Calculus Revisited
Part 1
A Self-Study Course

Lecture Notes

Center for Advanced Engineering Study

Herbert I. Gross

Massachusetts Institute of Technology

CALCULUS REVISITED
PART 1
A Self-Study Course

LECTURE NOTES

Herbert I. Gross

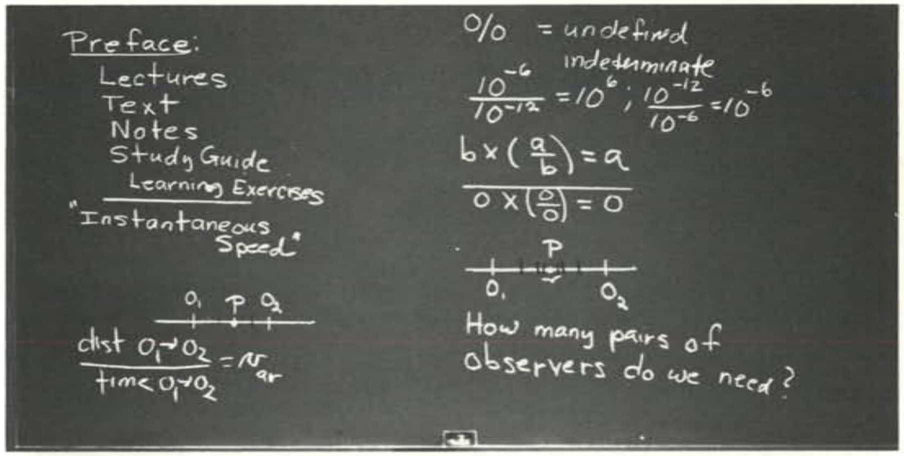
Center for Advanced Engineering Study
Massachusetts Institute of
Technology

