x_1 \Delta x + \Delta x^2)$$

$$\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = 3x_1^2 + 3x_1 \Delta x + \Delta x^2$$

$$\therefore f'(x_1) = 3x_1^2; f'(x) = 3x^2$$

Generalization

Let  $f(x) = x^n$   
 $n$  a positive whole number

$$f(x_1 + \Delta x) - f(x_1) = (x_1 + \Delta x)^n - x_1^n$$

$$= n x_1^{n-1} \Delta x + \Delta x^2 \left( \frac{n(n-1)}{2} x_1^{n-2} + \dots + \Delta x^{n-2} \right)$$

$$\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = n x_1^{n-1} + \Delta x \left( \frac{n(n-1)}{2} x_1^{n-2} + \dots + \Delta x^{n-2} \right)$$

$$f'(x_1) = n x_1^{n-1}$$

$$f(x) = x^n \rightarrow f'(x) = n x^{n-1}$$

Example 2

$f, g$  differentiable at  $x_1$   
 (i.e.,  $f'(x_1)$  and  $g'(x_1)$  exist)

Define  $h$  by

$$h(x) = f(x) + g(x)$$

[ $x$  dom  $f \cap \text{dom } g$ ]

Then:  $h'$  exists

$$h'(x) = f'(x) + g'(x)$$

Proof

$$h(x_1 + \Delta x) = f(x_1 + \Delta x) + g(x_1 + \Delta x)$$

$$h(x_1 + \Delta x) - h(x_1) = [f(x_1 + \Delta x) + g(x_1 + \Delta x)] - [f(x_1) + g(x_1)]$$

$$\therefore \frac{h(x_1 + \Delta x) - h(x_1)}{\Delta x} = \left[ \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \right] + \left[ \frac{g(x_1 + \Delta x) - g(x_1)}{\Delta x} \right]$$

$$\therefore h'(x_1) = f'(x_1) + g'(x_1)$$

Beware of "Self-Evident"

$$[f(x)g(x)]' \neq f'(x)g'(x)$$

$$f(x) = x+1 \rightarrow f'(x) = 1$$

$$g(x) = x-1 \rightarrow g'(x) = 1$$

$$f(x)g(x) = x^2 - 1 \rightarrow [f(x)g(x)]' = 2x$$

$$\therefore [f(x)g(x)]' \neq f'(x)g'(x)$$

In fact, if

$$h(x) = f(x)g(x)$$

then:

$$h(x_1+\Delta x) - h(x_1) = f(x_1+\Delta x)g(x_1+\Delta x) - f(x_1)g(x_1)$$

$$= f(x_1+\Delta x)g(x_1+\Delta x) - f(x_1)g(x_1+\Delta x) + f(x_1)g(x_1+\Delta x) - f(x_1)g(x_1)$$

$$\therefore \frac{h(x_1+\Delta x) - h(x_1)}{\Delta x} = \left[ \frac{f(x_1+\Delta x) - f(x_1)}{\Delta x} \right] g(x_1+\Delta x) + f(x_1) \left[ \frac{g(x_1+\Delta x) - g(x_1)}{\Delta x} \right]$$

$$h'(x_1) = f'(x_1)g(x_1) + f(x_1)g'(x_1)$$

$$(1(x_1-1) + (x_1+1)) = 2x_1$$

Summary

The basic definition:

$$f'(x_1) = \lim_{\Delta x \rightarrow 0} \left[ \frac{f(x_1+\Delta x) - f(x_1)}{\Delta x} \right]$$

never changes. But it can be manipulated to yield "convenient" "recipes".

⊛ It is not true always that  $\lim_{t \rightarrow a} g(t) = g(a)$

For example, let  $g(t) = \frac{t^2-1}{t-1}$

$\lim_{t \rightarrow 1} g(t) = 2$ ,  $g(1)$  doesn't exist! (0/0)

In our present case:

$$g(x_1+\Delta x) - g(x_1) = \left[ \frac{g(x_1+\Delta x) - g(x_1)}{\Delta x} \right] \Delta x$$

$$\lim_{\Delta x \rightarrow 0} [g(x_1+\Delta x) - g(x_1)] = \underbrace{g'(x_1)}_{\neq 0} \lim_{\Delta x \rightarrow 0} \Delta x = 0$$

$$\therefore \lim_{\Delta x \rightarrow 0} g(x_1+\Delta x) = g(x_1)$$

For a quotient

we can show that if  $h(x) = \frac{f(x)}{g(x)}$  then

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Example:

$$f(x) = x^{-n}, \quad n, \text{ positive integer}$$

$$= \frac{1}{x^n}$$

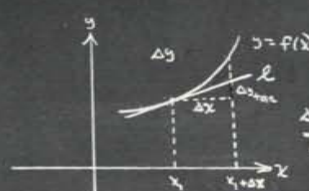
$$f'(x) = \frac{x^n(0) - (1)(nx^{n-1})}{x^{2n}} = -nx^{-n-1}$$

$$= -n x^{-n-1}$$

2.020 Approximations and Infinitesimals


34 min.

Approximations and Infinitesimals



$\Delta y_{\text{tan}} = \left(\frac{dy}{dx}\right)_{x=x_1} \Delta x$

$(4.01)^3 = 64.481201$



$x_1 = 4 \quad \Delta x = .01$

$y = x^3$

$\frac{dy}{dx} = 3x^2 = 48$

$\Delta y_{\text{tan}} = 48(.01) = .48$

64.48

$y = x^3$

$\left(\frac{dy}{dx}\right)_{x=x_1} = 3x_1^2$

$\Delta y_{\text{tan}} = 3x_1^2 \Delta x$

$\Delta y = (x_1 + \Delta x)^3 - x_1^3$

$= \boxed{3x_1^2 \Delta x + 3x_1 \Delta x^2 + \Delta x^3}$

$\frac{\Delta y}{\Delta x} = \left(\frac{dy}{dx}\right)_{x=x_1} + k$

$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \left[ \left(\frac{dy}{dx}\right)_{x=x_1} + \lim_{\Delta x \rightarrow 0} k \right]$

$\frac{dy}{dx} = \left(\frac{dy}{dx}\right)_{x=x_1} + \lim_{\Delta x \rightarrow 0} k$

$\boxed{\lim_{\Delta x \rightarrow 0} k = 0}$

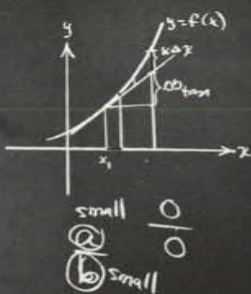


$$\Delta y = \underbrace{\left(\frac{dy}{dx}\right)}_{z=z_1} \Delta x + k \Delta x$$

$\Delta y \approx \Delta z$

$$\Delta y = \frac{dy}{dx} \Delta x + k \Delta x$$

where  $\lim_{\Delta x \rightarrow 0} k = 0$



Example:

$$y = x^3$$

$$\frac{dy}{dx} = 3x^2$$

$$\rightarrow dy = 3x^2 dx$$

$$\frac{\Delta y}{\Delta x} = 3x^2 \Delta x$$

$$\Delta y = \left(\frac{dy}{dx}\right)_{x=x_1} \Delta x + k \Delta x$$

where  $\lim_{\Delta x \rightarrow 0} k = 0$

$$f(x_1 + \Delta x) - f(x_1) =$$

$$f'(x_1) \Delta x + k \Delta x$$

where  $\lim_{\Delta x \rightarrow 0} k = 0$

$$\frac{\Delta y}{\Delta t} = \left(\frac{dy}{dx}\right)_{x=x_1} \frac{\Delta x}{\Delta t} + k \frac{\Delta x}{\Delta t} \quad \left\{ \lim_{\Delta x \rightarrow 0} k = 0 \right.$$

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta t} = \left[ \lim_{\Delta x \rightarrow 0} \left(\frac{dy}{dx}\right)_{x=x_1} \right] \left[ \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} \right]$$

$$+ \left[ \lim_{\Delta t \rightarrow 0} k \right] \left[ \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} \right]$$

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} + 0 \frac{dx}{dt}$$

$\frac{16}{4} = \frac{19}{5} \quad \frac{26}{5} \quad \frac{49}{8}$

$\left(\frac{dy}{dx}\right)$  is one symbol. It is not  $dy \div dx$



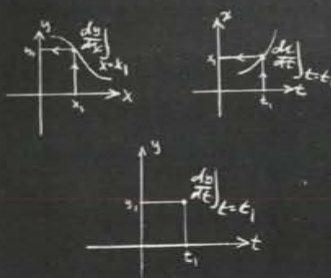
$$m_t = \left(\frac{dy}{dx}\right) = \frac{\Delta y}{\Delta x}$$

$dy \neq \Delta y$

2.030 Composite Functions and the Chain Rule

39 min.

Composite Functions and The Chain Rule



$$\left(\frac{dy}{dt}\right)_{t=t_1} = \left(\frac{dy}{dx}\right)_{x=x_1} \left(\frac{dx}{dt}\right)_{t=t_1}$$

$$\rightarrow \frac{\Delta y}{\Delta t} = \left(\frac{\Delta y}{\Delta x}\right)_{x=x_1} \left(\frac{\Delta x}{\Delta t}\right)_{t=t_1} + \underbrace{0}_{(\lim_{\Delta t \rightarrow 0} k = 0)}$$

$$\rightarrow \frac{\Delta y}{\Delta t} = \frac{\Delta y}{\Delta x} \frac{\Delta x}{\Delta t}$$

Example

Find  $\frac{dy}{dx}$  if

$y = (x^2 + 1)^2$ 

$$\left\{ \begin{array}{l} y = u^2 \\ u = x^2 + 1 \end{array} \right.$$

$$\frac{dy}{dx} = 2(x^2 + 1) \cdot 2x = 4x(x^2 + 1)$$

$y = (z^2 + 1)^2$ 

$$\left\{ \begin{array}{l} y = f(z) \\ z = g(t) \end{array} \right.$$

$$\frac{dy}{dz} = 2z$$

$$\frac{dz}{dt} = g'(t)$$

$$\frac{dy}{dt} = \frac{dy}{dz} \frac{dz}{dt} = \frac{f'(z)}{g'(t)}$$



$$\Delta y = f'(a)\Delta t + k_1 \Delta t$$

$$\Delta x = g'(a)\Delta t + k_2 \Delta t$$

$$\lim_{\Delta t \rightarrow 0} k_1 = \lim_{\Delta t \rightarrow 0} k_2 = 0$$

$$\frac{\Delta y}{\Delta x} = \frac{[f'(a) + k_1] \Delta t}{[g'(a) + k_2] \Delta t}$$

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \frac{f'(a)}{g'(a)}$$

$$\frac{d^2y}{dx^2} + \frac{d^2y/dt^2}{d^2x/dt^2} = \frac{d^2y}{dt^2}$$

$$y = f(t) \rightarrow d^2y/dt^2$$

$$\boxed{\frac{dx}{dt}} g'(t) \leftarrow x = g(t) \rightarrow d^2x/dt^2$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left( \frac{f'(t)}{g'(t)} \right)$$

$$= \frac{d}{dt} \left[ \frac{f'(t)}{g'(t)} \right] \frac{dt}{dx}$$

$$= \frac{[g'(t)f''(t) - f'(t)g''(t)] dt}{[g'(t)]^2 dx}$$

$$y = f(u)$$

$$u = g(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

Suppose  $y = x^2$ :

$$\frac{dy}{dy} = \frac{dy}{du} \frac{du}{dy}$$

$$1 = \left( \frac{dy}{du} \right) \left( \frac{du}{dy} \right)$$

Example:

$$\left. \begin{array}{l} y = t^4 \\ x = t^2 \end{array} \right\} \frac{dy}{dt} = 4t^3$$

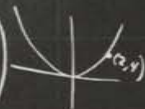
$$\frac{dx}{dt} = 2t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t^3}{2t} = 2t^2$$

$$\left. \begin{array}{l} y = t^2 \\ z = t \\ y = z^2 \end{array} \right\}$$

check:  $y = z^2$

$$\frac{dy}{dz} = 2z$$



2.040 Differentiation of Inverse Functions

28 min.

Differentiation

Inverse Functions:

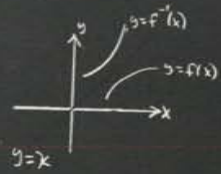
Find  $\frac{dy}{dx}$  if  $y = x^{\frac{1}{3}}$

$x = y^3$

$\frac{dx}{dy} = 3y^2$

$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{3y^2} = \frac{1}{3y^2} \cdot y^{\frac{2}{3}} \cdot y^{\frac{2}{3}}$

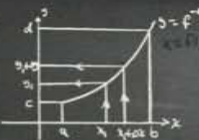
$= \frac{1}{3}(y^{\frac{2}{3}})^{-2} = \frac{1}{3}y^{-\frac{2}{3}}$



$\frac{dy}{du} = \frac{dy}{dx} \frac{dx}{du}$

$u = y$

$1 = \left(\frac{dy}{dx}\right)\left(\frac{dx}{dy}\right)$



$(f^{-1})'(y_1) = \lim_{\Delta y \rightarrow 0} \left[ \frac{f^{-1}(y_1 + \Delta y) - f^{-1}(y_1)}{\Delta y} \right]$

$= \lim_{\Delta x \rightarrow 0} \left[ \frac{y_1 + \Delta y - y_1}{f(y_1 + \Delta y) - f(y_1)} \right]$

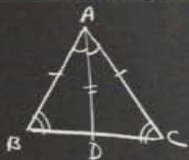
$\lim_{\Delta x \rightarrow 0} \left[ \frac{\Delta y}{f(y_1 + \Delta y) - f(y_1)} \right] =$

$\left. \frac{1}{\frac{dx}{dy}} \right|_{y=y_1}$

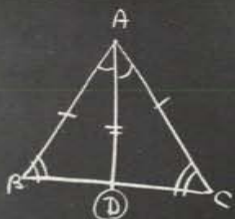
$\frac{dy}{dx} \Big|_{x=x_1} = \frac{1}{\left. \frac{dx}{dy} \right|_{y=y_1}}$

ASIDES:

(1) "Proof" vs  
"Intuition"



Statement-Reason

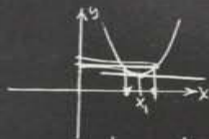


Statement-Reason

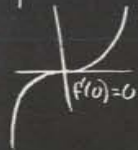
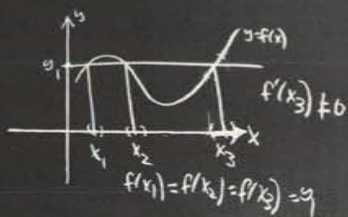
(2) How likely is  
it that  $f$   
is 1-1?

$$\left\{ \begin{array}{l} \lim_{x \rightarrow a} f(x) = f(a) \\ \frac{0}{0} \end{array} \right.$$

$$f'(x_1) = 0$$



"Local" vs "Global"



2.045 Implicit Differentiation

40 min.

Implicit Differentiation

$y = f(x)$ ;  $\frac{dy}{dx} = ?$

$x^8 + x^6 y^4 + y^6 = 3$

Assume  $y = y(x)$   
Such that  
 $x^8 + x^6 y^4 + y^6 = 3$

$x^2 = 4$ ;  $x = -2, 2$   
 $2x = 0$ ;  $x = 0$   
 $x^2 = 4$  means  $\{x, x^2 = 4\} = \{-2, 2\}$   
 $x^2 - 1 = (x+1)(x-1)$   
 $2x = (x+1) + 1(x-1)$   
 $\rightarrow 2x$

$xy = 1 \rightarrow y = \frac{1}{x} (x \neq 0)$   
 $x \frac{dy}{dx} + y = 0$   
 $\frac{dy}{dx} = -\frac{y}{x}$   
 $x = 2$   
 $y = \frac{1}{2}$   
 $\frac{dy}{dx} = -\frac{1/2}{2} = -\frac{1}{4}$

$x^2 + y^2 = 25$   
Assume  $y = y(x)$   
Such that  
 $x^2 + y^2 = 25$   
 $2x + 2y \frac{dy}{dx} = 0$   
 $\frac{dy}{dx} = -\frac{x}{y}$

$y(x) = \pm \sqrt{25 - x^2}$   
 $y_1(x) = +(25 - x^2)^{\frac{1}{2}}$   
 $\frac{dy_1}{dx} = \frac{1}{2}(25 - x^2)^{-\frac{1}{2}}(-2x)$   
 $= \frac{-x}{(25 - x^2)^{\frac{1}{2}}}$   
 $= \frac{-x}{y_1}$

$y_2(x) = -(25 - x^2)^{\frac{1}{2}}$   
 $\frac{dy_2}{dx} = -\frac{1}{2}(25 - x^2)^{-\frac{1}{2}}(-2x)$   
 $= \frac{x}{(25 - x^2)^{\frac{1}{2}}}$   
 $= \frac{x}{-y_2}$



Aside:

$$y = z^{\frac{p}{q}}, p, q \text{ integers}$$

$$y^q = z^p$$

$$q y^{q-1} \frac{dy}{dz} = p z^{p-1}$$

$$\frac{dy}{dz} = \frac{p z^{p-1}}{q y^{q-1}}$$

$$= \frac{p z^{p-1}}{q (z^{\frac{p}{q}})^{q-1}}$$

$$= \frac{p}{q} z^{\frac{p-1}{q} - \frac{p(q-1)}{q}} = \frac{p}{q} z^{\frac{p-1-pq+pq}{q}} = \frac{p}{q} z^{\frac{p-1-pq+pq}{q}}$$

Find the equation of the line tangent to  $x^8 + x^6 y^4 + y^6 = 3$  at the point  $(1, 1)$

$$8x^7 + 6x^5 y^4 + 4x^6 y^3 \frac{dy}{dx} + 6y^5 \frac{dy}{dx} = 0$$

$$\left. \frac{dy}{dx} = \frac{-(8x^7 + 6x^5 y^4)}{4x^6 y^3 + 6y^5} \right|_{x=1, y=1}$$

$$\frac{dy}{dx} = \frac{-14}{10} = -\frac{7}{5}$$

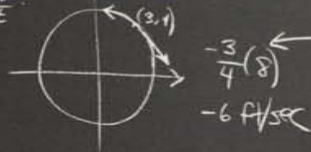
$$\frac{y-1}{x-1} = -\frac{7}{5}$$

$$7x + 5y = 12$$

## Related Rates

A particle moves along the curve  $x^2 + y^2 = 25$  ( $x, y$  in feet). At  $(3, 4)$ ,  $\frac{dx}{dt} = 8 \text{ ft/sec}$

Find  $\frac{dy}{dt}$



Assume "near"  $(3, 4)$   
 $x$  and  $y$  are differentiable functions of  $t$ . Then:

$$x^2 + y^2 = 25 \rightarrow$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}$$

$$\frac{dy}{dx} = 0 \iff x^5 (8x^2 + 6y^4) = 0$$

$$\iff x = 0$$

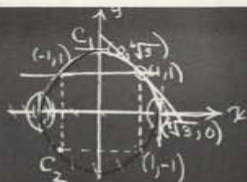
$$\frac{dx}{dy} = 0 \iff y^3 (4x^6 + 6y^2) = 0$$

$$\iff y = 0$$

$$x = 0 \rightarrow y = \pm\sqrt{3}$$

$$y = 0 \rightarrow x = \pm\sqrt{3}$$

$$(x^8 + x^6 y^4 + y^6 = 3)$$



$$C_1: x^8 + x^6 y^4 + y^6 = 3 \quad y \geq 0$$

$$C_2: x^8 + x^6 y^4 + y^6 = 3 \quad y < 0$$

2.050 Continuity

22 min.