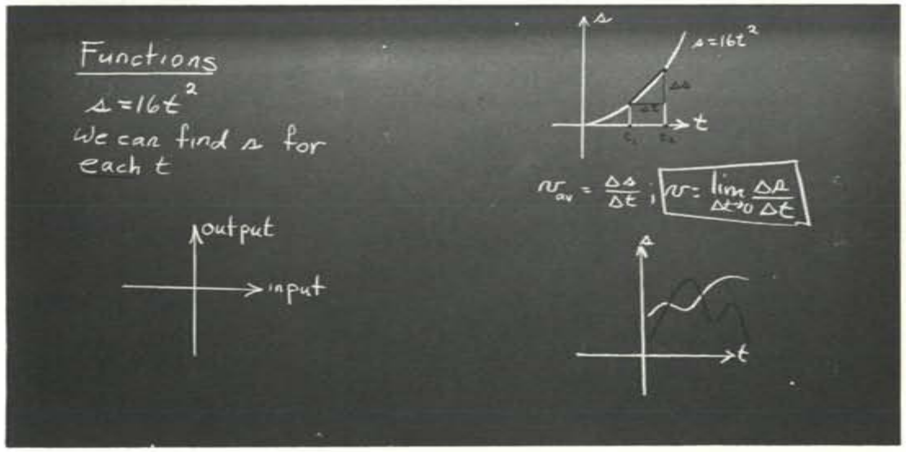
Table of Contents

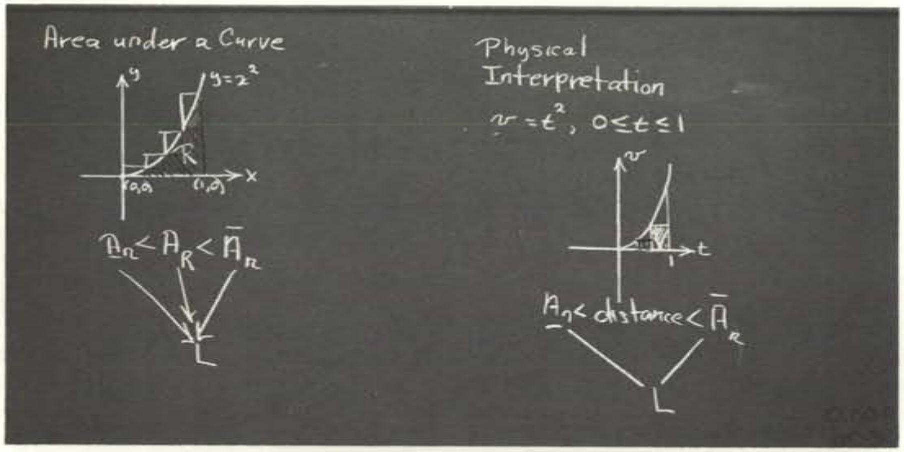
Block VII: Infinite Series

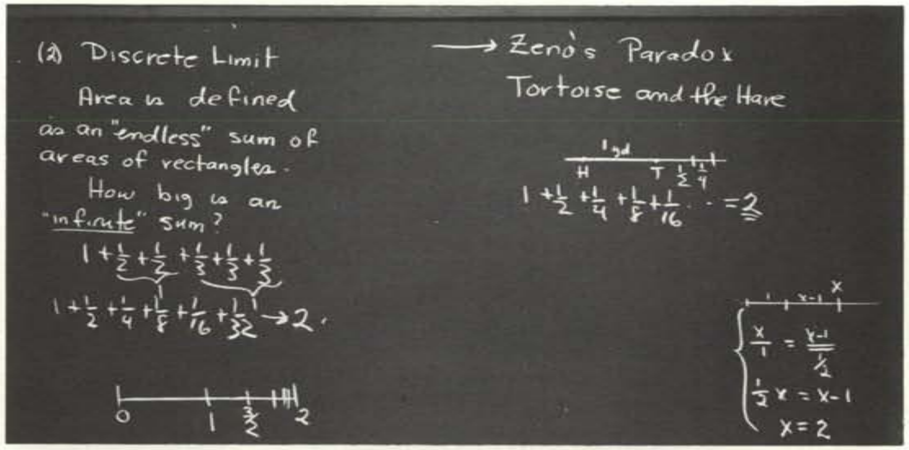
Block I: Sets, Functions, and Limits

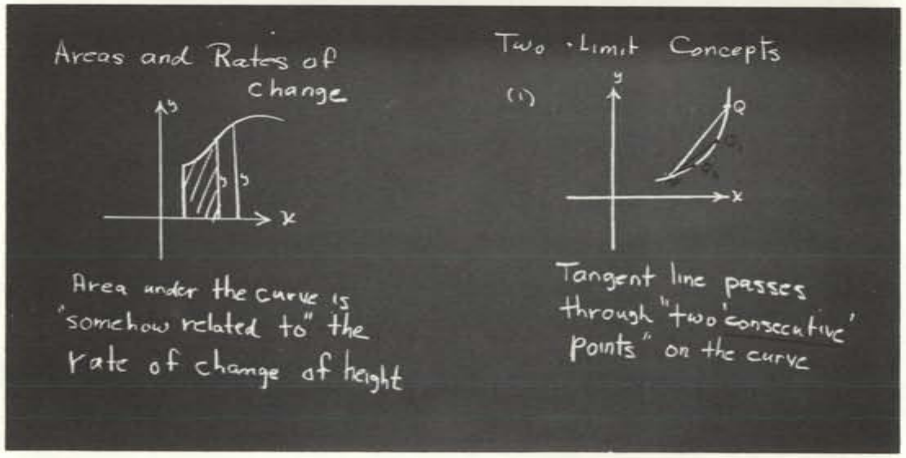
0.000 Preface **32 min.**

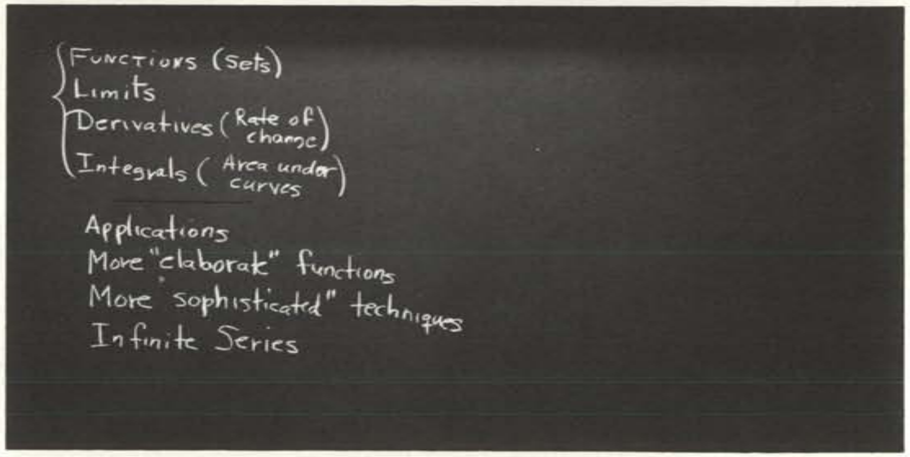




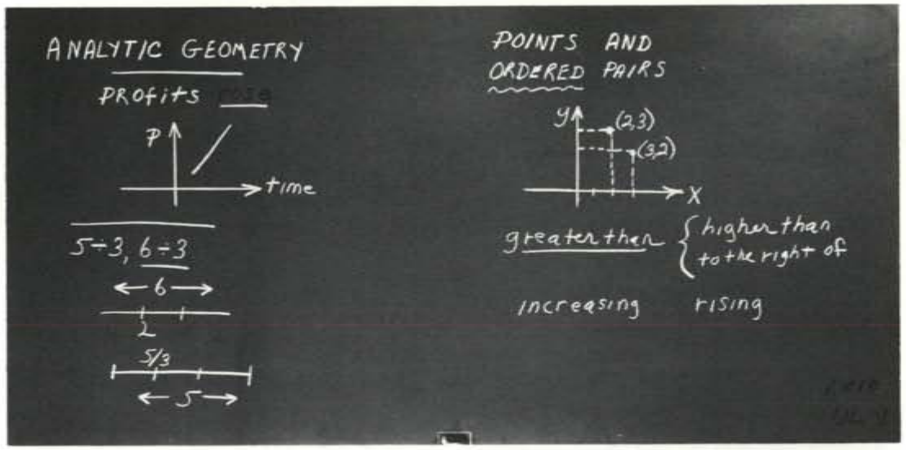

Area under a Curve
y = x²
Physical Interpretation
v = t², 0 ≤ t ≤ 1
A_R < distance < Ā_R
L

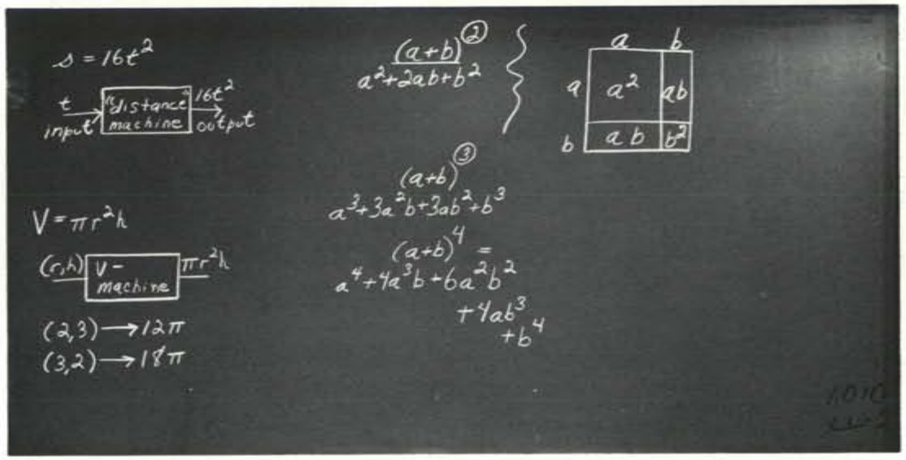

(2) Discrete Limit
Area is defined as an "endless" sum of areas of rectangles.
How big is an "infinite" sum?
1 + 1/2 + 1/4 + 1/8 + 1/16 + 1/32 → 2.
→ Zeno's Paradox
Tortoise and the Hare
1 + 1/2 + 1/4 + 1/8 + 1/16 ... = 2
1/2 x = x − 1
x = 2

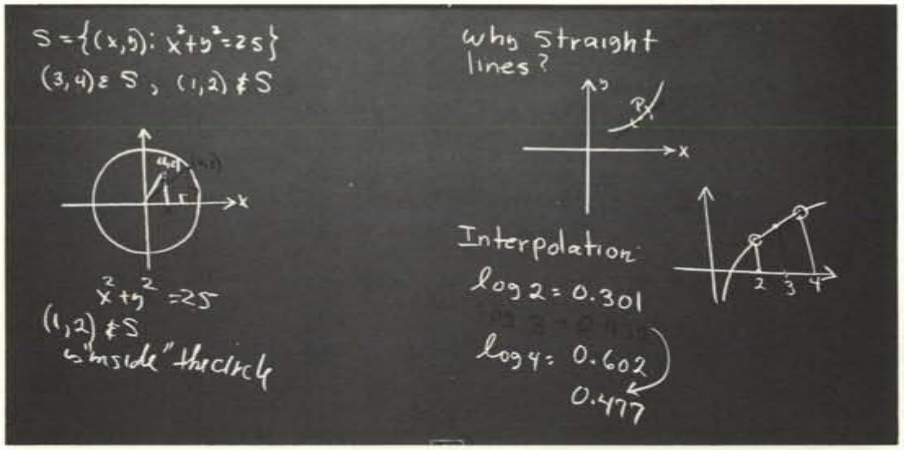

Areas and Rates of change
Area under the curve is "somehow related to" the rate of change of height
Two Limit Concepts
(1)
Tangent line passes through "two 'consecutive' points" on the curve


Functions (Sets)
Limits
Derivatives (Rate of change)
Integrals (Area under curves)
Applications
More "elaborate" functions
More "sophisticated" techniques
Infinite Series