Continuity

Does  $\lim_{x \rightarrow a} f(x) = f(a)$  ?

(1)  $f(a)$  must be defined. E.g.

Let  $f(x) = \frac{x^2 - 1}{x - 1}$

Then:

$$\lim_{x \rightarrow 1} f(x) = 2$$

But  $f(1) = \frac{0}{0}$  which is undefined.

Pictorially:

$\frac{x^2 - 1}{x - 1} = x + 1$  except when  $x = 1$

$\therefore g(x) = \begin{cases} \frac{x^2 - 1}{x - 1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$

$\therefore g(x) = x + 1$

(2)  $f(x)$  is "near"  $f(a)$  when  $x$  is "near"  $a$

I.e., curve  $y = f(x)$  is "unbroken" in a neighborhood of  $x = a$ .

Definition:

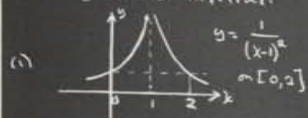
(1)  $f$  is called continuous at  $x = a$  if  $\lim_{x \rightarrow a} f(x) = f(a)$

(2)  $f$  is called continuous on the interval  $I$  if  $\lim_{x \rightarrow a} f(x) = f(a)$  for each  $a \in I$ .



### Geometric Ideas

(1) Cont. functions assume their max. and min values on any closed interval.

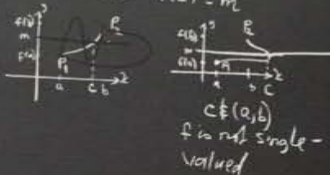


### (2) Intermediate Value Theorem

$f$  cont on  $[a, b]$ ,  $f(a) < f(b)$

Let  $m$  be such that  $f(a) < m < f(b)$

Then we can find  $c \in (a, b)$  such that  $f(c) = m$



### Analytic Ideas

(1)  $f, g$  cont at  $x=a$ .

$$h(x) = f(x) + g(x)$$

$$\lim_{x \rightarrow a} h(x) = \lim_{x \rightarrow a} [f(x) + g(x)]$$

$$= \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

$$= f(a) + g(a)$$

$$\lim_{x \rightarrow a} h(x) = h(a)$$

$\therefore$  Sum of two continuous functions is a continuous function.

(2) Differentiable  $\rightarrow$  Continuous

$$f(x) - f(a) = \left[ \frac{f(x) - f(a)}{x - a} \right] (x - a)$$

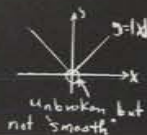
$$\lim_{x \rightarrow a} [f(x) - f(a)] = \lim_{x \rightarrow a} \left[ \frac{f(x) - f(a)}{x - a} \right] \lim_{x \rightarrow a} (x - a)$$
$$= f'(a) \cdot 0 = 0$$

$$\therefore \lim_{x \rightarrow a} f(x) = f(a)$$

Pictorially:

Smooth  $\rightarrow$  Unbroken

Unbroken  $\rightarrow$  Smooth



2.060 Curve Plotting

31 min.

Curve Plotting  
(with and without  
calculator)

$y = x^2$

$y = x^2$

$y < 0 // 1$

$f(x) = x^2 = f(-x)$

$(-x, f(-x))$   $(x, f(x))$

$y = x^2$   
 $y = (-x)^2 = x^2$

ASIDE:  
Even and Odd  
Functions

$f(x) = f(-x) \rightarrow f$  is even

$f(x) = -f(-x) \rightarrow f$  is odd

$(-1, -1)$   $(1, 1)$

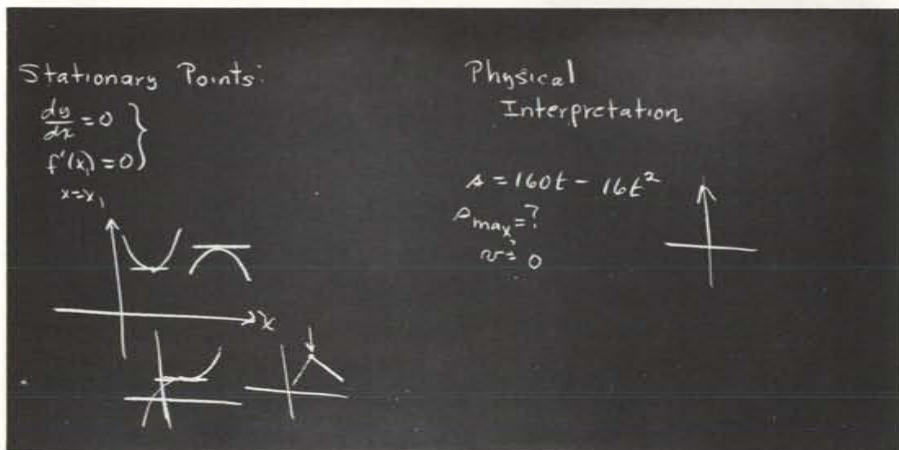
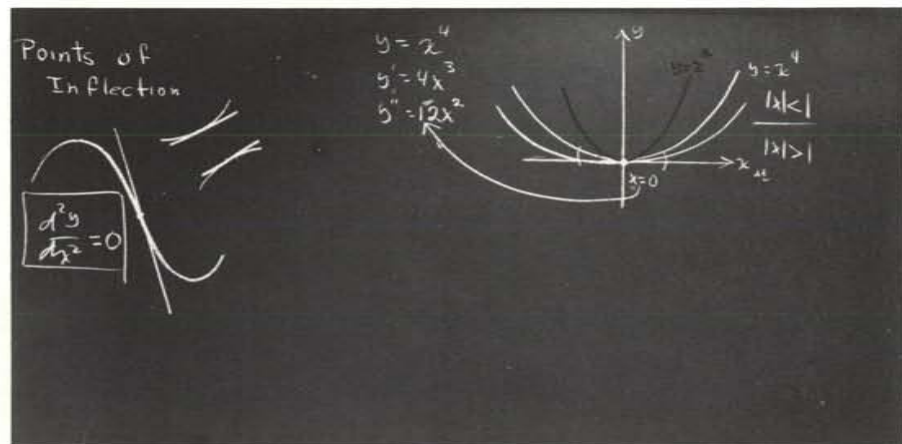
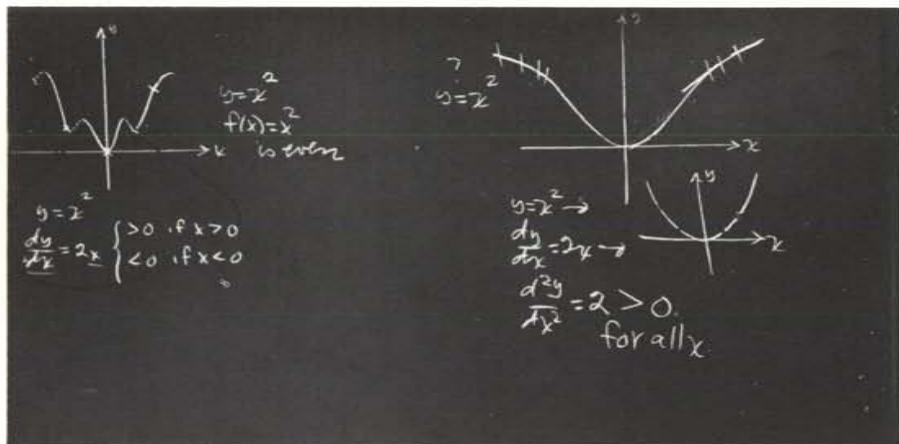
$(-1, -1)$   $(1, 1)$

Example:

$y = x^4 + x^2$   
 $y = (-x)^4 + (-x)^2 = x^4 + x^2$  }  $f(x) = f(-x)$

$y = x^3 + x$   
 $y = (-x)^3 + (-x) = -x^3 - x$  }  $f(x) = -f(-x)$

$f(x) = \frac{f(x) + f(-x)}{2}$  (even) +  $\frac{f(x) - f(-x)}{2}$  (odd)



2.070 Maxima-Minima

34 min.

Maxima-Minima  
High Points-Low Points

Fundamental Theorem:  
Suppose  $f(c) \geq f(x)$   
for all  $x \in N_\delta(c)$  and suppose  
 $f'(c)$  exists. Then  $f'(c) = 0$ .

$x < c \rightarrow f(x) < f(c)$   
 $x > c \rightarrow f(x) < f(c)$

Cautions:

(1) Beware of false  
converses:

(2) Beware if  $f'(c)$  doesn't exist

(3) Beware of Endpoints

$f(x) = x^2$  dom  $f = (2, 3)$   
 $2 < x < 3$

$f'(x) = 2x$   
 $= 0 \iff x = 0$

$f'(x)$  exists for all  $x \in \text{dom } f$

$f(x) = x^2$ , dom  $f = [2, 3]$   
 $2 \leq x \leq 3$

$f'(x) = 0$  No!

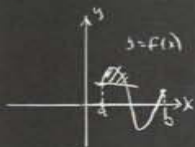
$f'$  non-existent? No!

If  $f$  has a max or min  
it does not occur in  $(a, b)$

$x \in N_\delta(c)$  ~~subset~~  
 $c - \delta < x < c + \delta$



Graphical Interpretation



SUMMARY:

Suppose  $f$  is continuous on  $[a, b]$ .  
Then to find candidates  $c$  such that  $f(c)$  is a minimum (maximum):

- (1) Find all  $c$  such that  $f'(c) = 0$
- (2) Find all  $c$  for which  $f'(c)$  fails to exist
- (3) Check  $x=a$  and  $x=b$

$$V = 30\pi x^2 - \pi x^3$$

$$0 < x < 30$$

$$\frac{dV}{dx} = 60\pi x - 3\pi x^2$$

$$\frac{d^2V}{dx^2} = 60\pi - 6\pi x$$

$$\frac{dV}{dx} = 0 \Leftrightarrow x = 0, 20$$

$$\frac{d^2V}{dx^2} = 0 \Leftrightarrow x = 10 \quad V = V(x)$$

$$(20, 4000\pi) \quad 0 < x < 30$$

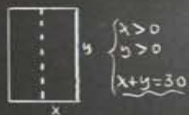


$$V = \pi x^2 y \quad \begin{cases} x, y > 0 \\ x + y = 30 \end{cases}$$

$$\frac{dV}{dx} = 2\pi xy + \pi x^2 \frac{dy}{dx} \quad 1 + \frac{dy}{dx} = 0$$

$$= 2\pi xy - \pi x^2 \quad \left( \frac{dy}{dx} = -1 \right)$$

$$= 0 \Leftrightarrow x = 2y$$



$$V = \pi x^2 y$$

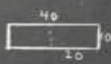
$$= \pi x^2 (30 - x)$$

$$= 30\pi x^2 - \pi x^3, \quad 0 < x < 30$$

$$V'(x) = 60\pi x - 3\pi x^2$$

$$= 0 \Leftrightarrow x = 0, x = 20$$

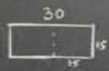
$$y = 10$$



$$A_1 = 400$$

$$V_1 = \pi(20)^2(10)$$

$$= 4000\pi$$



$$A_2 = 450$$

$$V_2 = \pi(15)^2(15)$$

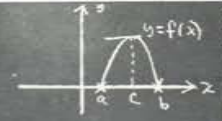
$$= 3375\pi$$

$$V = \pi x^2 y$$

2.080 Rolle's Theorem and its Consequences

30 min.


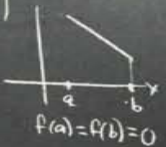
Rolle's Theorem and its Consequences



$[a, b]$   
 $(a, b)$


Cautions:

(1) There may be several  $c$ 's

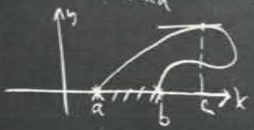
Let  $f$  be defined and continuous on  $[a, b]$  and differentiable in  $(a, b)$ . Suppose also  $f(a) = f(b) = 0$ . Then  $f'(c) = 0$  for at least one  $c \in (a, b)$ .

(2) Curve must be "smooth"

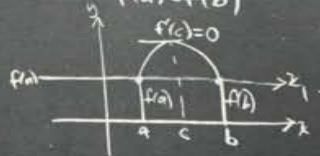


$f'(c) \neq 0$  doesn't exist

(3)  $f$  must be single-valued



(4) (Aside)  $f(a) = f(b) = 0$  is too restrictive

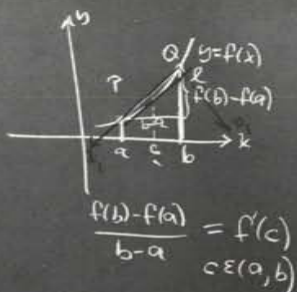


$f(a) = f(b) = f(c) = 0$

The Mean Value Theorem

Let  $f$  be continuous on  $[a, b]$  and differentiable in  $(a, b)$ . Then there exists a number  $c \in (a, b)$  such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$



(2)  $a \leq x \leq b$   
 $f'(x) \equiv g'(x) \rightarrow$   
 $f(x) - g(x) \text{ is constant}$



$$F(x) = f(x) - g(x)$$

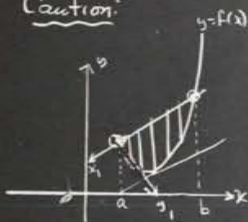
$$F'(x) = f'(x) - g'(x)$$

$$= 0$$

$$F(x) = C$$

$$\underline{f(x) - g(x) = C}$$

Caution:



Mean Value Theorem  
 Supplies  
 Rigor to Intuition

Example:

(1)  $F'(x) \equiv 0 \rightarrow F(x) = C$   
 $\rightarrow a \neq b \rightarrow F(a) = F(b)$

$$\boxed{\frac{F(b) - F(a)}{b - a} = F'(c)}$$

$$F(b) - F(a) = 0$$

$$F(b) = F(a)$$

2.090 Inverse Differentiation

42 min.

Inverse Differentiation

$f \rightarrow \begin{array}{|c|} \hline D\text{-} \\ \hline \text{machine} \\ \hline \end{array} \rightarrow f'$

$D(f), f'; D(f(x)), f'(x)$

$\begin{array}{ccc} \text{dom } D & \xrightarrow{D} & \text{Range } D \\ \left\{ \begin{array}{l} x^2+c \\ x^2+1 \\ f(x) \\ x^3 \end{array} \right\} & & \left\{ \begin{array}{l} 2x \\ 3x^2 \end{array} \right\} \end{array}$

$D$  is not 1-1

$\left\{ x^2+c \right\} \xrightarrow{D^{-1}\text{-machine}} \leftarrow 2x$

Suppose  $D(f(x)) = 2x$   
 we know  $D(x^2) = 2x$   
 $\therefore f(x) - x^2 = \text{constant}$   
 or:  $f(x) \in \{x^2+c\}$

In general:  
 Suppose we are given  $f(x)$   
 Let  $E_f = \{f(x)+c\}$   
 then  
 $\begin{array}{ccc} f(x) & \xrightarrow{D\text{-machine}} & f'(x) \\ E_f & & \end{array}$   
 is 1-1

So:  
 $D^{-1}(f(x)) = \left\{ G(x) : G'(x) = f(x) \right\}$   
 (Implicitly)  
 $= \left\{ F(x) + c : F'(x) = f(x) \right\}$   
 (Explicitly)

Example: Let  $h(x) = \frac{x\sqrt{x^2+1}}{x(x^2+1)^{\frac{3}{2}}}$   
 $h'(x) = \frac{1}{2}x(x^2+1)^{-\frac{3}{2}}(2x) + (x^2+1)^{-\frac{3}{2}}$   
 $= \frac{x^2}{\sqrt{x^2+1}} + \sqrt{x^2+1}$   
 $= \frac{2x^2+1}{\sqrt{x^2+1}}$

$\therefore D^{-1}\left(\frac{2x^2+1}{\sqrt{x^2+1}}\right) = \left\{ \frac{x\sqrt{x^2+1} + c}{\sqrt{x^2+1}} \right\}$

But even if we didn't know this, we would still have:  
 $D^{-1}\left(\frac{2x^2+1}{\sqrt{x^2+1}}\right) = \left\{ G(x) : G'(x) = \frac{2x^2+1}{\sqrt{x^2+1}} \right\}$



### Some Recipes

$$(1) D^{-1}(x^n) = \frac{x^{n+1}}{n+1} + C$$

(since  $D(\frac{x^{n+1}}{n+1} + C) = x^n (n+1)$ )

Example: Determine  $D^{-1}(x^2)$ .

$$D(x^3) = 3x^2$$

$$\left. \begin{aligned} \frac{1}{3} D(x^3) &= x^2 \\ D(\frac{1}{3} x^3) &= \frac{1}{3} D(x^3) \end{aligned} \right\} D(cf) = cD(f)$$

$$\therefore D(\frac{1}{3} x^3) = x^2$$

$$D^{-1}(x^2) = \left\{ \frac{1}{3} x^3 + C \right\}$$

### Beware of non-constant factors

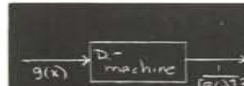
$$D^{-1}[(x^2+1)^7] = ?$$

$$D[(x^2+1)^7] = 8(x^2+1)^7(2x) = 16x(x^2+1)^7$$

$$\frac{1}{16x} D[(x^2+1)^7] = (x^2+1)^7$$

$$D\left[\frac{(x^2+1)^7}{16x}\right] \neq (x^2+1)^7$$

$$D\left[\frac{1}{16}(x^2+1)^7\right] \neq (x^2+1)^7$$



$$g'(x) \equiv \frac{1}{g^2(x)}$$

$$g^2(x)g'(x) \equiv 1$$

$$\begin{cases} 3g^2(x)g'(x) \equiv 3 \\ [g^3(x)]' \equiv 3 \equiv [3x]' \end{cases}$$

$$g^3(x) = 3x + C$$

$$g(x) = \sqrt[3]{3x+C}$$

### Traditional Notation

$$D^{-1}(f(x)) = \int f(x) dx$$

$$(1) \int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

$$(2) \int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx$$

$$(3) \int k f(x) dx = k \int f(x) dx$$

### Inverse Chain Rule

$$D[f(g(x))] = f'(g(x))g'(x)$$

$$D\left[\frac{1}{3}(x^2+1)^3\right] = (x^2+1)^2 \cdot 2x$$

$$D\left[\frac{1}{6}(x^2+1)^3\right] = (x^2+1)^2 \cdot 2x \cdot \frac{1}{3}$$

$$D^{-1}\left[\frac{1}{3}(x^2+1)^2\right] = \frac{1}{9}(x^2+1)^3 + C$$

$$D^{-1}\left[\frac{1}{3}(x^2+1)^2\right] = \frac{1}{9}(x^2+1)^3 + C$$

$$(2) D^{-1}[f(x) + g(x)] = D^{-1}(f(x)) + D^{-1}(g(x))$$

$$D^{-1}(x^5 + x^3) = ? \quad \left\{ \begin{aligned} D(f+g) &= Df + Dg \end{aligned} \right.$$

$$\begin{cases} D(\frac{1}{6}x^6) = x^5 \\ D(\frac{1}{4}x^4) = x^3 \end{cases}$$

$$D(\frac{1}{6}x^6) + D(\frac{1}{4}x^4) = x^5 + x^3$$

$$D(\frac{1}{6}x^6 + \frac{1}{4}x^4) = x^5 + x^3$$

$$D^{-1}(x^5 + x^3) = \left\{ \frac{1}{6}x^6 + \frac{1}{4}x^4 + C \right\}$$

### Why $\int f(x) dx$ ?

Why not  $\int f(x) dx$ ?