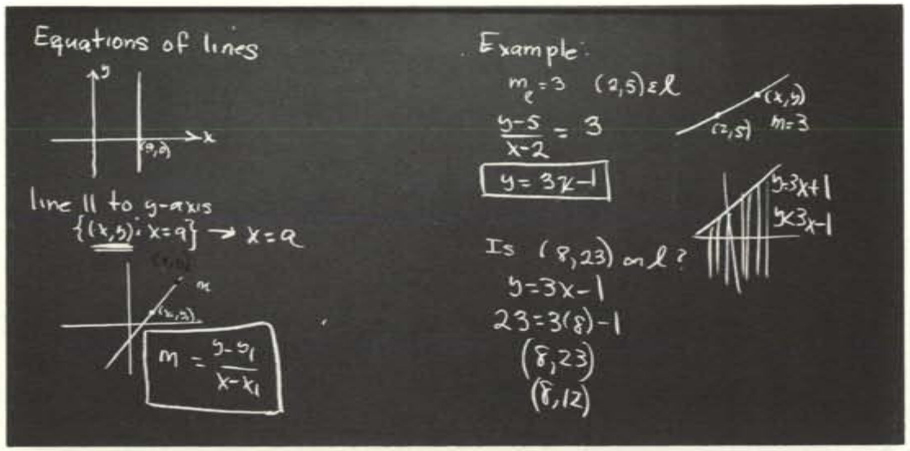
1.010 Analytic Geometry **37 min.**



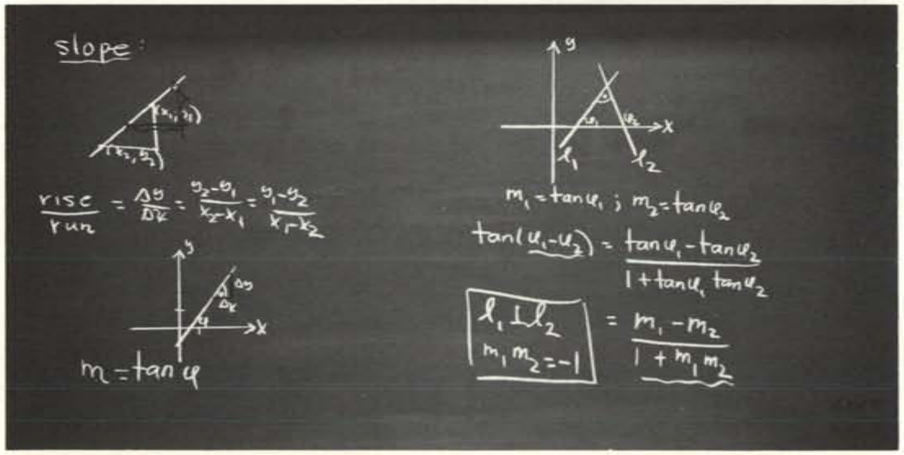




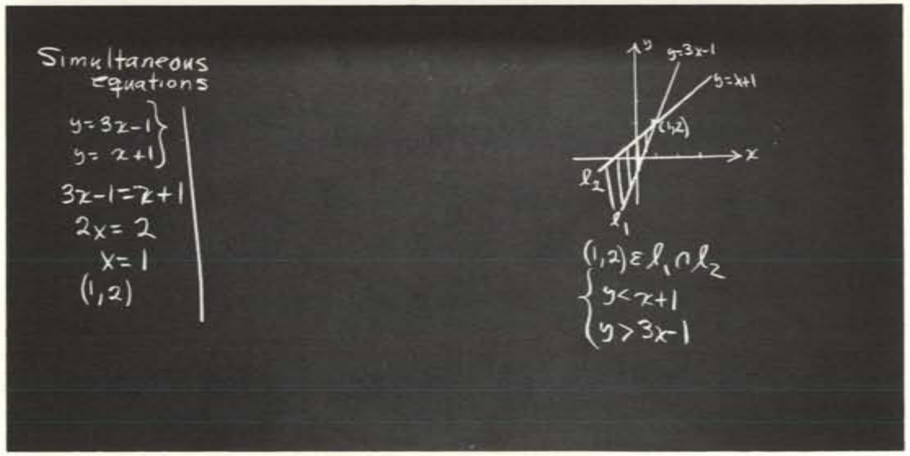
$S = \{(x,y): x^2+y^2=25\}$
$(3,4) \in S$, $(1,2) \notin S$
$x^2+y^2=25$
$(1,2) \notin S$
"inside" the circle
why straight lines?
Interpolation
$\log 2 = 0.301$
$\log 4 = 0.602$
0.477



Equations of lines
line ∥ to y-axis
$\{(x,y): x=a\} \to x=a$
$m = \frac{y-y_1}{x-x_1}$
Example:
$m_\ell = 3$ $(2,5) \in \ell$
$\frac{y-5}{x-2} = 3$
$y = 3x-1$
$y=3x+1$
$y<3x-1$
Is $(8,23)$ on ℓ?
$y=3x-1$
$23=3(8)-1$
$(8,23)$
$(8,12)$

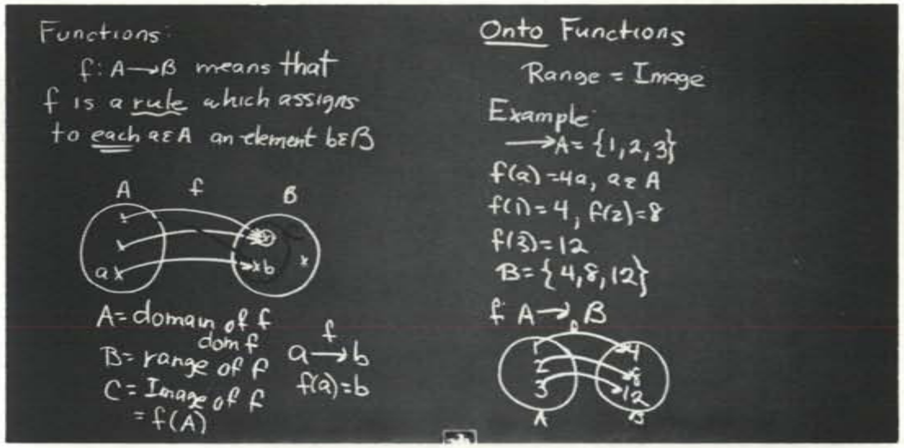


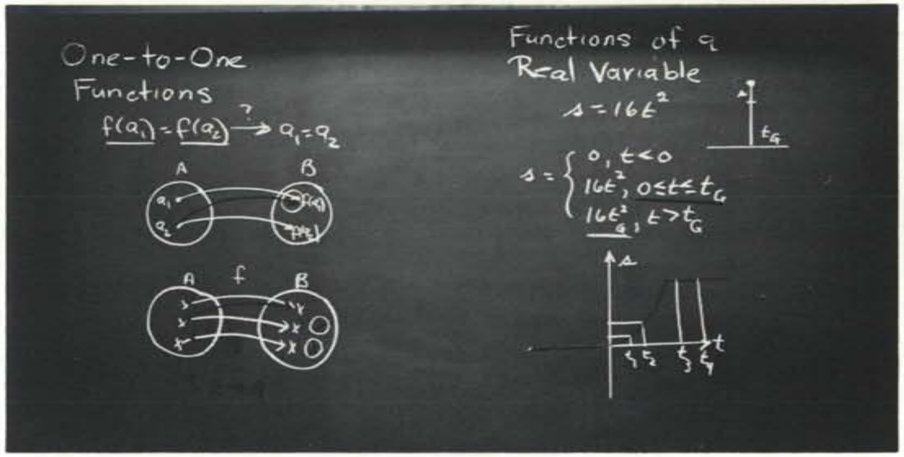
slope:
$\frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2-y_1}{x_2-x_1} = \frac{y_1-y_2}{x_1-x_2}$
$m = \tan\varphi$
$m_1 = \tan\varphi_1$; $m_2 = \tan\varphi_2$
$\tan(\varphi_1-\varphi_2) = \frac{\tan\varphi_1-\tan\varphi_2}{1+\tan\varphi_1\tan\varphi_2}$
$\ell_1 \perp \ell_2$
$m_1 m_2 = -1$
$= \frac{m_1-m_2}{1+m_1m_2}$