A box labeled "D-machine" has an input arrow from the left labeled "f(x)/x^2" and an output arrow to the right labeled "f(x)dx".

$$g'(x) = \frac{1}{[g(x)]^2}$$

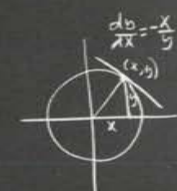
Let  $y = g(x)$

$$\frac{dy}{dx} = \frac{1}{y^2}$$

$$y^2 dy = dx$$

$$3y^2 dy = 3dx$$

$$y^3 = 3x + C, y = \sqrt[3]{3x+C}$$



### Example

$$\frac{dy}{dx} = -\frac{x}{y}$$

and when  $x=3, y=4$

$$y dy = -x dx$$

$$2y dy = -2x dx$$

$$y^2 = -x^2 + C$$

$$x^2 + y^2 = C$$

$$9 + 16 = C$$

$$x^2 + y^2 = 25$$

$$\left. \begin{aligned} g'(x) &\equiv -\frac{x}{g(x)} \\ g(3) &= 4 \end{aligned} \right\}$$

$$g(x)g'(x) \equiv -x$$

$$2g(x)g'(x) \equiv -2x$$

$$[g^2(x)]' \equiv [-x^2]$$

$$[g(x)]^2 = -x^2 + C$$

$$g(x) = \pm \sqrt{C - x^2}$$

2090 LC-4

2.100 The "Definite" Indefinite Integral

29 min.

The "Definite" Indefinite Integral

$$D^{-1}(f(x)) = \int f(x) dx$$

$$= \{G(x) + C \mid G' = f\}$$

$$= \{F(x) + C \mid F' = f\}$$

$f(x) = x^2$   
 $\left[\frac{1}{3}x^3\right]' = x^2$   
 $f(x) = \frac{1}{3}x^3 + C$   
 Suppose  $f(x_0) = y_0$   
 $\therefore C = y_0 - \frac{1}{3}x_0^3$   
 $f(x) = \frac{1}{3}x^3 + y_0 - \frac{1}{3}x_0^3$

$\frac{dy}{dx} = x^2$   
 $y = \frac{1}{3}x^3 + C$   
 $y_0 = \frac{1}{3}x_0^3 + C$   
 $y = \frac{1}{3}x^3 + y_0 - \frac{1}{3}x_0^3$

Generalization

$$\frac{dy}{dx} = f(x) \quad a \leq x \leq b$$

Assume  $G'(x) = f(x)$

$$y(x) = G(x) + C$$

$$y(a) = G(a) + C$$

$$\therefore C = y(a) - G(a)$$

$$y(x) = G(x) + y(a) - G(a)$$

$$y(b) = G(b) + y(a) - G(a)$$

$$y(b) - y(a) = G(b) - G(a)$$

$$\Delta y \Big|_{x=a}^{x=b} = G(x) \Big|_{x=a}^{x=b}, \quad G' = f$$

Notice that if  $H' = f$  then

$$\Delta y \Big|_{x=a}^{x=b} = H(b) - H(a)$$

For in this case  $H' = G'$

$$H(x) = G(x) + C_1$$

$$H(b) = G(b) + C_1$$

$$H(a) = G(a) + C_1$$

$$\underline{H(b) - H(a) = G(b) - G(a)}$$

Geometric Interpretation

Physical Interpretation

$$v = t^2 \quad 0 \leq t \leq 1$$

$$v = \frac{dx}{dt}$$

$$x = \frac{1}{3}t^3 + C$$

when  $t=0$ , let  $x = x_0 (=x(0))$

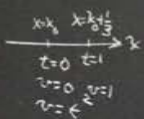
$$\therefore C = x_0$$

$$x(t) = \frac{1}{3}t^3 + x_0$$

$$x(1) = \frac{1}{3} + x_0$$

$$x(0) = x_0$$

$$\Delta x \Big|_{t=0}^{t=1} = \frac{1}{3}$$



Generalization

$$v = f(t), \quad a \leq t \leq b$$

$$G' = f$$

$$\therefore x(t) = G(t) + C$$

$$x(a) = G(a) + C$$

$$\therefore x(t) = G(t) + x(a) - G(a)$$

$$x(b) = G(b) + x(a) - G(a)$$

$$x(b) - x(a) = G(b) - G(a)$$

$$\Delta x \Big|_{t=a}^{t=b} = G(t) \Big|_{t=a}^{t=b}$$

$$G' = f$$

$$G' = f$$

Once we invent

$$\int f(x) dx \text{ to denote } \{G(x); G' = f\}$$

Why not invent

$$\int_a^b f(x) dx \text{ to denote}$$

$$G(b) - G(a) ?$$

$$G(b) = \int_a^b f(x) dx$$

$$G(a) = \int_a^a f(x) dx$$

$$G(b) - G(a) = \int_a^b f(x) dx - \int_a^a f(x) dx$$

Summary

Suppose

$$\frac{dy}{dx} = f(x), \quad a \leq x \leq b$$

Then

$$\Delta y \Big|_{x=a}^{x=b} = \int_a^b f(x) dx$$

$$= G(x) \Big|_{x=a}^{x=b}$$

$$= G(b) - G(a)$$

$$\text{where } G' = f$$

Note

$$\Delta x \Big|_{t=a}^{t=b} \text{ is a}$$

displacement (net)

$$v = t - 1, \quad 0 \leq t \leq 2$$

$$x = \frac{1}{2}t^2 - t + C$$

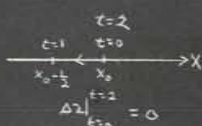
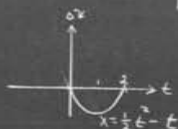
$$x(2) = 2 - 2 + C = C$$

$$x(0) = C$$

$$x(2) - x(0) = 0$$

$$\Delta x \Big|_{t=0}^{t=2} = \frac{1}{2}t^2 - t \Big|_{t=0}^{t=2}$$

$$= 0$$



$$\text{distance travelled} = \frac{1}{2}t^2 - t$$

Block III: The Circular Functions

3.010 Circular Functions

35 min.

Circular Functions

$x^2 + y^2 = 1$   
 $P(x, y)$   
 $\sin t = \overrightarrow{RP} = y$   
 $\cos t = \overrightarrow{OR} = x$

number  $\rightarrow \sin \rightarrow \sin t$

$$\begin{cases} \sin 0 = 0 \\ \cos \frac{\pi}{2} = 0 \\ \sin \frac{\pi}{2} = 1 \\ \cos 0 = 1 \end{cases}$$

$\sin^2 t + \cos^2 t = 1$   
 $y^2 + x^2 = 1$

RADIAN MEASURE

$\neq$  QOS

$x^2 + y^2 = 1$   
 $P(x, y)$   
 $\sin t = \overrightarrow{RP}$   
 $\sin(\frac{\pi}{2} \text{ rad}) = 1$   
 $\sin \frac{\pi}{2}$

$x^2 + y^2 = 1$   
 $P(x, y)$   
 $x = \cos t$   
 $y = \sin t$

$x = \cos t$   
 $y = \sin t$



$f(x) = \sin x$   
 $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$   
 $= \lim_{\Delta x \rightarrow 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x}$   
 $= \lim_{\Delta x \rightarrow 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x}$   
 $= \lim_{\Delta x \rightarrow 0} \left[ \cos x \left( \frac{\sin \Delta x}{\Delta x} \right) + \frac{\cos \Delta x - 1}{\Delta x} \sin x \right]$

$\lim_{t \rightarrow 0} \frac{\sin t}{t} = 1$

$\frac{\sin t \cos t}{2} < \left(\frac{t}{2\pi}\right)\pi < \frac{t}{2}$   
 $\cos t < \frac{t}{\sin t} < \frac{1}{\cos t}$   
 $\lim_{t \rightarrow 0} \frac{t}{\sin t} = 1$

$x = \sin kt$   
 $\frac{dx}{dt} = k \cos kt$   
 $\frac{d^2x}{dt^2} = -k^2 \sin kt$

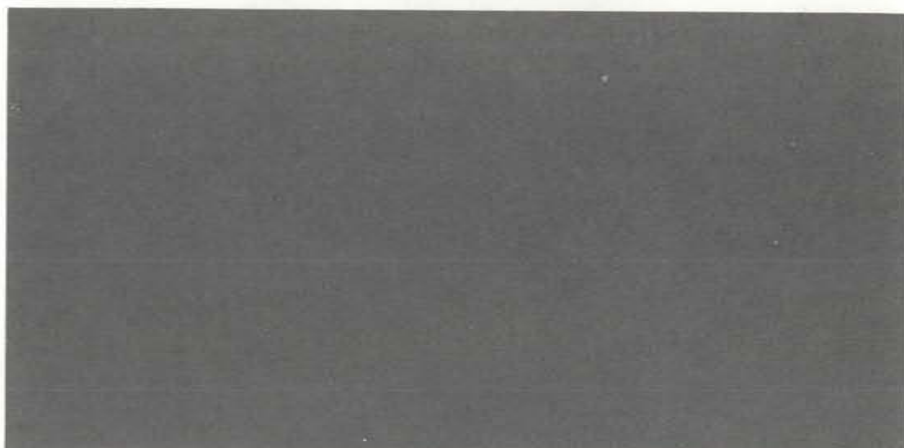
$\int \cos u \, du = \sin u + C$   
 $\int \sin u \, du = -\cos u + C$   
 $\frac{d(\sin^n x)}{dx} = n \sin^{n-1} x \cos x$   
 $\int \sin^n x \cos x \, dx = \frac{1}{n+1} \sin^{n+1} x + C$

$\frac{d^2x}{dt^2} = -k^2 x$

$\lim_{t \rightarrow 0} \frac{\sin t}{t} = 1$   
 $\lim_{t \rightarrow 0} \frac{1 - \cos t}{t} =$   
 $\lim_{t \rightarrow 0} \frac{1 - \cos^2 t}{t(1 + \cos t)} =$   
 $\lim_{t \rightarrow 0} \left[ \frac{\sin t}{t} \right] \left[ \frac{\sin t}{1 + \cos t} \right]$   
 $1 \times 0$   
 $y = \sin x \rightarrow \frac{dy}{dx} = \cos x$

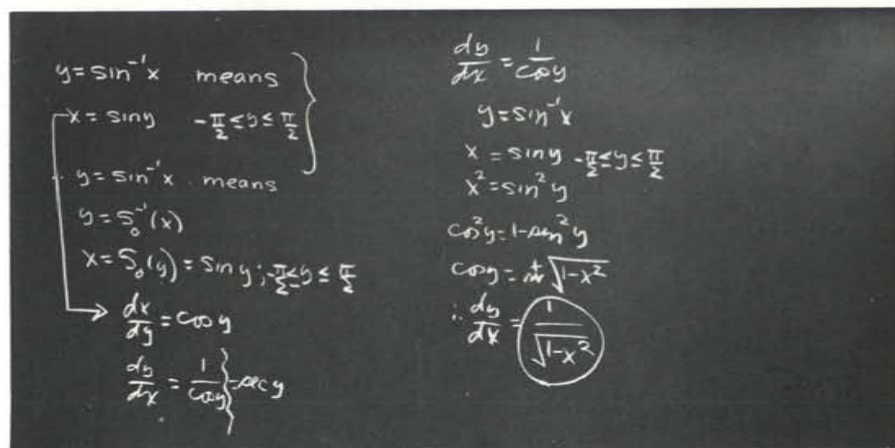
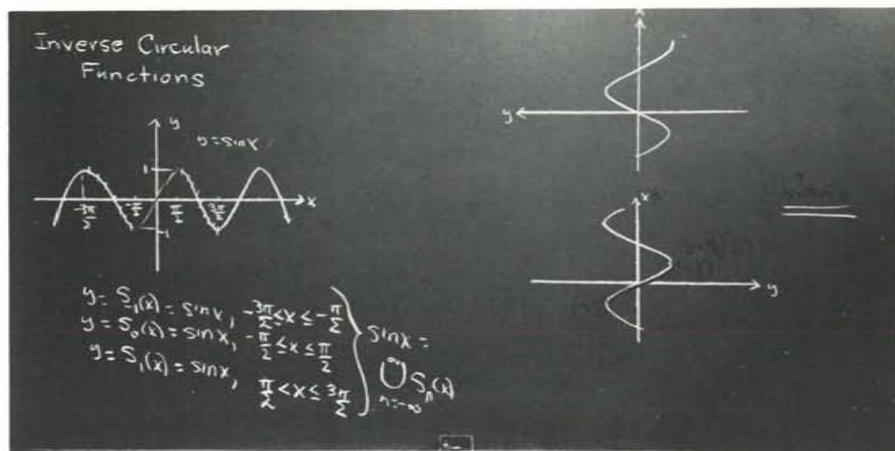
$y = \begin{cases} f(u) \\ \sin u \end{cases}$   
 $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$   
 $= \cos u \frac{du}{dx}$   
 $\cos x = \sin\left(\frac{\pi}{2} - x\right)$   
 $\frac{d(\cos x)}{dx} = -\cos\left(\frac{\pi}{2} - x\right)$   
 $= -\sin x$

3010  
LC-4



3.020 Inverse Circular Functions

26 min.



$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$



$$\sin \theta = x$$

$$\cos \theta = \sqrt{1-x^2}$$

$$- \sin \theta d\theta = \frac{-dx}{\sqrt{1-x^2}}$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos \theta d\theta}{\cos \theta}$$

$$= \theta + C = \sin^{-1} x + C$$

$$y = \cos^{-1} x$$



$$\rightarrow y + (\frac{\pi}{2} - y) = \frac{\pi}{2}$$

$$\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}$$

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$$

$$\begin{aligned} \frac{d(\cos^{-1} x)}{dx} &= \frac{d}{dx} \left( \frac{\pi}{2} - \sin^{-1} x \right) \\ &= - \frac{d}{dx} (\sin^{-1} x) = \frac{-1}{\sqrt{1-x^2}} \end{aligned}$$

Block IV: The Definite Integral

4.010 2-dimensional Area

36 min.

(2-dimensional) Area  
or  
"Calculus Revisited"  
Revisited

Area  
↓  
Integral Calculus | Diff'l Calculus  
Greece, 600 B.C. | 1680 A.D.

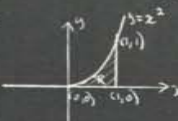
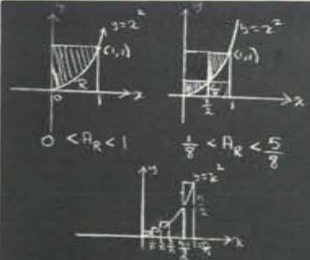
Axioms for Area

- (1)  $A = bh$ , for a rectangle
- (2)  $R \subset S$  implies  $A_R \leq A_S$
- (3)  $R = R_1 \cup \dots \cup R_n$  then  
 $A_R = A_{R_1} + \dots + A_{R_n} = \sum_{k=1}^n A_{R_k}$

Method of Exhaustion:

Given a region  $R$ , we "squeeze it" between two networks of rectangles.

Example  
We wish to determine the area,  $A_R$ , of the region  $R$  where:

For each  $n$ :

$$L_n < A_R < U_n$$

$$\lim_{n \rightarrow \infty} L_n = \lim_{n \rightarrow \infty} U_n$$

(since  $U_n - L_n = \frac{1}{n}$ )  
 $\therefore \lim_{n \rightarrow \infty} (U_n - L_n) = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$ )

$$\therefore A_R = \lim_{n \rightarrow \infty} U_n = \lim_{n \rightarrow \infty} L_n$$

$$L_{1000} = 0.3328335$$

$$U_{1000} = 0.3338335 \leftarrow A_R = 0.33$$

$$U_n = \left(\frac{1}{n}\right)^2 \frac{1}{2} + \left(\frac{2}{n}\right)^2 \frac{1}{4} + \dots + \left(\frac{n-1}{n}\right)^2 \frac{1}{2}$$

$$= \frac{1}{n^2} [1^2 + \dots + n^2]$$

$$L_n = \frac{1}{n^2} [1^2 + \dots + (n-1)^2]$$



$$u_n = \frac{1}{n^3} [n(n+1)(2n+1)]$$

$$= \frac{1}{n^3} \left( \frac{n+1}{n} \right) \left( \frac{2n+1}{n} \right)$$

$$= \frac{1}{n^2} \left( 1 + \frac{1}{n} \right) \left( 2 + \frac{1}{n} \right)$$

$$> \frac{1}{n^2} (1)(2)$$

$$\frac{L_n}{n^2} > \frac{1}{3} \leftarrow u_n$$

Similarly  $A_n$

$$L_n = \frac{1}{n^2} \left( 1 - \frac{1}{n} \right) \left( 2 - \frac{1}{n} \right)$$

For each  $n$ ,  $L_n < \frac{1}{3}$ ,  $L_n < L_{n+1}$

$$\lim_{n \rightarrow \infty} L_n = \frac{1}{3}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} = \frac{1}{3}, \lim_{n \rightarrow \infty} \frac{1}{n^2 + \frac{1}{n^3}} = \frac{1}{3}$$

### Generalization

Let  $f$  be continuous on  $[a, b]$  (and non-decreasing) and define  $R$  by:



Partition  $[a, b]$  into  $n$  equal parts

$$a = x_0 < x_1 < \dots < x_n = b; \Delta x = \frac{b-a}{n}$$

and define:

$$U_n = f(x_1)\Delta x + \dots + f(x_n)\Delta x = \sum_{k=1}^n f(x_k)\Delta x$$

$$L_n = f(x_0)\Delta x + \dots + f(x_{n-1})\Delta x = \sum_{k=0}^{n-1} f(x_k)\Delta x$$

Then: (1)  $\lim_{n \rightarrow \infty} U_n = \lim_{n \rightarrow \infty} L_n$   
 (since  $U_n - L_n = [f(x_n) - f(x_0)] \Delta x$ )  $\left. \begin{array}{l} = A_n = \lim_{n \rightarrow \infty} U_n \\ = \lim_{n \rightarrow \infty} L_n \end{array} \right\} = \int_a^b f(x) dx$

### Aside #1

Let  $c_k \in [x_{k-1}, x_k]$

$$\text{Form: } \sum_{k=1}^n f(c_k)\Delta x$$

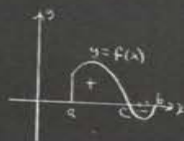
Then  $L_n < \sum_{k=1}^n f(c_k)\Delta x < U_n$

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k)\Delta x = \left. \begin{array}{l} \lim_{n \rightarrow \infty} U_n \\ \lim_{n \rightarrow \infty} L_n \end{array} \right\} = \int_a^b f(x) dx$$

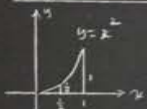


### Aside #2

We can remove the restriction that  $f$  be non-negative if we replace area by net area.



### Trapezoidal Approximations



$$A_{\frac{1}{2}} = \frac{1}{2} \left[ \frac{1}{2} + \frac{1}{2} \right] \frac{1}{2} = \frac{1}{2} + \frac{1}{8} = \frac{5}{8} = 0.625$$



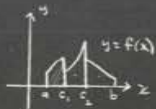
$$A_{\frac{1}{3}} = \frac{1}{3} \left[ \frac{1}{4} + \frac{1}{2} + \frac{1}{4} + \frac{1}{9} + \frac{4}{9} + \frac{1}{4} + 1 \right]$$

$$= \frac{1}{12} [11 + 4 + 4 + 9 + 9 + 16]$$

$$= \frac{44}{12} = \frac{11}{3} \approx 3.667$$

### Definition


$f$  is called piecewise continuous on  $[a, b] \iff f$  is continuous except at a finite number of points where it has "jump" discontinuities.



400

4.020 Marriage of Differential & Integral Calculus 30 min.

The "Marriage" of Differential and Integral Calculus



$f(x) \Delta x < \Delta A < f(x+\Delta x) \Delta x$   
 $f(x) < \frac{\Delta A}{\Delta x} < f(x+\Delta x)$   
 $\swarrow \quad \searrow$   
 $f(x)$

$\frac{dA}{dx} = f(x)$  or  $\frac{dA}{dx} = f(x)$

First Fundamental Theorem of Integral Calc.

Suppose we know explicitly a function  $G$  such that  $G' = f$ . Then  $A(x) = G(x) + C$ .

Since  $A(a) = 0$ , we have:  
 $0 = A(a) = G(a) + C; \therefore C = -G(a)$


$\therefore A(x) = G(x) - G(a)$

Letting  $x = b$   
 $A_R = A(b) = G(b) - G(a); G' = f$

But  $A_R = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x$

That is, we can compute  $\lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x$  by use of "inverse" derivatives.

Example:



$A_R = \lim_{n \rightarrow \infty} \sum_{k=1}^n \sin\left(\frac{k\pi}{2n}\right) \frac{\pi}{2n} = 1$

$= G\left(\frac{\pi}{2}\right) - G(0)$ , where  $G(x) = \sin x$   
 $= -\cos x \Big|_0^{\pi/2} = 0 - (-1) = 1$

Aside

we earlier used  $\int_a^b f(x) dx$  to denote  $G(b) - G(a)$  where  $G' = f$

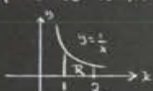
Hence, we have shown that  $A_R = \int_a^b f(x) dx = G(b) - G(a); G' = f$

Historically,  $\int_a^b f(x) dx$  was "invented" to denote  $\lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x$

Then by the First Fundamental Theorem,  $\int_a^b f(x) dx = G(b) - G(a); G' = f$

Second Fundamental  
Thm of Integral Calc.

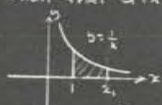
Suppose we want  $A_R$   
where



$$A_R = \int_1^2 \frac{dx}{x} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{1}{x_k}\right) \Delta x$$
$$= G(2) - G(1) \text{ where } G'(x) = \frac{1}{x}$$

But I don't know (explicitly)  
such a  $G$ !

In fact, because we can  
"pinpoint"  $A_R$ , we can "construct"  
 $G$  such that  $G'(x) = \frac{1}{x}$ . Namely



$$G(x) = A(x) = \int_1^x \frac{dy}{y} \rightarrow$$
$$G'(x) = A'(x) = \frac{1}{x}$$

(Aside:  $G(x) = \ln x \therefore A_R = \ln 2$   
and we may, therefore, compute  
 $\ln 2$  as an infinite sum)

In general

Let  $f$  be continuous  
on  $[a, b]$ . Define  $G$  by:

$$G(x) = \int_a^x f(x) dx, \quad x \in [a, b]$$
$$= \lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x$$

Then:

$$G'(x) = f(x)$$

Summary:

(1) First Fund. Thm allows us  
to compute  $\lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x$   
provided we can find  $G$  such that  
 $G' = f$ . In this case:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x = G(b) - G(a)$$

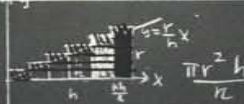
(2) Second Fund. Thm allows us,  
given  $f$ , to construct  $G$  such  
that  $G' = f$ . Namely:

$$G(x) = \int_a^x f(x) dx \quad (= \lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x)$$

4.030 3-dimensional Area (Volume)

42 min.

3-dimensional Area



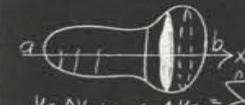
(1)  $V_{\text{cyl}} = Ah$

(2)  $R \approx S \rightarrow V_R \approx V_S$


(3)  $R = \bigcup_{k=1}^n R_k$   
 $V_R = \sum_{k=1}^n V_{R_k}$

$\frac{r}{n} \left( \frac{h}{n} \right) = \frac{kr}{n}$   
 $\pi \left( \frac{kr}{n} \right)^2 \frac{h}{n} = \frac{\pi r^2 h}{n^3} k^2$   
 $U_n = \frac{\pi r^2 h}{n^3} \sum_{k=1}^n k^2$   
 $= \frac{\pi r^2 h}{n^3} \frac{n(n+1)(2n+1)}{6}$   
 $= \frac{1}{6} \pi r^2 h \left( 1 + \frac{1}{n} \right) \left( 2 + \frac{1}{n} \right)$   
 $\lim_{n \rightarrow \infty} U_n = \frac{1}{3} \pi r^2 h$


Solids of Revolution



$V = \Delta V_1 + \dots + \Delta V_n = \sum_{k=1}^n \Delta x_k$   
 $A(M_k) \Delta x_k \leq \Delta V_k \leq A(M_k) \Delta x_k$   
 $\sum_{k=1}^n A(M_k) \Delta x_k \leq V \leq U_n = \sum_{k=1}^n A(M_k) \Delta x_k$   
 $V = \lim_{n \rightarrow \infty} U_n = \int_a^b f(x) dx$   
 $A(M) \Delta x \leq \Delta V \leq A(M) \Delta x$   
 $A(M) \leq \frac{\Delta V}{\Delta x} \leq A(M) \quad \frac{dV}{dx} = A(x)$   
 $V = \int_a^b A(x) dx = G(b) - G(a)$



$A(x) = \pi y^2 = \pi f^2(x)$   
 $V_R = \int_a^b \pi f^2(x) dx$





Cylindrical Shells

$A(x) = \pi(r^2)$   
 $= \frac{\pi r^2}{h} x$   
 $V = \frac{\pi r^2}{h} \int_0^h x^2 dx$   
 $= \frac{\pi r^2}{h} \left[ \frac{1}{3} x^3 \right]_0^h$   
 $= \frac{1}{3} \pi r^2 h$

$\pi(x+\Delta x)^2 f(x) \leq \Delta V \leq \pi(x+\Delta x)^2 f(x+\Delta x)$   
 $\frac{\pi(2x\Delta x + \Delta x^2) f(x)}{\Delta x} \leq \Delta V \leq \frac{\pi(2x\Delta x + \Delta x^2) f(x+\Delta x)}{\Delta x}$   
 $\pi(2x + \Delta x) f(x) \leq \frac{\Delta V}{\Delta x} \leq \pi(2x + \Delta x) f(x + \Delta x)$

$y = 2x - x^2 \quad 0 \leq x \leq 2$

$x^2 - 2x + y = 0$   
 $x = \frac{2 \pm \sqrt{4 - 4y}}{2}$   
 $= \frac{2 \pm \sqrt{4 - 4y}}{2}$   
 $= 1 \pm \sqrt{1 - y}$

$A(y) = \pi(1 + \sqrt{1 - y})^2 - \pi(1 - \sqrt{1 - y})^2$   
 $= 4\pi \sqrt{1 - y}$   
 $V = 4\pi \int_0^1 (1 - y)^{1/2} dy$   
 $= 4\pi \left[ -\frac{2}{3} (1 - y)^{3/2} \right]_0^1$   
 $= \frac{8\pi}{3}$

$V_y = \int_0^2 2\pi x(2x - x^2) dx$   
 $= 2\pi \int_0^2 (2x^2 - x^3) dx$   
 $= 2\pi \left[ \frac{2}{3} x^3 - \frac{1}{4} x^4 \right]_0^2$   
 $= 2\pi \left( \frac{16}{3} - 4 \right)$   
 $= \frac{8\pi}{3} \left( > 2 \int_0^1 2\pi x \sqrt{1 - x^2} dx \right)$

$\pi x f(x) \leq \frac{dV}{dx} \leq 2\pi x f(x)$   
 $\frac{dV}{dx} = 2\pi x f(x)$   
 $V = \int_a^b 2\pi x f(x) dx = G(b)$   
 $G' = f^{-1} \circ G$

$V = \int_0^h 2\pi x y dx$   
 $= \int_0^h 2\pi x y dx$

$V = \int_0^h 2\pi x \left( \frac{r}{h} x \right) dx$   
 $= \frac{2\pi r}{h} \int_0^h x^2 dx$   
 $= \frac{2\pi r}{h} \left[ \frac{1}{3} x^3 \right]_0^h$   
 $= \frac{2}{3} \pi r h^2$

Check  
 $\pi h^2 r - \frac{1}{3} \pi h^2 r = \frac{2}{3} \pi h^2 r$   
 $\frac{2}{3} \pi h^2 r \neq \frac{1}{3} \pi r^2 h$   
 (equality holds only if  $\frac{2}{3} h = r$ )

$f$  cont on  $[a, b]$   
 $a = x_0 < x_1 < \dots < x_n = b$   
 $Q = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n f(c_k) \Delta x_k$   
 $c_k \in [x_{k-1}, x_k]$   
 $\Delta x_k = x_k - x_{k-1}$

$Q = \int_a^b f(x) dx$   
 $= G(b) - G(a), G' = f$

403

4.040 1-dimensional Area (Arc Length)

36 min.


1-dimensional Area  
(arc length)

Axiom #1  
We can measure the length of any straight line segment.

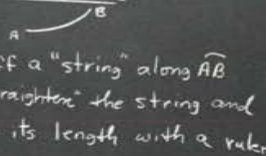
Axiom #2  
The length of the whole equals the sum of the lengths of the parts.

However, it need not be true that  $R \subset S \rightarrow L_R \leq L_S$ .

For example:




Intuitive Approach

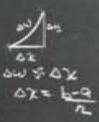


Lay off a "string" along  $\widehat{AB}$ . Then "straighten" the string and measure its length with a ruler.

Analytical Approach: Trial #1



$y=f(x)$




$\frac{\Delta y}{\Delta x}$   
 $\Delta y \approx \Delta x$   
 $\Delta x = \frac{b-a}{n}$

$A_n = \lim_{n \rightarrow \infty} U_n$  but only because

$\begin{cases} L_n < A_n < U_n \\ \text{and } \lim_{n \rightarrow \infty} L_n = \lim_{n \rightarrow \infty} U_n \end{cases}$

However, if we let  $\Delta y \rightarrow 0$  and define  $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n \Delta x = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{(b-a)}{n}$  then  $\int_a^b = b-a$  not  $\omega$

Analytical Approach: Trial #2



$\frac{\Delta y}{\Delta x}$

Now let  $\Delta y \approx \Delta x$ , and define  $L_n^b$  by

$$L_n^b = \lim_{n \rightarrow \infty} \sum_{k=1}^n \Delta a$$

$$= \lim_{n \rightarrow \infty} \sum_{k=1}^n \sqrt{\Delta x^2 + \Delta y^2}$$

"The" three questions

→ (1) Does the limit,  $L_a^b$  exist?  
(2) If so, how do we compute it?  
(3) How does  $L_a^b$  compare with our intuitive ideas about arc length?

Answer to Question (2):

$$L_a^b = \lim_{n \rightarrow \infty} \sum_{k=1}^n \sqrt{\Delta x^2 + \Delta y^2}$$

$$\sqrt{\Delta x^2 + \Delta y^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x$$

If  $f$  is differentiable on  $[a, b]$  we may invoke MVT to conclude

$$\frac{\Delta y}{\Delta x} = f'(c_k), c_k \in [x_{k-1}, x_k]$$



$$\therefore L_a^b = \lim_{n \rightarrow \infty} \sum_{k=1}^n \sqrt{1 + [f'(c_k)]^2} \Delta x$$

Now if  $f'$  is continuous, so also are  $[f'(x)]^2$ ,  $1 + [f'(x)]^2$  and  $\sqrt{1 + [f'(x)]^2}$ . Thus, in this case

$$L_a^b = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

(which may be hard to evaluate)

Summary

If  $f$  is differentiable on  $[a, b]$  and  $f'$  is continuous on  $[a, b]$  then

$$L_a^b = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Suppose  $\Delta w_k = g(c_k) \Delta x + \alpha_k \Delta x$   
Then:  $w = \sum_{k=1}^n \Delta w_k$   
 $= \sum_{k=1}^n g(c_k) \Delta x + \sum_{k=1}^n \alpha_k \Delta x$

$$\therefore w - \sum_{k=1}^n g(c_k) \Delta x = \sum_{k=1}^n \alpha_k \Delta x$$

$$w - \int_a^b g(x) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n \alpha_k \Delta x \neq 0$$

Situation #1:  $\alpha_k = c(\neq 0)$  for all  $k$

$$\begin{aligned} \text{Then } \lim_{n \rightarrow \infty} \sum_{k=1}^n \alpha_k \Delta x &= \lim_{n \rightarrow \infty} \sum_{k=1}^n c \Delta x \\ &= \lim_{n \rightarrow \infty} c \left( \sum_{k=1}^n \Delta x \right) \\ &= c(L-a) \neq 0 \end{aligned}$$

Situation #2:  $\alpha_k = \beta \Delta x$

$$\begin{aligned} \text{Then } \sum_{k=1}^n \alpha_k \Delta x &= \sum_{k=1}^n \beta (\Delta x)^2 \\ &= \sum_{k=1}^n \beta \left(\frac{b-a}{n}\right)^2 \\ &= n \beta \left(\frac{b-a}{n}\right)^2 \\ &= \beta \frac{(b-a)^2}{n} \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} \sum_{k=1}^n \alpha_k \Delta x = 0$$

$$\therefore w = \int_a^b g(x) dx$$

in this situation

In other words, we have let  $\Delta w = \Delta \epsilon$

$$\text{and assumed that } w = \sum_{k=1}^n \Delta w \approx \sum_{k=1}^n \Delta \epsilon$$

what we have shown is that

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \Delta \epsilon (= L_a^b) \text{ exists. We have not}$$

shown that this limit is  $w$ !

In essence, how do we know if all the error has been squeezed out?

This is precisely what Question (3) is all about

Generalization of Question (3)

Suppose  $w$  is any function defined on  $[a, b]$  and we assume that  $\Delta w = g(c_k) \Delta x$  where  $g$  is some "intuitive" function defined on  $[a, b]$

$$\text{Then: } w = \sum_{k=1}^n \Delta w \approx \sum_{k=1}^n g(c_k) \Delta x$$

and if  $g$  is continuous on  $[a, b]$ ,  $\lim_{n \rightarrow \infty} \sum_{k=1}^n g(c_k) \Delta x$  exists and is denoted by  $\int_a^b g(x) dx$ .

The question is:

$$\text{Does } w = \int_a^b g(x) dx?$$

In general, if  $\Delta w_k = g(c_k) \Delta x + \alpha_k \Delta x$ , and for each  $k$ ,  $\lim_{\Delta x \rightarrow 0} \alpha_k = 0$

then:  $w = \int_a^b g(x) dx$

In our problem, then, we must show that  $\Delta w_k - \sqrt{1 + [f'(c_k)]^2} \Delta x$  is a higher order differential

$$\Delta \epsilon < \Delta w < \overline{AB} + \overline{BC}$$

$$\overline{AB} = \sqrt{1 + [f'(c_k)]^2} \Delta x$$

$$\overline{BC} = \epsilon \Delta x, \lim_{\Delta x \rightarrow 0} \epsilon = 0$$

$$\Delta \epsilon = \sqrt{1 + [f'(c_k)]^2} \Delta x$$

$$\geq \sqrt{1 + [f'(c_k)]^2} \Delta x$$

$$\sqrt{1 + [f'(c_k)]^2} \Delta x \leq \Delta w \leq \sqrt{1 + [f'(c_k)]^2} \Delta x + \epsilon \Delta x$$

$$\therefore \Delta w = \sqrt{1 + [f'(c_k)]^2} \Delta x + \alpha \Delta x$$

where  $\alpha < \epsilon$

$$\lim_{\Delta x \rightarrow 0} \alpha < \lim_{\Delta x \rightarrow 0} \epsilon = 0$$



Block V: Transcendental Functions

5.010 Logarithms without Exponents

34 min.

Logarithms Without Exponents

$$\frac{dm}{dt} = km$$

$$\frac{dm}{m} = k dt$$

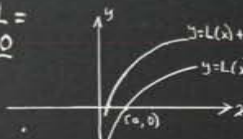
$$\int \frac{dm}{m} = kt + C$$

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$   
 $n = -1$   
 $D(x^{-1}) \rightarrow 0 \neq \frac{1}{x}$

Problem:  
 To determine  $L(x)$  such that  $L'(x) = \frac{1}{x}$ .  
 (This is the case of  $\int x^n dx$  with  $n = -1$ )

Differential Calculus Approach

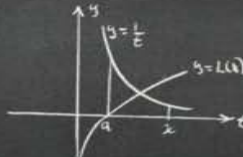
$y = L(x)$   $dm/L = k > 0$   
 $y' = \frac{1}{x}$   
 $y'' = -\frac{1}{x^2}$



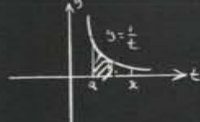
A graph showing two curves,  $y = L(x)$  and  $y = L(x) + c$ , on a coordinate system. The origin is marked as  $(0,0)$ . The curves are increasing and concave down, starting from the origin.

Integral Calculus Approach

Pick  $a > 0$  and once picked "fix" it.



A graph showing the function  $y = \frac{1}{t}$  and the function  $y = L(x)$  on a coordinate system. The x-axis is labeled  $t$  and the y-axis is labeled  $y$ . The curve  $y = \frac{1}{t}$  is a hyperbola in the first quadrant. The curve  $y = L(x)$  is a straight line passing through the point  $(a, 0)$  on the x-axis.