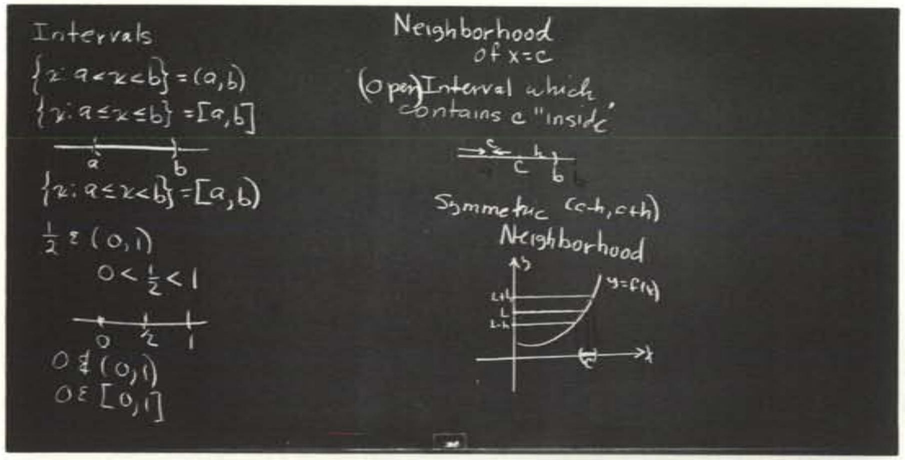


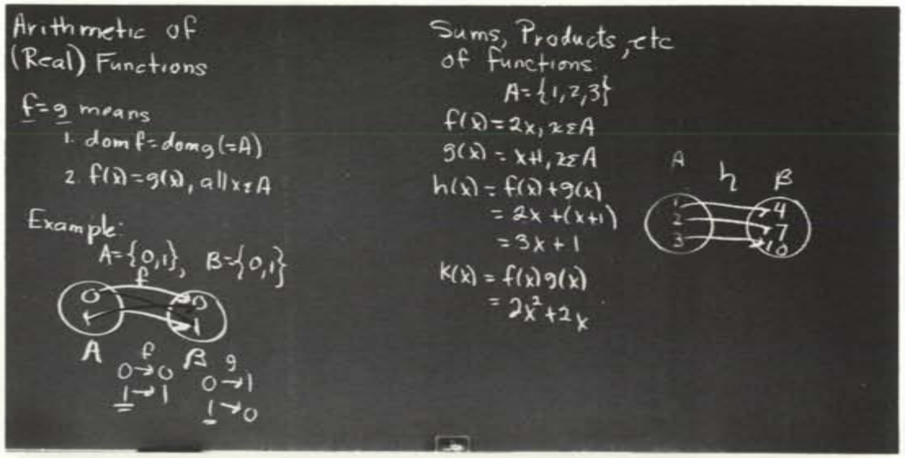
Simultaneous equations
$y=3x-1$
$y=x+1$
$3x-1=x+1$
$2x=2$
$x=1$
$(1,2)$
$y=3x-1$
$y=x+1$
$(1,2) \in \ell_1 \cap \ell_2$
$y<x+1$
$y>3x-1$

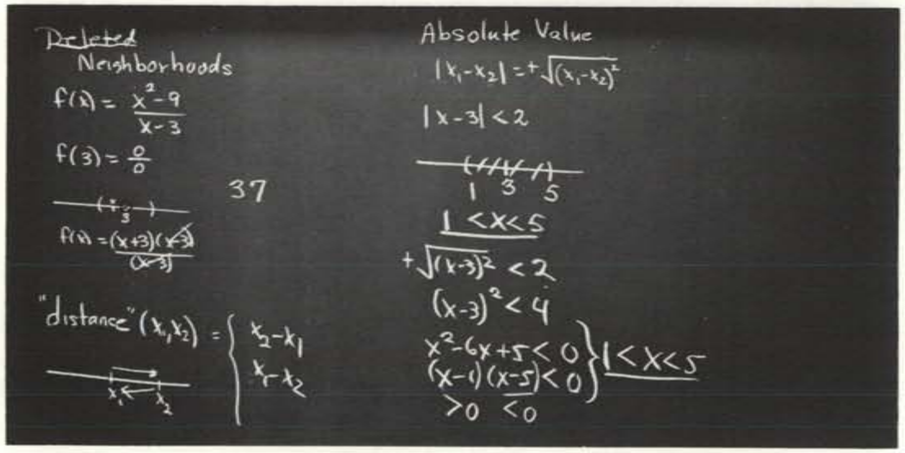
1.020 Functions **39 min.**

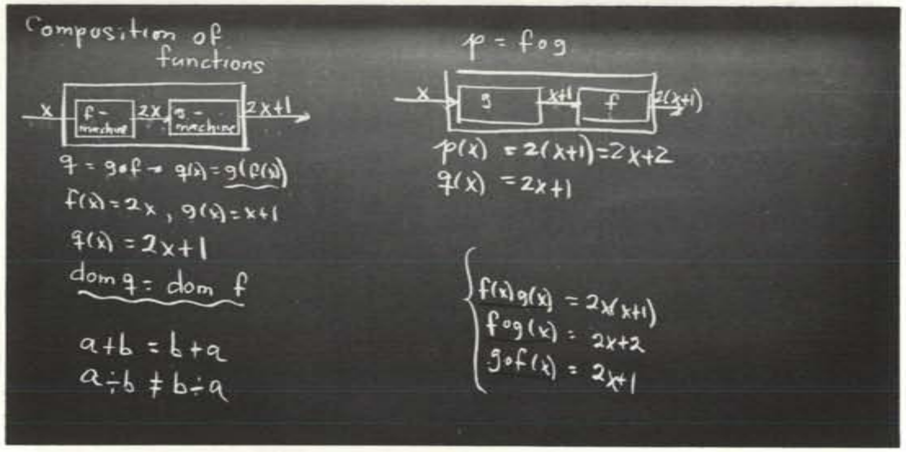




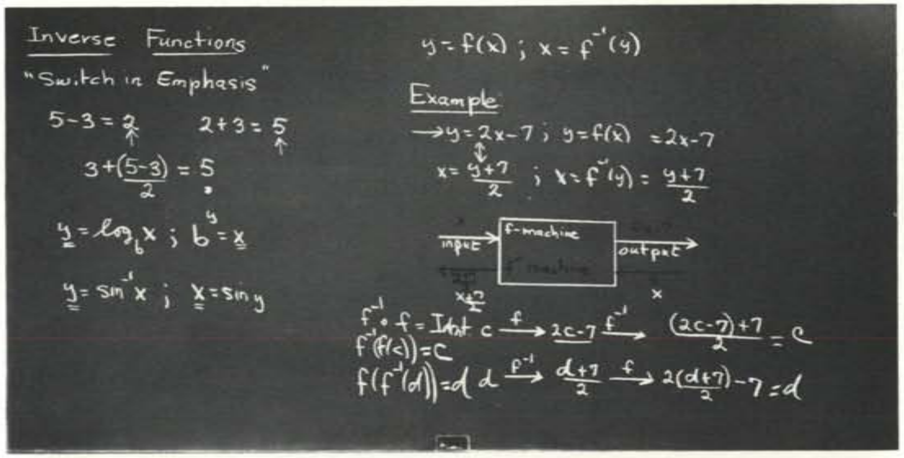

Intervals
$\{x: a<x<b\} = (a,b)$
$\{x: a\le x\le b\} = [a,b]$
$\{x: a\le x<b\} = [a,b)$
$\frac{1}{2} \in (0,1)$
$0 < \frac{1}{2} < 1$
$0 \notin (0,1)$
$0 \in [0,1]$
Neighborhood of x=c
(open) Interval which contains c "inside"
Symmetric (c-h,c+h) Neighborhood
y=f(x)

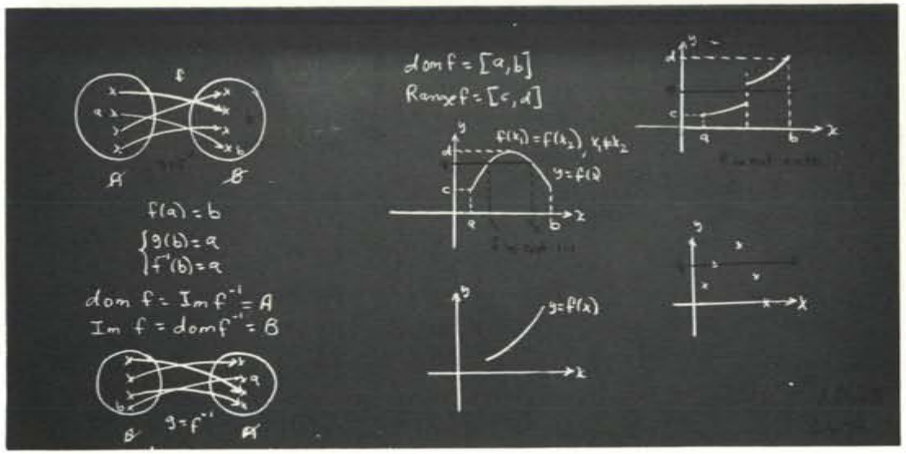

Arithmetic of (Real) Functions
f=g means
1. dom f = dom g (=A)
2. f(x)=g(x), all x ε A
Example:
$A=\{0,1\}$, $B=\{0,1\}$
Sums, Products, etc of functions
$A=\{1,2,3\}$
f(x)=2x, x ε A
g(x) = x+1, x ε A
h(x) = f(x)+g(x)
= 2x + (x+1)
= 3x + 1
k(x) = f(x)g(x)
$= 2x^2+2x$