A graph showing the function  $y = \frac{1}{t}$  on a coordinate system. The x-axis is labeled  $t$  and the y-axis is labeled  $y$ . The curve  $y = \frac{1}{t}$  is a hyperbola in the first quadrant. A shaded region is shown under the curve from  $t = a$  to  $t = x$ .

Define  $L(x)$  by:

$$L(x) = \int_a^x \frac{dt}{t}$$

$\left[ \int_a^x f(t) dt \right]' = f(x)$



### Logarithmic Functions

$f$  is called logarithmic

$$\text{if } f(x_1 x_2) = f(x_1) + f(x_2)$$

for all  $x_1, x_2$  in dom  $f$

If  $f$  is logarithmic,

then (1)  $f(1) = 0$ , since

$$f(1) = f(1 \cdot 1) = f(1) + f(1)$$

(2) If  $f(\frac{1}{x})$  is defined

then  $f(\frac{1}{x}) = -f(x)$ , since

$$0 = f(1) = f(x \cdot \frac{1}{x}) = f(x) + f(\frac{1}{x})$$

$$(3) f(\frac{x}{b}) = f(x) - f(b), \text{ since}$$

$$f(\frac{x}{b}) = f(x \cdot \frac{1}{b}) = f(x) + f(\frac{1}{b}) = f(x) - f(b)$$

$$(4) f(x^n) = n f(x) \text{ for any positive integer } n$$

$$\text{For: } f(x^n) = f(\underbrace{x \cdot \dots \cdot x}_{n \text{ times}}) = \underbrace{f(x) + \dots + f(x)}_{n \text{ times}} = n f(x)$$

How is this related to  $L(x)$ ?

### Example

$$y = uv$$

$$\ln y = \ln(uv)$$

$$= \ln u + \ln v$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{u} \frac{du}{dx} + \frac{1}{v} \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{y}{u} \frac{du}{dx} + \frac{y}{v} \frac{dv}{dx}$$

$$= v \frac{du}{dx} + u \frac{dv}{dx}$$

with "traditional" logarithms, if the base is  $b$  then

$$\log_b b = 1$$

Thus if  $e$  is to be the "base" for  $\ln x$ , then

$$\ln e = 1$$

Let's compare

$L(bx)$  with  $L(b) + L(x)$

$$\frac{dL(bx)}{dx} = \frac{dL(bx)}{d(bx)} \frac{d(bx)}{dx} = \frac{1}{bx} \cdot b = \frac{1}{x}$$

$$\frac{d(L(b) + L(x))}{dx} = \frac{1}{x}$$

$$\therefore L(bx) = L(b) + L(x) + C$$

If  $z=1$ :

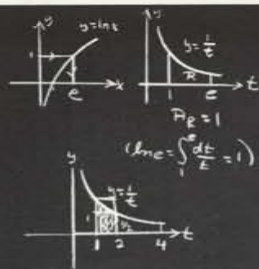
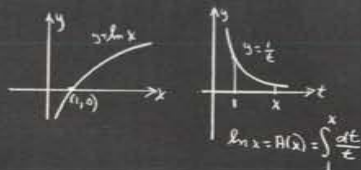
$$L(b) = L(b) + L(1) + C$$

$$\therefore C = -L(1) \quad \therefore L(1) = 0$$

$\therefore L(x)$  is logarithmic  $\leftrightarrow$   
 $L(1) = 0$

Define  $\ln z$  to be this member of the family,  $L(x) + C$

That is:  $\frac{d(\ln x)}{dx} = \frac{1}{x}$  and  $\ln(1) = 0$



$$\frac{1}{2} < \ln 2 < 1$$

$$\ln e = 1$$

$$\ln 4 = \ln(2^2) = 2 \ln 2 > 1$$

$$\therefore \ln 2 < \ln e < \ln 4 \quad \frac{d \ln x}{dx} = \frac{1}{x} > 0$$

$$2 < e < 4$$

$$\checkmark \frac{dm}{dt} = km$$

$$\frac{dm}{m} = k dt$$

$$\int \frac{dm}{m} = \int k dt + C$$

$$\ln m = kt + C$$

$$m = e^{kt+C} = e^{kt} \cdot e^C$$

5.020 Inverse Logarithms

21 min.

INVERSE  
Logarithms

$y = a^x \quad a > 0$   
 $\frac{d \ln a}{da} = \frac{1}{a}$

$y = \ln^{-1} x$

Claim

$$\ln^{-1}(x_1 + x_2) = (\ln^{-1} x_1, \ln^{-1} x_2)$$

$$y_1 = \ln^{-1} x_1, y_2 = \ln^{-1} x_2$$

$$\therefore x_1 = \ln y_1$$

$$x_2 = \ln y_2$$

$$\therefore x_1 + x_2 = \ln y_1 + \ln y_2$$

$$= \ln(y_1 y_2)$$

$$\therefore \ln^{-1}(x_1 + x_2) = y_1 y_2$$

Find  $\frac{dy}{dx}$  if  $y = \ln^{-1} x$

$$y = \ln^{-1} x \rightarrow x = \ln y$$

$$\frac{dx}{dy} = \frac{1}{y} \quad \boxed{\frac{dy}{dx} = y}$$

$$\therefore \frac{d(\ln^{-1} x)}{dx} = \ln^{-1} x$$

$$\int \frac{dy}{y} = \int dx$$

↑

Notation

$\ln^{-1} x$  is usually abbreviated by  $(e^x)$

This matches the identification of  $\ln x$  with  $\log_e x$

$$\therefore \frac{d(e^x)}{dx} = e^x$$

$$\frac{de^u}{dx} = \frac{de^u}{du} \frac{du}{dx}$$

$$= e^u \frac{du}{dx}$$

$$\int e^{-x^2} dx = ?$$

$$\int 2xe^{-x^2} dx = ?$$

$$u = -x^2$$

$$du = -2x dx$$

$$\int 2xe^{-x^2} dx =$$

$$\int -e^u du = -e^u + C = -e^{-x^2} + C$$

$$\int e^{-x^2} d(-x^2) = e^{-x^2} + C$$

$$\int -2xe^{-x^2} dx = e^{-x^2} + C$$

5020  
16-3

$$y'' - 5y' + 6y = 0$$

Try  $y_1 = e^{rx}$

$$y_1' = re^{rx}$$

$$y_1'' = r^2 e^{rx}$$

$$r^2 e^{rx} - 5r e^{rx} + 6e^{rx} = 0$$

$$e^{rx} (r^2 - 5r + 6) = 0$$

$$\therefore r = 2 \text{ or } r = 3$$

$$y = e^{2x}$$

$$y' = 2e^{2x}$$

$$y'' = 4e^{2x}$$

$$y'' - 5y' + 6y = 0$$

$$y = e^{3x}$$

$$y' = 3e^{3x}$$

$$y'' = 9e^{3x}$$

$$y'' - 5y' + 6y = 0$$

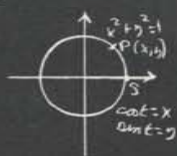
In general if  $a$  and  $b$  are constants  $y_1 = e^{rx}$   
"transforms"  $y'' + ay' + by = 0$   
into  $r^2 + ar + b = 0$

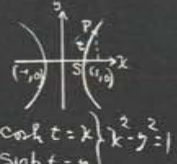
5020  
16-4

5.030 What a Difference a Sign Makes

27 min.


What a Difference a Sign Makes

$\begin{cases} x^2 + y^2 = 1 \\ \text{Circular Functions} \end{cases}$ 


$\begin{cases} x^2 - y^2 = 1 \\ \text{Hyperbolic Functions} \end{cases}$ 


ASIDE

(1)  $x^2 - y^2 = 1$  means  $x^2 + (iy)^2 = 1$  where  $i^2 = -1$



(2) Given  $a$  and  $b$   $\begin{matrix} a=1000 \\ b=4 \end{matrix}$

let  $x = \frac{a+b}{2}$   $\frac{a+b}{2} = 502$

$y = \frac{a-b}{2}$   $\frac{a-b}{2} = 498$

then  $x+y=a$   
 $x-y=b$

$\begin{cases} D(e^t) = e^t \\ D(e^{-t}) = -e^{-t} \\ D(e^t e^{-t}) = e^t - e^{-t} \\ D(e^t - e^{-t}) = e^t + e^{-t} \end{cases}$

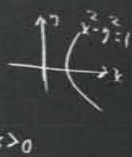
Let  $\begin{cases} C(t) = e^t + e^{-t} \\ S(t) = e^t - e^{-t} \end{cases}$

$\therefore \begin{cases} C'(t) = S(t) \\ S'(t) = C(t) \end{cases}$

$\begin{cases} C^2(t) = e^{2t} + 2 + e^{-2t} \\ S^2(t) = e^{2t} - 2 + e^{-2t} \end{cases}$

$C^2(t) - S^2(t) = 4$

$\left[ \frac{C(t)}{2} \right]^2 - \left[ \frac{S(t)}{2} \right]^2 = 1$

$\begin{cases} \cosh t = \frac{C(t)}{2} = \frac{e^t + e^{-t}}{2} \\ \sinh t = \frac{S(t)}{2} = \frac{e^t - e^{-t}}{2} \end{cases}$ 


$\begin{cases} x = \cosh t \\ y = \sinh t \end{cases} \rightarrow x^2 - y^2 = 1, x > 0$



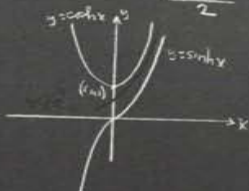
$$y = \tanh x$$

$$= \frac{\sinh x}{\cosh x}$$

$$\frac{d(\tanh x)}{dx} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$$
$$= \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

$$\int \operatorname{sech}^2 x = \tanh x + C$$

$$\begin{cases} y = \cosh x = \frac{e^x + e^{-x}}{2} \\ y' = \sinh x = \frac{e^x - e^{-x}}{2} \\ y'' = \cosh x = \frac{e^x + e^{-x}}{2} \end{cases}$$



$$x = \sinh kt \rightarrow x$$

$$\frac{dx}{dt} = k \cosh(kt)$$

$$\frac{d^2x}{dt^2} = k^2 \sinh kt$$
$$= k^2 x$$

Hyperbolic Functions  
are a solution to

$$\frac{d^2x}{dt^2} = k^2 x$$

Circular Functions

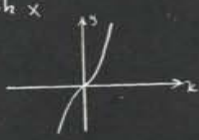
solve  $\frac{d^2x}{dt^2} = -k^2 x$

5.040 Inverse Hyperbolic Functions

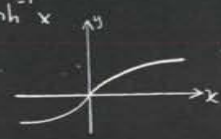
30 min.

Inverse Hyperbolic Functions

$y = \sinh x$



$y = \sinh^{-1} x$



$$\begin{cases} y = \sinh^{-1} x \\ x = \sinh y \end{cases}$$

$$\frac{dx}{dy} = \cosh y$$


$$\frac{dy}{dx} = \frac{1}{\cosh y}$$

$$\cosh^2 y - \sinh^2 y = 1$$

$$\cosh y = \pm \sqrt{1 + \sinh^2 y}$$


$$= \sqrt{1 + x^2}$$

$$\therefore \frac{d(\sinh^{-1} x)}{dx} = \frac{1}{\sqrt{1 + x^2}}$$

$$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + C$$


$$A(x) = \int_0^x \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x$$

$\rightarrow \int \frac{dx}{\sqrt{1+x^2}} = ?$



$\tan \theta = x$   
 $\sec^2 \theta d\theta = dx$   
 $\sqrt{1+x^2} = \sec \theta$   
 $\therefore \int \frac{dx}{\sqrt{1+x^2}} = \int \sec \theta d\theta$   
 $z = \tan \theta$   
 $1+x^2 = 1+\tan^2 \theta = \sec^2 \theta$   
 $\sec^2 \theta - \tan^2 \theta = 1$   
 $\cosh^2 \theta - \sinh^2 \theta = 1$

$\sinh \theta = x$   
 $\cosh \theta d\theta = dx$   
 $\sqrt{1+x^2} = \sqrt{1+\sinh^2 \theta} = \cosh \theta$   
 $\int \frac{dx}{\sqrt{1+x^2}} = \int d\theta = \theta + C = \sinh^{-1} x + C$

$$y = \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

$$x = \sinh y \\ = \frac{e^y - e^{-y}}{2}$$

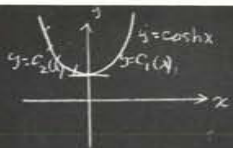
$$\therefore 2x = e^y - \frac{1}{e^y}$$

$$e^{2y} - 2xe^y - 1 = 0$$

$$\left(\frac{e^y}{2}\right)^2 - 2x\left(\frac{e^y}{2}\right) - 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$

$$e^y = x \pm \sqrt{x^2 + 1} \\ y = \ln(x + \sqrt{x^2 + 1})$$



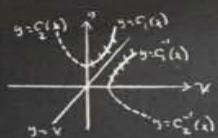
Define

$$c_1(x) = \cosh x, x \geq 0$$

$$c_2(x) = \cosh x, x \leq 0$$

Then  $c_1$  and  $c_2$  are each 1-1

$\therefore c_1^{-1}, c_2^{-1}$  exist.



$$\cosh^{-1} x \equiv c_1^{-1}(x)$$

That is,

$$y = \cosh^{-1} x \text{ means}$$

$$x = \cosh y \\ \text{and } y \geq 0$$

(don't  $\cosh^{-1} x$  is  $x \geq 1$ )

Find  $\frac{dy}{dx}$  if  $y = \cosh^{-1} x$

$$x = \cosh y, y \geq 0$$

$$\frac{dx}{dy} = \sinh y, y \geq 0$$

$$\frac{dy}{dx} = \frac{1}{\sinh y}$$

$$\cosh^2 y - \sinh^2 y = 1$$

$$\sinh y = \sqrt{x^2 - 1}$$

$$y \geq 0 \rightarrow \sinh y \geq 0$$

$$\therefore \frac{d(\cosh^{-1} x)}{dx} = \frac{1}{\sqrt{x^2 - 1}}$$

$$\left\{ \begin{aligned} \frac{d c_1^{-1}(x)}{dx} &= \frac{1}{\sqrt{x^2 - 1}} \\ \frac{d c_2^{-1}(x)}{dx} &= -\frac{1}{\sqrt{x^2 - 1}} \end{aligned} \right.$$

$$\frac{d c_2^{-1}(x)}{dx} = -\frac{1}{\sqrt{x^2 - 1}}$$

Block VI: More Integration Techniques

6.010 Some Basic Recipes


30 min.

Some Basic Recipes

Pick a differentiable function  $G$ ; say  $G' = f$   
 Then:  $\int f(x) dx = G(x) + C$

---

Given  $f$  the required  $G$  may not exist in "familiar" form  
 For example:  $f(x) = e^{-x^2}$



Objective of this Block is to find "recipes" for finding  $G(x)$  for various types of  $f(x)$

$$\int u^n dx = \begin{cases} \frac{u^{n+1}}{n+1} + C, n \neq -1 \\ \ln|u| + C, n = -1 \end{cases}$$

$$\int (\sin x)^n d(\sin x) = \begin{cases} \frac{1}{n+1} \sin^{n+1} x + C, n \neq -1 \\ \ln|\sin x| + C, n = -1 \end{cases}$$

$$\therefore \int \sin^n x \cos x dx = \begin{cases} \frac{1}{n+1} \sin^{n+1} x + C, n \neq -1 \\ \ln|\sin x| + C, n = -1 \end{cases}$$

$$\int \sin^n x dx$$

$$\int \cos^2 x dx = \int \cos^2 x \cos x dx = \int (1 - \sin^2 x) \cos x dx$$

$$\int \cos^2 \theta d\theta = \int (\cos^2 \theta) d\theta = \int \left( \frac{1 + \cos 2\theta}{2} \right) d\theta = \frac{1}{8} \int (+3\cos 2\theta + 3\cos^2 2\theta + \cos^2 2\theta) d\theta$$



### Sums and Differences of Squares

$$\int \frac{dx}{\sqrt{a^2-x^2}} \quad \begin{array}{c} a \\ \swarrow \searrow \\ \sqrt{a^2-x^2} \end{array}$$

$$a \sin \theta = x, \quad a \cos \theta = \sqrt{a^2-x^2}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a \cos \theta d\theta}{a \cos \theta} = \int d\theta = \theta + C = \sin^{-1} \frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} \quad \begin{array}{c} x \\ \swarrow \searrow \\ \sqrt{x^2-a^2} \end{array}$$

$$\begin{aligned} a \csc \theta &= x \\ -a \csc \theta \cot \theta d\theta &= dx \\ \sqrt{x^2-a^2} &= a \cot \theta \end{aligned} \quad \left\{ \begin{array}{l} \csc^2 \theta - \cot^2 \theta = 1 \\ \csc^2 \theta - \sinh^2 \theta = 1 \end{array} \right.$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \int -\csc \theta d\theta$$

$$\begin{aligned} a \cosh \theta &= x \\ dx &= a \sinh \theta d\theta \\ \sqrt{x^2-a^2} &= \sqrt{a^2(\cosh^2 \theta - 1)} \\ &= a \sinh \theta \end{aligned} \quad \left\{ \begin{array}{l} \int \frac{dx}{\sqrt{x^2-a^2}} = \theta + C \\ = \cosh^{-1} \frac{x}{a} + C \end{array} \right.$$

### Completing the Square

$$\begin{aligned} ax^2+bx+c &= \\ a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) &= \\ a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}\right) &= \\ a\left(x+\frac{b}{2a}\right)^2+\left(c-\frac{b^2}{4a}\right) &= \end{aligned}$$

$$\int \frac{dx}{ax^2+bx+c} =$$

$$\int \frac{dx}{a\left(x+\frac{b}{2a}\right)^2+c} =$$

$$\frac{1}{a} \int \frac{dx}{\left(x+\frac{b}{2a}\right)^2+\frac{c}{a}} =$$

$$\frac{1}{a} \int \frac{dx}{\left(x+\frac{b}{2a}\right)^2+k^2} = \frac{1}{a} \int \frac{du}{u^2+k^2}$$

$$\left. \begin{array}{l} \text{Let } u = x + \frac{b}{2a} \\ du = dx \end{array} \right\}$$

6.020 Partial Fractions

32 min.

Partial Fractions

Technique applies to  $\int \frac{P(x)}{Q(x)} dx$  where  $P$  and  $Q$  are polynomials in  $x$  (deg  $P$  < deg  $Q$ )

(1) We can handle linear and quadratic denominators

(2) Theoretically, every real polynomial can be factored into linear and quadratic terms

We can't factor  $x^2+1$  without using non-real numbers

$$\begin{aligned} x^2+1 &= x^2-i^2 = (x+i)(x-i) \\ x^4+1 &= (x^2+1)^2 - 2x^2 \\ &= (x^2+1+\sqrt{2}x)(x^2+1-\sqrt{2}x) \end{aligned}$$

$$\int \frac{dx}{(x-1)(x^2+1)} = ?$$

$$\frac{1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$1 = A(x^2+1) + (Bx+C)(x-1)$$

$$\begin{cases} A+B=0 \\ C-B=0 \\ A-C=1 \end{cases} \quad A+C=0$$

$$\therefore A = \frac{1}{2}, C = -\frac{1}{2}$$

$$B = \frac{1}{2}$$

$$\frac{1}{(x-1)(x^2+1)} = \frac{1}{2} \left[ \frac{1}{x-1} - \frac{x+1}{x^2+1} \right]$$

$$= \frac{1}{2} \left( \frac{1}{x-1} \right) - \frac{1}{2} \left( \frac{x}{x^2+1} \right) - \frac{1}{2} \left( \frac{1}{x^2+1} \right)$$

$$\therefore \int \frac{dx}{(x-1)(x^2+1)} = \frac{1}{2} \int \frac{dx}{x-1} - \frac{1}{2} \int \frac{x dx}{x^2+1} - \frac{1}{2} \int \frac{dx}{x^2+1}$$

$$= \frac{1}{2} \ln|x-1| - \frac{1}{4} \ln|x^2+1| - \frac{1}{2} \tan^{-1} x + C$$

### Polynomial Identities

$$a_2 x^2 + a_1 x + a_0 = b_2 x^2 + b_1 x + b_0$$

(1) Let  $x=0$ :  $\therefore a_0 = b_0$

$$\therefore x(a_2 x + a_1) = x(b_2 x + b_1)$$

$$\therefore a_2 x + a_1 = b_2 x + b_1$$

(2) Let  $x=1$ :  $\therefore a_1 = b_1$ , etc

$$2a_2 x + a_1 = 2b_2 x + b_1$$

$$2a_2 = 2b_2$$

### Beware!

In general

$$a_1 u_1 + a_2 u_2 = b_1 u_1 + b_2 u_2$$

does not imply that

$$a_1 = b_1 \text{ and } a_2 = b_2$$

Example

$$5(x) + 6\left(\frac{x}{2}\right) = 3(x) + 10\left(\frac{x}{2}\right)$$

$$\text{But } 5 \neq 3$$

$$6 \neq 10$$

"It has been discovered  
that..."

$$z = \tan \frac{x}{2} \quad \begin{array}{c} \sqrt{1+z^2} \\ \hline z \\ \hline 1 \end{array}$$

$$dz = \frac{1}{2} \sec^2 \frac{x}{2} dx \quad \left. \begin{array}{l} dx = 2 dz \\ \uparrow \\ 1+z^2 \end{array} \right\}$$

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} \quad \left. \begin{array}{l} \sin y = \frac{2z}{1+z^2} \leftarrow \\ = 2 \frac{z}{\sqrt{1+z^2}} \frac{1}{\sqrt{1+z^2}} \end{array} \right\}$$

$$\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \frac{1-z^2}{1+z^2} - \frac{z^2}{1+z^2} = \frac{1-2z^2}{1+z^2}$$

$$\int \sec x dx =$$

$$\int \frac{dx}{\cos x} =$$

$$\int \frac{2 dz}{1+z^2} \left( \frac{1+z^2}{1-z^2} \right) =$$

$$2 \int \frac{dz}{1-z^2}$$

## 6.030 Integration by Parts

26 min.

## Integration by Parts

$$d(uv) = u dv + v du$$

$$u dv = d(uv) - v du$$

$$\int u dv = uv - \int v du (+C)$$

Example:

$$\text{Let } u = x, \quad v = \sin x \\ \frac{du}{dx} = 1, \quad \frac{dv}{dx} = \cos x$$

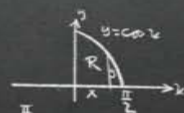
Then

$$\int u dv = \int x \cos x dx \leftarrow$$

$$\int v du = \int \sin x dx \leftarrow$$

$$\int x \cos x dx = x \sin x - \int \sin x dx \\ = x \sin x + \cos x + C$$

$$\text{(check: } x \cos x + \sin x - \sin x = x \cos x)$$



$$V_3 = 2\pi \int_0^{\pi/2} x \cos x dx \\ = 2\pi [G(\frac{\pi}{2}) - G(0)], \quad G(x) = x \cos x \\ = 2\pi (x \sin x + \cos x) \Big|_0^{\pi/2} = 2\pi (\frac{\pi}{2} - 1)$$

$$\int \frac{x \cos x dx}{u} \quad \left\{ \begin{array}{l} u = x, \quad dv = \cos x dx \\ du = dx, \quad v = \sin x (+C) \end{array} \right.$$

$$= x \sin x - \int \sin x dx$$

$$= x \sin x + \cos x + C$$

$$\int \frac{x \cos x dx}{u} \quad \left\{ \begin{array}{l} u = \cos x, \quad dv = x dx \\ du = -\sin x dx, \quad v = \frac{1}{2} x^2 \end{array} \right.$$

$$= \frac{1}{2} x^2 \cos x + \frac{1}{2} \int x^2 \sin x dx$$

$$\int x^2 \sin x dx = 2 \int x \cos x dx - 2 \cos x$$

$$\int x^2 \sin x dx \quad \left\{ \begin{array}{l} u = x^2, \quad dv = \sin x dx \\ du = 2x dx, \quad v = -\cos x \end{array} \right.$$

$$= -x^2 \cos x + 2 \int x \cos x dx$$

$$\int x \cos x dx = x \sin x + \cos x + C$$

$$\therefore \int x^2 \sin x dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$



$$\int \frac{\tan^{-1} x}{x} dx$$

$$\frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}$$

$$u = \tan^{-1} x, dv = dx$$

$$du = \frac{dx}{1+x^2}, v = x$$

$$\int \tan^{-1} x dx =$$

$$x \tan^{-1} x - \int \frac{x dx}{1+x^2} =$$

$$x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$$

$$u = \tan^{-1} x \quad dv = dx$$

$$du = \frac{dx}{1+x^2} \quad v = x + C_1$$

$$\int \tan^{-1} x dx = (x + C_1) \tan^{-1} x$$

$$- \int \frac{(x + C_1) dx}{1+x^2}$$

$$= x \tan^{-1} x + C_1 \tan^{-1} x$$

$$- \int \frac{x dx}{1+x^2} - C_1 \tan^{-1} x$$

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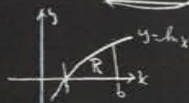
$$\int \ln x dx$$

$$u = \ln x, dv = dx$$

$$du = \frac{1}{x} dx, v = x + C$$

$$\int \ln x dx = x \ln x - \int dx$$

$$= x \ln x - x + C$$



$$R = \int_a^b \ln x dx = \left. x \ln x - x \right|_a^b$$

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6.040 Improper Integrals

29 min.

Improper Integrals

Find the flaw!

$$\int_{-1}^1 \frac{dx}{x^2} = G(1) - G(-1)$$

$$G(x) = -\frac{1}{x} \Rightarrow G'(x) = \frac{1}{x^2}$$

$$G(1) = -1, G(-1) = 1$$

$$\therefore \int_{-1}^1 \frac{dx}{x^2} = -2$$

$$\left(\frac{1}{x^2} \geq 0 \therefore \int_{-1}^1 \frac{dx}{x^2} \geq 0\right)$$

Key Point

$$\int_a^b f(x) dx = G(b) - G(a)$$

$$G' = f$$

requires that  $f$  be (piecewise) continuous on  $[a, b]$

$\frac{1}{x^2} = \infty$  when  $x=0, 0 \in [-1, 1]$


Definition #1

$\int_a^b f(x) dx$  is called improper of the first kind  $\leftrightarrow f$  is infinite for at least one  $c \in [a, b]$

If  $c$  is the only point in  $[a, b]$  at which  $f$  is infinite we define

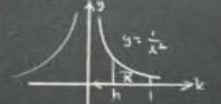
$$\int_a^b f(x) dx = \lim_{h \rightarrow 0^+} \left[ \int_a^{c-h} f(x) dx + \int_{c+h}^b f(x) dx \right]$$

Pictorially:



Area of an infinite region  $\infty \times 0$

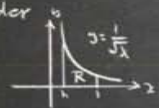
The question centers about whether  $h \rightarrow 0$  "faster than"  $f(c+h) \rightarrow \infty$ . In our example this didn't happen.



$$A_R = \int_h^1 \frac{dx}{x^2} = \left. -\frac{1}{x} \right|_h^1 = \frac{1}{h} - 1$$

$$\lim_{h \rightarrow 0^+} A_R = \lim_{h \rightarrow 0^+} \left( \frac{1}{h} - 1 \right) = \infty - 1 = \infty$$

On the other hand,  
Consider



$$A_R = \int_1^h \frac{dx}{\sqrt{x}} = 2\sqrt{x} \Big|_1^h$$

$$= 2 - 2\sqrt{h}$$

$$\lim_{h \rightarrow 0^+} A_R = 2 - 2\sqrt{0} = 2$$

$$\int_0^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x} \Big|_0^1 = 2$$

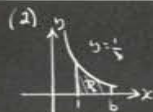
### Definition #2

If  $f$  is infinite at  $x=c$ ,  
 $c \in [a, b]$  then the improper  
integral  $\int_a^b f(x) dx$  is convergent

i.f.  $\lim_{h \rightarrow 0^+} \left[ \int_a^c f(x) dx + \int_c^b f(x) dx \right]$  exists

otherwise it is called  
divergent

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UG3



$$A_R = \int_1^b \frac{dx}{x} = \ln b$$

$$\lim_{b \rightarrow \infty} A_R = \infty = \int_1^{\infty} \frac{dx}{x}$$

$$V_R = \pi \int_1^b \left(\frac{1}{x}\right)^2 dx$$

$$= \pi \left(-\frac{1}{x}\right) \Big|_1^b$$

$$= \pi \left(1 - \frac{1}{b}\right)$$

$$\lim_{b \rightarrow \infty} V_R = \pi$$

Infinite Area but finite volume!

### Computational Aside

We do not have to be  
able to compute  $\int_a^{\infty} f(x) dx$   
to determine its convergence

For example, consider  
 $\int_1^{\infty} e^{-x^2} dx$ . For  $x \geq 1$ ,  $x^2 \geq x$

$$\therefore -x^2 \leq -x$$

$$\therefore \int_1^b e^{-x^2} dx \leq \int_1^b e^{-x} dx$$

$$= -e^{-x} \Big|_1^b$$

$$= e^{-1} - e^{-b}$$



$$\leq \frac{1}{e} - e^{-b}$$



$$A_R = \int_1^b \frac{dx}{x^2} = -\frac{1}{x} \Big|_1^b$$

$$\lim_{b \rightarrow \infty} A_R = \lim_{b \rightarrow \infty} \left(-\frac{1}{b} + 1\right)$$

$$= \lim_{b \rightarrow \infty} (1 - \frac{1}{b}) = 1$$

### Definition #3

$$\int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx$$

where  $f$  is continuous for  $x \geq a$   
is called improper of the  
second kind

### Examples

(i) Note that  $\frac{1}{x^2}$  and  $\frac{1}{\sqrt{x}}$   
( $x \geq 0$ ) are inverses.



$$\int_1^{\infty} \frac{dx}{x^2} = \lim_{b \rightarrow \infty} \left[ \int_1^b \frac{dx}{x^2} \right]$$

$$= \lim_{b \rightarrow \infty} \left(-\frac{1}{x}\right) \Big|_1^b$$

$$= \lim_{b \rightarrow \infty} \left[-\frac{1}{b} + 1\right]$$

$$= 1$$

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UG3

### Summary

The limits of  
 $\int_a^{\infty} f(x) dx$ ,  $\int_{-\infty}^b f(x) dx$ ,  $\int_{-\infty}^{\infty} f(x) dx$

give us warning to  
beware

However always  
examine  $\int_a^b f(x) dx$   
for "infinities" of  $f$

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Block VII: Infinite Series

7.010 Many Versus Infinite

26 min.

Many Versus Infinite

$$N = 10^{10} = 10,000,000,000$$

$N+1, N+2, N+3, \dots$

$N$  is no nearer the end of the number system than is 1.

Additional Examples

① 1, 3, 2, 5, 7, 4, 9, 11, 6, ...

No matter where we stop (even at  $10^{10}$ ) there are twice as many "odds" as "evens"

②

$$1 + (-1) + 1 + (-1) + \dots$$
$$\left[ \begin{array}{l} [1 + (-1)] + [1 + (-1)] + \dots = 0 \\ [1 + (-1) + 1] + [(-1) + 1] + \dots = 1 \end{array} \right.$$

Notice the need for order as well as the terms.

Note:

Our intuition is defied because it doesn't apply!

Why deal with infinite sums? Because we need them!

$$\rightarrow A_R = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(k) \Delta x$$

How shall we add infinitely many terms?

Consider  $a_1 = \frac{1}{2}, a_2 = \frac{1}{4}, a_3 = \frac{1}{8}$  and we want

$$a_1 + a_2 + a_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$$



$$A_1 = \frac{1}{2}$$

$$A_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$A_3 = A_2 + a_3 = \frac{3}{4} + \frac{1}{8} = \frac{7}{8}$$

$$= a_1 + a_2 + a_3$$

(Do not confuse the  $a$ 's and  $A$ 's... The sequence of numbers being added to  $a_1, a_2, a_3$  ( $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ ) while the partial sums in the sequence  $a_1, a_2, a_3$  ( $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}$ ). The sum is the number  $A_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$ )

Generalized, if  $a_n = \frac{1}{2^n}$

then  $A_n = a_1 + \dots + a_n = \frac{1}{2} + \dots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$

$$= 1 - \frac{1}{2^n}$$

For example:

$$\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{1024} = 1 - \frac{1}{1024}$$

So, how about  $\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} + \frac{1}{2^{n+1}} + \dots$  ENDLESSLY?

$$A_n = 1 - \frac{1}{2^n}$$

$$\lim_{n \rightarrow \infty} A_n = 1$$

Define  $\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n \rightarrow \infty} \left( \frac{1}{2} + \frac{1}{4} + \dots \right) = 1$

Notice same limit theorems as before apply.

Example:

$$\lim_{n \rightarrow \infty} \left( \frac{2n+3}{5n+7} \right) = \lim_{n \rightarrow \infty} \left[ \frac{2 + \frac{3}{n}}{5 + \frac{7}{n}} \right] = \frac{2}{5}$$

$$\sum_{n=1}^{\infty} \left( \frac{2n+3}{5n+7} \right) = \infty \text{ since}$$

"after a while" each term behaves like  $\frac{2}{5}$ . That is,

$$\frac{5}{12} + \frac{7}{17} + \frac{9}{22} + \frac{11}{27} + \dots$$

That is:  $\sum_{n=1}^{\infty} a_n$  converges  $\rightarrow \lim_{n \rightarrow \infty} a_n = 0$

(For if  $\lim_{n \rightarrow \infty} a_n = L \neq 0$ , then

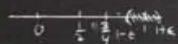
$$\sum_{n=k}^{\infty} a_n \text{ "behaves like" } \sum_{n=k}^{\infty} L$$

On the other hand  $\lim_{n \rightarrow \infty} a_n = 0$  is not enough to guarantee the convergence of  $\sum_{n=1}^{\infty} a_n$

Example

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \dots$$

Pictorially:

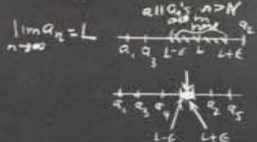


After "a while" all  $s_n$  are within  $\epsilon$  of 1

Basic Definition

The infinite sequence  $b_1, b_2, b_3, \dots = \{b_n\}$  is said to converge to the limit  $L$  (written  $\lim_{n \rightarrow \infty} b_n = L$ )  $\leftrightarrow$   $\forall \epsilon > 0, \exists$  can find  $N(\epsilon)$  such that  $n > N \rightarrow |a_n - L| < \epsilon$

Again Pictorially



limit: infinite sequence

as last term: finite sequence

Limit replaces infinitely many points by a finite number of points ~ a "dot"

## 7.020 Positive Series

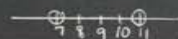
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Positive SeriesOrdering:

$$S = \{11, 8, 9, 7, 10\}$$

$$= \{7, 8, 9, 10, 11\}$$

$7 \leq x, 11 \geq x$  where  $x \in S$   
 $\therefore 7$  is a lower bound for  $S$   
 $11$  is an upper bound for  $S$



$7$  is the greatest lower bound for  $S$   
 $11$  is the least upper bound for  $S$

These results are more subtle for infinite sets:

Examples

(1) Let  $S = \{a_n: a_n = \frac{1}{n+1}\}$

$\therefore S = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

$1$  is lub for  $S$ , but  $1 \notin S$

(2) Let  $S = \{a_n: a_n = \frac{1}{n}\}$

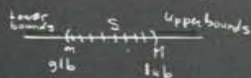
$\therefore S = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$

$0$  is glb for  $S$ , but  $0 \notin S$

Formal Definitions

(#1)  $M$  is called an upper bound for  $S \iff M \geq x$ , for all  $x \in S$

(#2)  $M$  is called a least upper bound for  $S$  if (1)  $M$  is an upper bound for  $S$  and (2)  $L < M \rightarrow L$  is not u.b. for  $S$  ( $M$  need not belong to  $S$ )



(#3) A set is bounded if it has both an upper and a lower bound

Key Property

Every bounded set has a glb and lub

(#4) A sequence  $\{a_n\}$  is called monotonic non-decreasing if

$$a_n \leq a_{n+1} \text{ for all } n$$

(That is:  $a_1 \leq a_2 \leq a_3 \leq \dots \leq a_n \leq a_{n+1} \leq \dots$ )

For such sequences, two possibilities exist:

(i)  $\{a_n\}$  has no upper bound

In this case we write

$$\lim_{n \rightarrow \infty} a_n = \infty$$

(An example is:

$$1, 2, 3, \dots, n, \dots \text{ where } a_n = n)$$

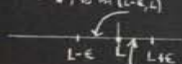
(2)  $\{a_n\}$  has an upper bound

Then  $\lim_{n \rightarrow \infty} a_n = L$  where

$$L = \text{lub}\{a_n\}$$

Proof

at least one  $a_n$  is in  $(L-\epsilon, L)$



$\exists a_n$  in  $(L-\epsilon, L)$

$$n > N \rightarrow a_n \geq a_N$$

$$\therefore n > N \rightarrow L - \epsilon < a_n \leq a_N \leq L$$

$$\rightarrow |a_n - L| < \epsilon$$

$$\rightarrow \lim_{n \rightarrow \infty} a_n = L$$

### Positive Series

If  $a_n \geq 0$  for each  $n$ , then  $\sum_{n=1}^{\infty} a_n$  is called a positive series.

In this case the sequence of partial sums is monotonic non-decreasing.

Therefore:

If  $\sum_{n=1}^{\infty} a_n$  is a positive series, it either diverges to  $\infty$  or it converges to the limit  $L$  where  $L$  is the lub for the sequence of partial sums.

### Ratio Test

Given  $\sum_{n=1}^{\infty} a_n$ , form the sequence  $\left\{ \frac{a_{n+1}}{a_n} \right\} = \{u_n\}$

(For example, given  $\sum_{n=1}^{\infty} \frac{10^n}{n!}$ ,  $u_n$  is given by  $\frac{10^{n+1}}{(n+1)!} \div \frac{10^n}{n!} = \frac{10}{n+1}$ )

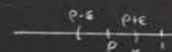
Now, assuming  $\lim_{n \rightarrow \infty} u_n$  exists, call it  $p$ . Then:

$\sum_{n=1}^{\infty} a_n$  converges if  $p < 1$   
diverges if  $p > 1$

If  $p = 1$ , test fails

Note Even if  $u_n < 1$  for every  $n$ ,  $p$  may still equal 1. For example,  $\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$ , but  $\frac{n}{n+1} < 1$

Geometric "Proof" for  $p < 1$



There exists  $N$  such that  $n > N \rightarrow \frac{a_{n+1}}{a_n} < r < 1$

$$\therefore \frac{a_{n+1}}{a_n} < r \therefore a_{n+1} < r a_n$$

$$\frac{a_{n+2}}{a_{n+1}} < r \therefore a_{n+2} < r a_{n+1} < r^2 a_n$$

$$\sum_{k=1}^{\infty} a_k < a_n \left( \sum_{k=0}^{\infty} r^k \right) \text{ convergent geometric series}$$

### Comparison Test

Suppose  $\sum_{n=1}^{\infty} c_n$  is a convergent positive series and  $0 \leq u_n \leq c_n$  for each  $n$

Then  $\sum_{n=1}^{\infty} u_n$  also converges

Proof

$$\text{Let } T_n = c_1 + \dots + c_n \quad S_n = u_1 + \dots + u_n$$

$$\lim_{n \rightarrow \infty} T_n = T = \text{lub}\{T_n\}$$

$$\therefore S_n \leq T \text{ for each } n$$

$\therefore S_n$  is bounded (and monotonic non-decreasing)  $\therefore \lim_{n \rightarrow \infty} S_n$  exists

### Notes:

(1) The condition  $0 \leq u_n \leq c_n$  for all  $n$  can be weakened to  $0 \leq u_n \leq c_n$  for all  $n \geq N$  (i.e. for  $n$  "sufficiently large") since convergence depends on the "end" of the sequence

$$\rightarrow 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

(2) If  $u_n \geq d_n$  where  $\sum_{n=1}^{\infty} d_n$  is a positive divergent series then  $\sum_{n=1}^{\infty} u_n$  also diverges (since its convergence would imply  $\sum_{n=1}^{\infty} d_n$  converges)

### Integral Test

#### Series and Improper Integrals

Suppose there is a decreasing continuous function  $f(x)$  such that  $f(n) = u_n$  is the  $n^{\text{th}}$  term of the positive series  $u_1 + u_2 + \dots + \dots$

Then:  $\sum_{n=1}^{\infty} u_n$  converges

if and only if  $\int_1^{\infty} f(x) dx$  converges

"Proof"



$$u_1 + u_2 > \int_1^{n+1} f(x) dx$$

$$\therefore \int_1^{\infty} f(x) dx \text{ diverges} \rightarrow \sum_{n=1}^{\infty} u_n \text{ diverges}$$



$$\sum_{n=1}^{\infty} u_n < u_1 + \int_1^{\infty} f(x) dx \rightarrow \int_1^{\infty} f(x) dx \text{ converges} \rightarrow \sum_{n=1}^{\infty} u_n \text{ converges}$$

## 7.030 Absolute Convergence

21 min.

## Absolute Convergence

Consider:

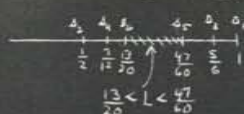
$$1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{(-1)^{n+1}}{n} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

we see:

- (1) Terms alternate in sign
- (2)  $n^{\text{th}}$  term  $\rightarrow 0$
- (3) Terms decrease in magnitude

Claim:  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  converges

## Geometric Proof



(It turns out that  $L = \ln 2$  but who'd have guessed it!)

SO WHAT? ...

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges, but}$$

because +'s cancel -'s  
not because the terms get  
 small fast enough!

That is, if we replace each  
 term by its magnitude, we

obtain  $\sum_{n=1}^{\infty} \frac{1}{n}$

which diverges

## Definition

1.  $\sum_{n=1}^{\infty} a_n$  is said to  
 converge absolutely  
 if  $\sum_{n=1}^{\infty} |a_n|$  converges

2. A series which  
 converges but not  
 absolutely is called  
 conditionally convergent

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  is conditionally  
 convergent



### The Subtlety of Conditional Convergence

The sum of a cond. conv. series depends on the order of the terms

#### Example

Divide the terms of  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n}$  into 2 "teams" (sets)

$$\rightarrow P = \left\{ 1, \frac{1}{5}, \frac{1}{5}, \frac{1}{20}, \dots \right\}$$

$$\rightarrow N = \left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{24}, \dots \right\}$$

Both  $\sum_{n=1}^{\infty} \frac{1}{2^{k-1}}$  and  $\sum_{n=1}^{\infty} \frac{1}{2^n}$  diverge to  $\infty$

Let us, for example, make the sum  $\frac{3}{2}$ . We start with P and write  $1 + \frac{1}{5} + \frac{1}{5} + \dots$  until the sum first exceeds (or equals)  $\frac{3}{2}$ . (This must happen since  $1 + \frac{1}{5} + \frac{1}{5} + \dots$  diverges to  $\infty$ )

In particular:

$$1 + \frac{1}{5} + \frac{1}{5} = \frac{23}{15} > \frac{3}{2}$$

We next "annex" members of N until the sum falls below  $\frac{3}{2}$

### Summary - So Far:

If  $\sum_{n=1}^{\infty} a_n$  is cond. conv., its limit exists, but the limit changes as the order of the terms is changed. That is, rearranging the terms changes the series.

This is not true in "finite" arithmetic!

The moral:

DON'T "MONEY" WITH CONDITIONAL CONVERGENCE

The beauty of absolute convergence is that the sum is the same for every rearrangement of the terms!

Details are left to the supplementary notes!

$$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} = \frac{31}{30} < \frac{3}{2}$$

Continue with P,

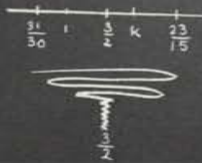
$$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{5} = \frac{247}{210} < \frac{3}{2}$$

$$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \frac{1}{13} =$$

$$\frac{131,093}{90,090} < \frac{3}{2}$$

$$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \frac{1}{15} =$$

$$\frac{2,056,485}{1,351,350} = k > \frac{3}{2}$$

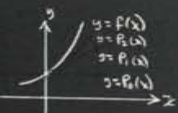


( $\frac{3}{2}$  was not important - though the arithmetic gets "messier" if we choose a larger number)

7.040 Polynomial Approximations

32 min.

Polynomial Approximations



$P_0(x) = f(0)$   
 $P_1(x) = f(0) + f'(0)x$   
 ( $y = mx + b$ )

Let  $P_2(x) = a_0 + a_1x + a_2x^2$

$P_2(0) = a_0 = f(0)$   
 $P_2'(0) = a_1 = f'(0)$   
 $P_2''(0) = 2a_2 = f''(0) \rightarrow a_2 = \frac{f''(0)}{2!}$

$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$

If  $P_n(x) = a_0 + a_1x + \dots + a_nx^n$ ,

$P_n^{(k)}(x) = n! a_k$   
 $\therefore P_n^{(k)}(0) = n! a_k = f^{(k)}(0)$   
 $\therefore a_k = \frac{f^{(k)}(0)}{k!}$

$\therefore P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k$   
 $= a_0 + a_1x + \dots + \frac{f^{(n)}(0)}{n!} x^n$

and we let  $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$  denote  $\lim_{n \rightarrow \infty} P_n(x) = P(x)$

Example:

Let  $f(x) = e^x$

Then  $f^{(n)}(x) = e^x$

$\therefore f^{(n)}(0) = 1 = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$

$\therefore P_n(x) = \sum_{k=0}^n \frac{x^k}{k!}$   
 $= 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$

$P_0(x) = 1$   
 $P_1(x) = 1 + x$   
 $P_2(x) = 1 + x + \frac{x^2}{2}$   
 $P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$

We would like to be able to compare  $f(x)$  with  $\lim_{n \rightarrow \infty} P_n(x)$

This involves 3 questions:

- Does  $\lim_{n \rightarrow \infty} P_n(x)$  exist?  $P(x) = ?$
- If so, does it equal  $f(x)$ ?
- Letting  $P(x) = \lim_{n \rightarrow \infty} P_n(x)$ , does  $P$  possess the "polynomial-properties" possessed by each  $P_n$ ?

### Counter-Examples

① Let  $f(x) = \frac{1}{1-x}$   
 $f'(x) = (1-x)^{-2}$   
 $f''(x) = 2(1-x)^{-3}$   
 $f'''(x) = 3!(1-x)^{-4}$   
 $f^{(n)}(x) = n!(1-x)^{-n-1}$   
 $\frac{f^{(n)}(0)}{n!} = 1$

$\therefore f(x) = \frac{1}{1-x} \rightarrow P_n(x) = 1 + x + x^2 + \dots + x^n$

Now,

$P(2) = -1$

$P(2) = \lim_{n \rightarrow \infty} P_n(2) = \infty$

② Let  $f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ 4 & \text{if } x > 2 \end{cases}$

$f(0) = 0, f'(0) = 0, f''(0) = 2$

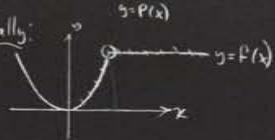
$f^{(n)}(0) = 0, n > 2$

$\therefore P_0(x) = 0, P_1(x) = 0, P_2(x) = x^2$

$P_n(x) = x^2, n > 2$

$\therefore P(x) = \lim_{n \rightarrow \infty} P_n(x) = x^2 \neq f(x)$

Pictorially:



②  $\sum \frac{|x|^n}{n!}$  yields

$u_n = \frac{|x|^n}{n!}, u_{n+1} = \frac{|x|^{n+1}}{(n+1)!}$

$\frac{u_{n+1}}{u_n} = \frac{|x|}{n+1}$

$\rho = \lim_{n \rightarrow \infty} \frac{|x|}{n+1} = 0 < 1$

for all x

$\therefore \sum \frac{x^n}{n!}$  converges for  $|x| < \infty$

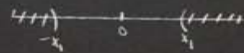
In general, given  $\sum a_n x^n$  there exists M such that

$\sum a_n x^n$  convs. abs. if  $|x| < M$   
 diverges if  $|x| > M$

Pictorially:

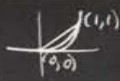
$\sum a_n x^n$

(If  $\sum a_n x^n$  converges then  $\sum a_n x^n$  converges abs. if  $|x| < M$ )



(If  $\sum a_n x^n$  diverges then  $\sum a_n x^n$  diverges if  $|x| > M$ )

③ Let  $P_n(x) = x^n$   
 $\text{dom } P_n = [0, 1]$



Then  $P(x) = \lim_{n \rightarrow \infty} P_n(x)$

$\frac{y}{P(x)} = \begin{cases} 0, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } x = 1 \end{cases}$

$\therefore$  Each  $P_n$  is continuous when  $x=1$  but  $P$  is not!

### The Ratio Test and Absolute Convergence:

$\sum_{n=0}^{\infty} a_n x^n$  converges if  $\sum_{n=0}^{\infty} |a_n x^n|$  does.

But we may test  $\sum |a_n x^n|$  for convergence by the ratio test etc.

#### Examples

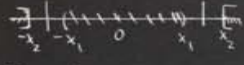
(i)  $\sum_{n=0}^{\infty} |x|^n$  converges  $\Leftrightarrow |x| < 1$

(Let  $u_n = |x|^n, u_{n+1} = |x|^{n+1}$   
 $\frac{u_{n+1}}{u_n} = |x| < 1$ )

$\therefore \rho = \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} |x| < 1$

$\therefore \sum_{n=0}^{\infty} x^n$  converges (absolutely) if  $|x| < 1$

$\sum a_n x^n$



Suppose  $\sum a_n x_1^n$  conv

$\sum a_n x_2^n$  div

### "Taylor's Theorem with Remainder"

Using integration by parts repeatedly (as shown in our text) it follows that if f and its first n+1 derivatives exist at x=0, then

$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + R_n(x)$

where  $R_n(x) = \frac{P_n(x)}{n!} = \int_0^x \frac{(x-t)^n}{n!} f^{(n+1)}(t) dt$

$f(x) = P(x) \Leftrightarrow \lim_{n \rightarrow \infty} R_n(x) = 0$



7.050 Uniform Convergence

28 min.

Uniform Convergence

Review: If  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$   
for all  $x \in [a, b]$  we say that  
 $\{f_n\}$  converges to  $f(x)$  on  $[a, b]$

Example: Suppose  $f_n(x) = \frac{n}{2n+1} x^2$

Then  $f(x) = \lim_{n \rightarrow \infty} f_n(x) = \frac{1}{2} x^2$

Hence  $\{\frac{n x^2}{2n+1}\}$  converges to  $\frac{1}{2} x^2$

Now  $\lim_{n \rightarrow \infty} f_n(2) = 2$        $\lim_{n \rightarrow \infty} f_n(4) = 8$

$n > N_1 \rightarrow |f_n(2) - 2| < \epsilon$        $n > N_2 \rightarrow |f_n(4) - 8| < \epsilon$

Of course,  $N_1$  and  $N_2$  need not be the same.

Two Basic Definitions

(1) Let  $\text{dom } f_n = [a, b]$ . We say  $\{f_n\}$  converges pointwise to  $f$  on  $[a, b] \iff \lim_{n \rightarrow \infty} f_n(x) = f(x)$  for each  $x \in [a, b]$ .

That is, given  $\epsilon > 0$  we can find  $N$  such that  $n > N$  implies  $|f_n(x) - f(x)| < \epsilon$  for a given  $x \in [a, b]$

In general, the choice of  $N$  depends on the choice of  $x$ , and there are infinitely many such choices in  $[a, b]$

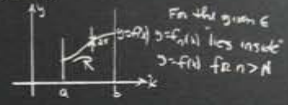
(2) If we can find one  $N$  such that  $n > N \rightarrow |f_n(x) - f(x)| < \epsilon$  for every  $x \in [a, b]$  we say that the convergence is uniform.

A Pictorial Interpretation



For the given  $\epsilon$ , there exists  $N$  such that if  $n > N$  then  $y = f_n(x)$  lies in shaded region.

Letting  $\epsilon$  be "arbitrary" small, we obtain:



For the given  $\epsilon$ ,  $f_n(x)$  lies inside  $y = f(x) \pm \epsilon$  for  $n > N$ .

Loosely speaking,  $y = f_n(x)$  looks like  $y = f(x)$  for sufficiently large values of  $n$ .

From our picture it should seem clear that if  $\{f_n\}$  converges uniformly to  $f$  on  $[a, b]$  and if each  $f_n$  is continuous on  $[a, b]$ , then  $f$  is also continuous on  $[a, b]$ .

(Note: This result, as we have already seen, need not be true for pointwise convergence. Recall our example in which  $f_n(x) = x^n$ ,  $\text{dom } f = [0, 1]$ )

Since  $f$  continuous at  $x = x_0$  means  $\lim_{x \rightarrow x_0} f(x) = f(x_0)$  and since  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$

we may rewrite (1) as:  
 $\lim_{x \rightarrow x_0} \left[ \lim_{n \rightarrow \infty} f_n(x) \right] = \lim_{n \rightarrow \infty} f_n(x_0) = \lim_{n \rightarrow \infty} \left[ \lim_{x \rightarrow x_0} f_n(x) \right]$



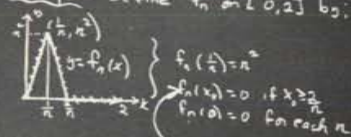
$$(2) \lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b f(x) dx$$

$$= \int_a^b \lim_{n \rightarrow \infty} f_n(x) dx$$

These two results are proven as theorems in the supplementary notes. A theorem about differentiation is also proved. (Differentiation is in a way more subtle than integration since it requires "smoothness" as well as "unbrokenness")

Note: (2) need not be true if the convergence is not uniform

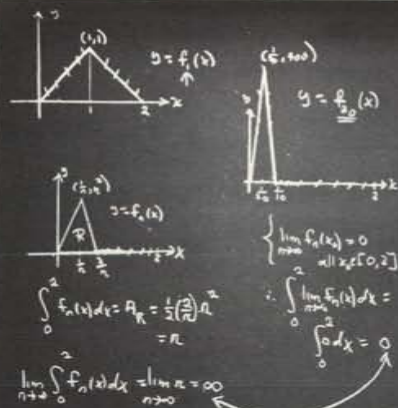
Example: Define  $f_n$  on  $[0, 2]$  by:



(a)  $\lim_{n \rightarrow \infty} f_n(0) = 0$

(b) If  $0 < \epsilon < 2$ , we may find  $N$  such that  $n > N \rightarrow \frac{2}{n} < \epsilon$

$\therefore n > N \rightarrow f_n(x) = 0$   
 $\therefore \lim_{n \rightarrow \infty} f_n(x) = 0$



### Application of Uniform convergence to series

Recall that  $\sum_{k=0}^{\infty} a_k x^k$  is an abbreviation for  $\lim_{n \rightarrow \infty} \left( \sum_{k=0}^n a_k x^k \right)$

That is,  $a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$  represents the limit of the sequence:  $a_0, a_0 + a_1 x, a_0 + a_1 x + a_2 x^2, a_0 + a_1 x + a_2 x^2 + a_3 x^3, \dots$  polynomials

Hence if  $\left\{ \sum_{k=0}^n a_k x^k \right\}$  converges uniformly to  $\sum_{k=0}^{\infty} a_k x^k$

then, for example,  $\sum_{k=0}^{\infty} a_k x^k$  must be continuous since each  $\sum_{k=0}^n a_k x^k$  is.

Also if  $[a, b]$  is included in the interval of uniform convergence then  $\int_a^b \left( \sum_{k=0}^{\infty} a_k x^k \right) dx = \sum_{k=0}^{\infty} \left( \int_a^b a_k x^k dx \right)$

Next time we shall show that within the interval of absolute convergence  $\left\{ \sum_{k=0}^{\infty} a_k x^k \right\}$  converges uniformly to  $\sum_{k=0}^{\infty} a_k x^k$ . In other words within the radius of convergence  $\sum_{k=0}^{\infty} a_k x^k$  "enjoys" the "usual" properties associated with  $\sum_{k=0}^{\infty} a_k x^k$ .

7.060 Uniform Convergence of Series

27 min.

Uniform Convergence of Series

Weierstrass M-Test

Suppose  $\sum_{n=1}^{\infty} M_n$  is a positive convergent series and that  $|f_n(x)| \leq M_n$  for each  $n$  and each  $x \in [a, b]$ .

Then  $\left\{ \sum_{k=1}^n f_k(x) \right\}$  converges uniformly to  $\sum_{k=1}^{\infty} f_k(x) (= f(x))$  on  $[a, b]$ .

Proof

$$\begin{aligned} \left| f(x) - \sum_{k=1}^n f_k(x) \right| &= \left| \sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^n f_k(x) \right| \\ &= \left| \sum_{k=n+1}^{\infty} f_k(x) \right| \\ &\leq \sum_{k=n+1}^{\infty} |f_k(x)| \\ &\leq \sum_{k=n+1}^{\infty} M_k \\ &\leq \epsilon \text{ for } n \text{ sufficiently large} \\ &\text{independently of } x \end{aligned}$$

An Example:

$$\cos x + \cos \frac{4x}{4} + \dots + \cos \left( \frac{n^2 x}{n^2} \right) + \dots = \sum_{n=1}^{\infty} \frac{\cos n^2 x}{n^2}$$

$\left| \frac{\cos(n^2 x)}{n^2} \right| \leq \frac{1}{n^2}$  for each  $n$  and all  $x$

$\sum \frac{1}{n^2}$  is a pos. conv. series

$\therefore \left\{ \sum_{k=1}^n \frac{\cos k^2 x}{k^2} \right\}$  conv. unif. to  $\sum_{n=1}^{\infty} \frac{\cos n^2 x}{n^2}$

$\therefore$  let us compute, say,

$$\int_0^{\pi} (\cos x + \cos \frac{4x}{4} + \dots) dx = \int_0^{\pi} \left( \sum_{n=1}^{\infty} \frac{\cos n^2 x}{n^2} \right) dx$$

By uniform convergence, we have:

$$\int_0^{\pi} \left( \cos x + \cos \frac{4x}{4} + \cos \frac{9x}{9} + \dots \right) dx = \int_0^{\pi} \sum_{n=1}^{\infty} \frac{\cos n^2 x}{n^2} dx = \sum_{n=1}^{\infty} \frac{\sin n^2 \pi}{n^2}$$

### Application to Power Series

Let  $|x| < |x_1| < R$  where

$\sum_{n=0}^{\infty} a_n x^n$  converges for  $|x| < R$

$$\lim_{n \rightarrow \infty} a_n x_1^n = 0 \quad \left( \begin{array}{l} \text{Since} \\ \sum a_n x_1^n \text{ converges} \end{array} \right) < \frac{1}{M}$$

$\therefore$  Given  $M > 0$ , there exists  $N$  such that  $n > N \rightarrow |a_n x_1^n| < M$

Key idea is:

$$\begin{aligned} |a_n x^n| &= |a_n \left(\frac{x_0}{x_1}\right)^n x_1^n| \\ &= |a_n x_1^n| \left|\frac{x_0}{x_1}\right|^n \\ &\leq M \left|\frac{x_0}{x_1}\right|^n \end{aligned}$$

Now  $\left|\frac{x_0}{x_1}\right|$  is a positive constant  $< 1$

$\therefore \sum_{n=0}^{\infty} M \left|\frac{x_0}{x_1}\right|^n$  is a positive convergent (geometric) series

$\therefore$  By Weierstrass M-test

$\sum_{n=0}^{\infty} a_n x^n$  is uniformly convergent

if  $|x| < R$  where  $R$  is the radius of convergence for  $\sum_{n=0}^{\infty} a_n x^n$

In other words

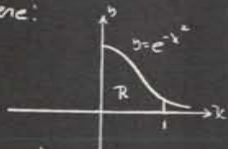
$$A_R = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} + \dots$$

$\approx 0.748$

### Example

Find the area of  $R$

where:



$A_R = \int_0^1 e^{-x^2} dx$ , but we do not know (explicitly)  $g(x)$  such that  $g'(x) = e^{-x^2}$

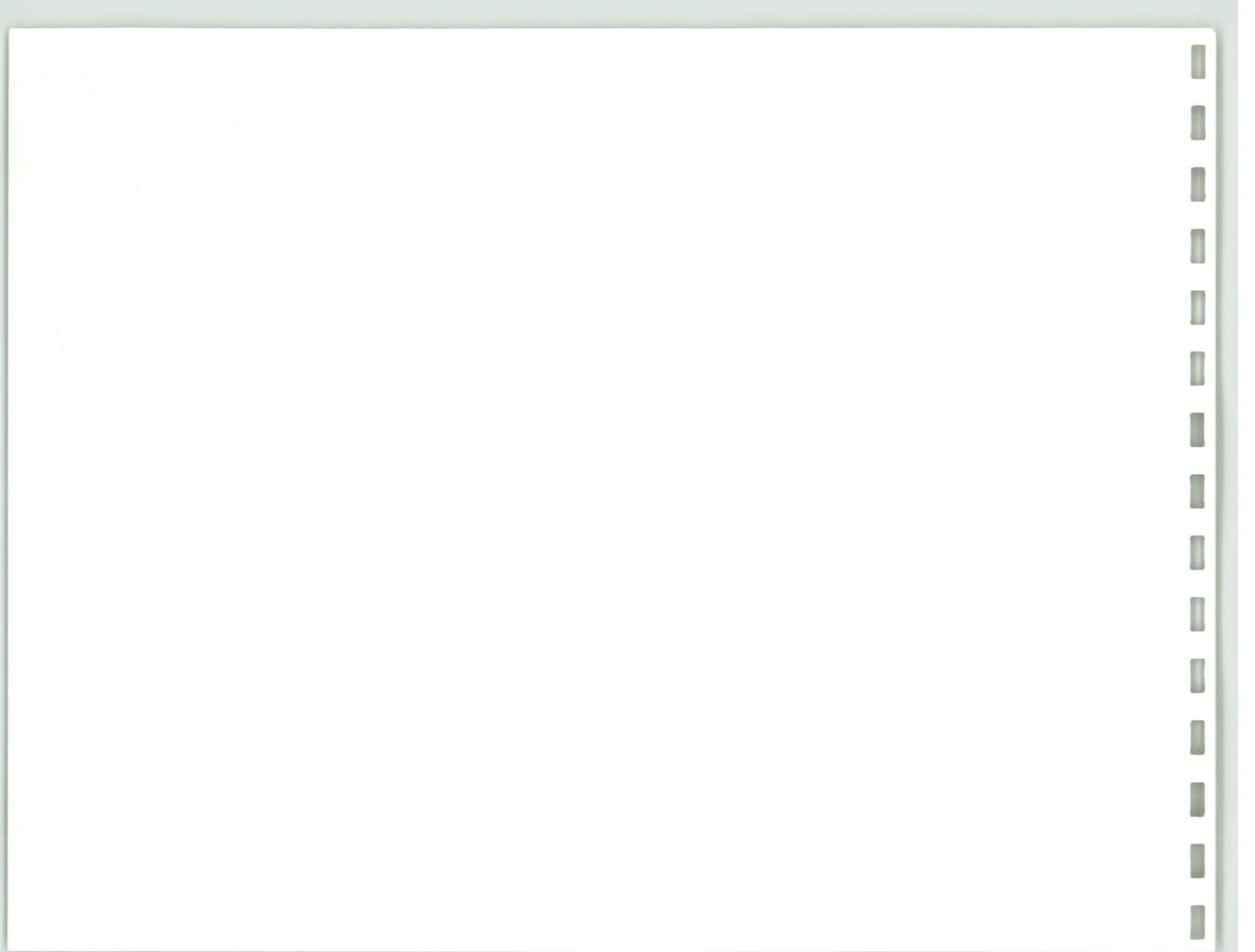
We do know:

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

for all  $x$

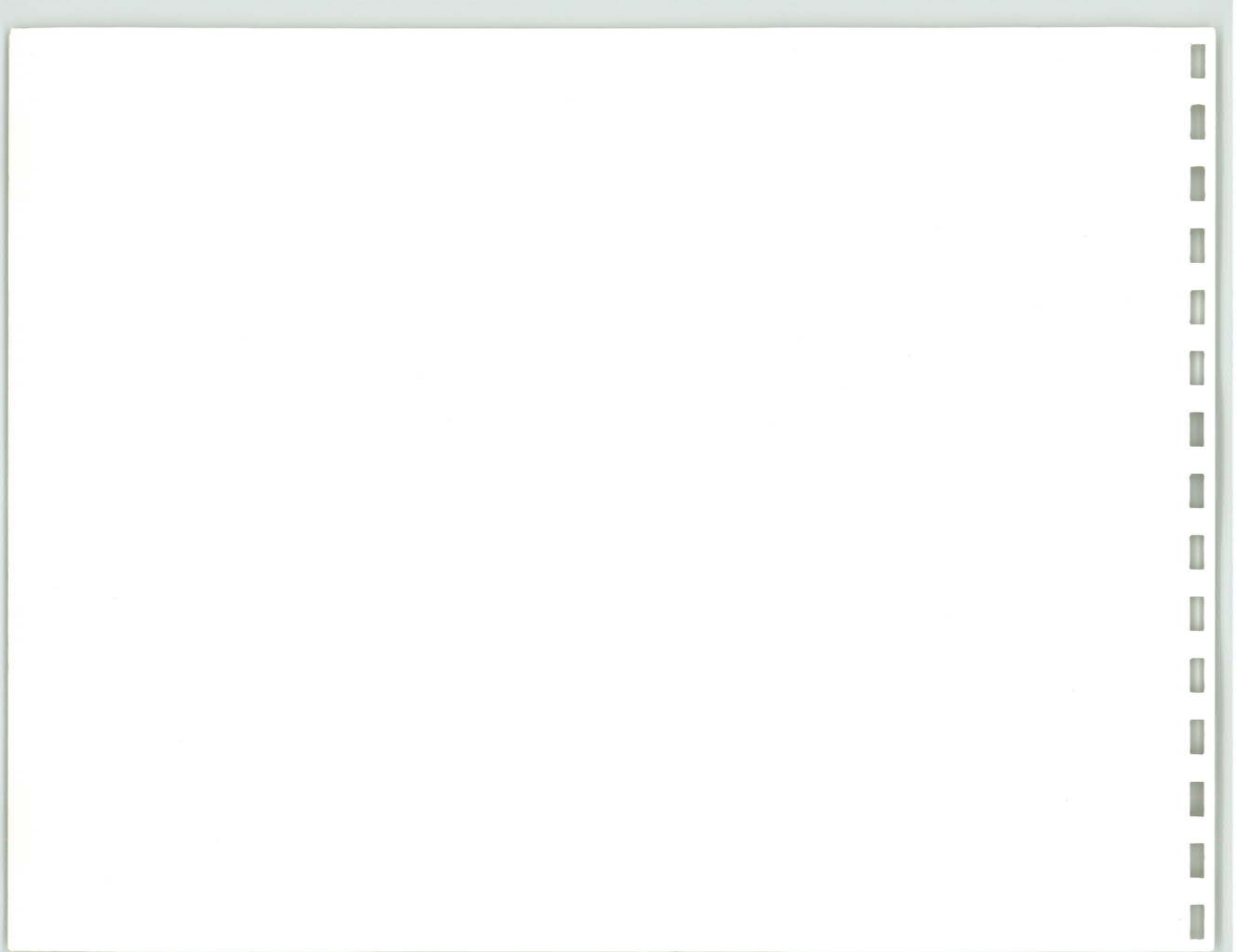
$$\therefore e^{-x} = 1 - x + \frac{x^2}{2!} - \dots + \frac{(-1)^n x^{2n}}{n!} + \dots$$

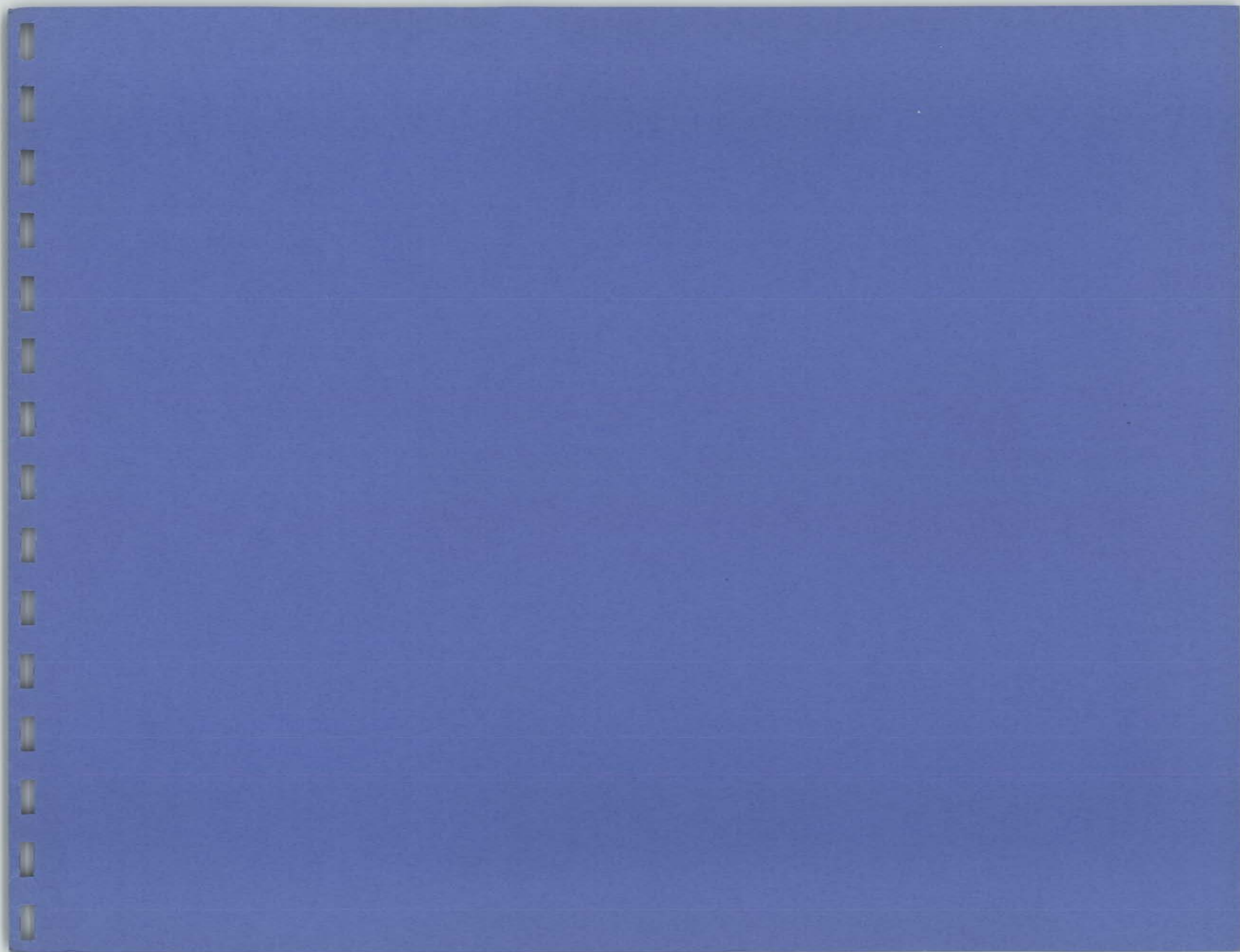
$$\begin{aligned} \therefore \int_0^1 e^{-x^2} dx &= \int_0^1 \left( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \right) dx \\ &= \sum_{n=0}^{\infty} \left( \int_0^1 \frac{(-1)^n x^{2n}}{n!} dx \right) \\ &\quad \text{(by unif. conv)} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!} \Big|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} \end{aligned}$$

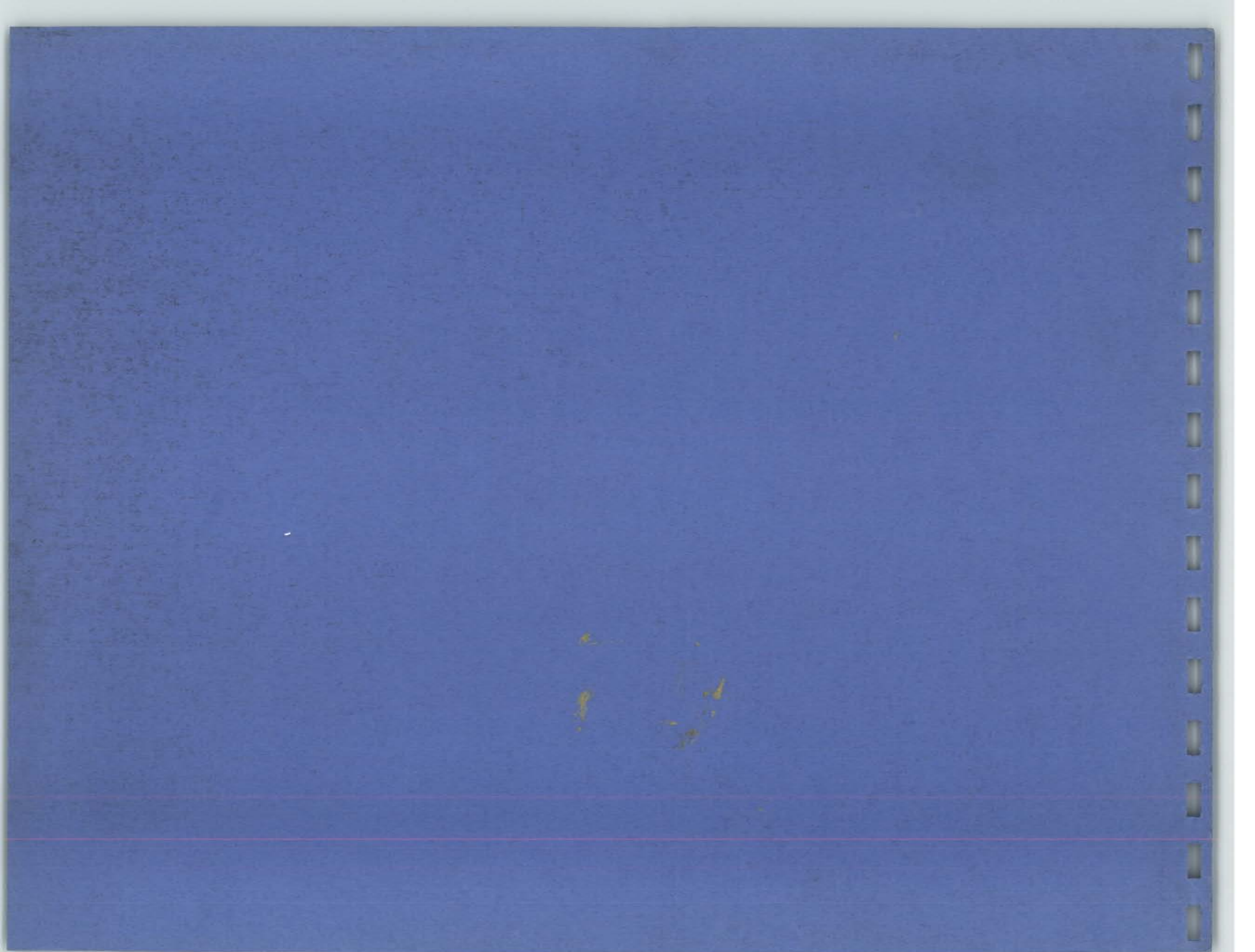














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