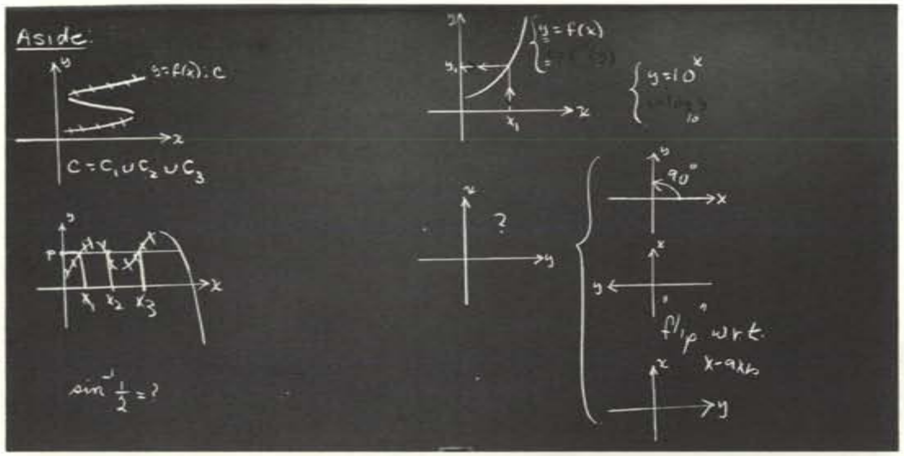

Deleted Neighborhoods
$f(x) = \frac{x^2-9}{x-3}$
$f(3) = \frac{0}{0}$
37
"distance" (x_1,x_2)
Absolute Value
$|x_1-x_2| = +\sqrt{(x_1-x_2)^2}$
$|x-3| < 2$
$1 < x < 5$
$+\sqrt{(x-3)^2} < 2$
$(x-3)^2 < 4$
$x^2-6x+5<0$
$(x-1)(x-5)<0$

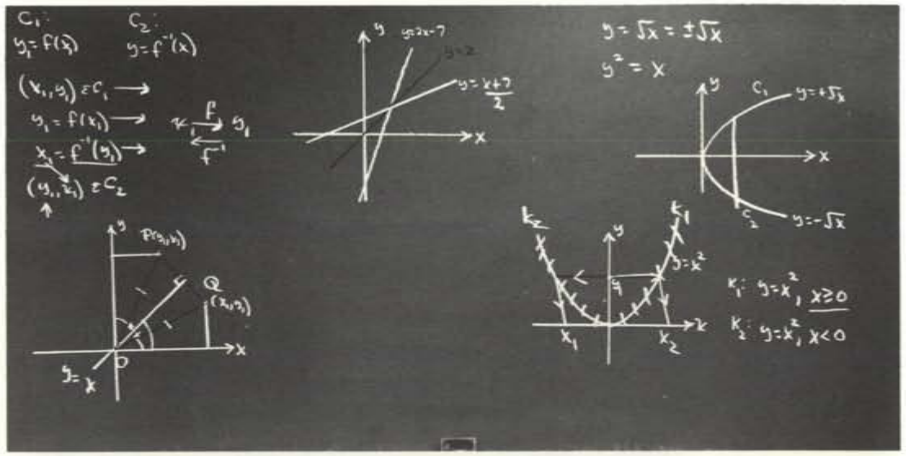

Composition of functions
f-machine
g-machine
$q = g\circ f \rightarrow q(x) = g(f(x))$
f(x) = 2x, g(x) = x+1
q(x) = 2x+1
dom q = dom f
a+b = b+a
a÷b ≠ b÷a
$p = f\circ g$
p(x) = 2(x+1) = 2x+2
q(x) = 2x+1
f(x)g(x) = 2x(x+1)
$f\circ g(x)$ = 2x+2
$g\circ f(x)$ = 2x+1

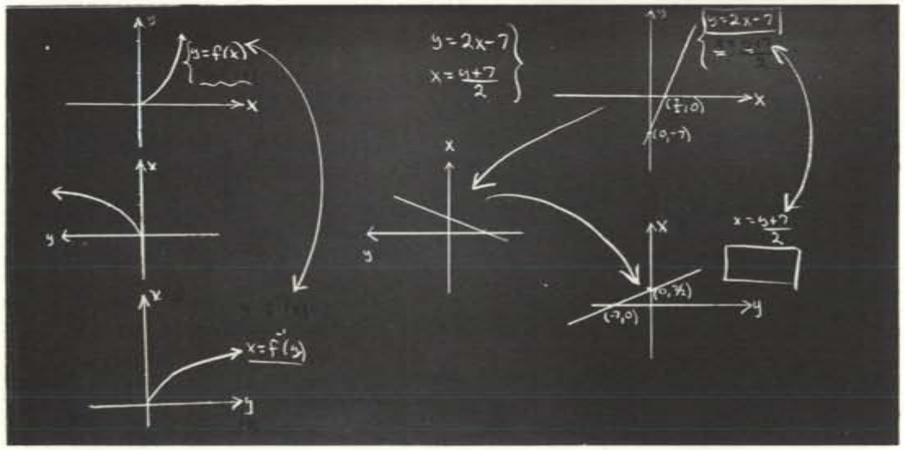
1.025 Inverse Functions 40 min.

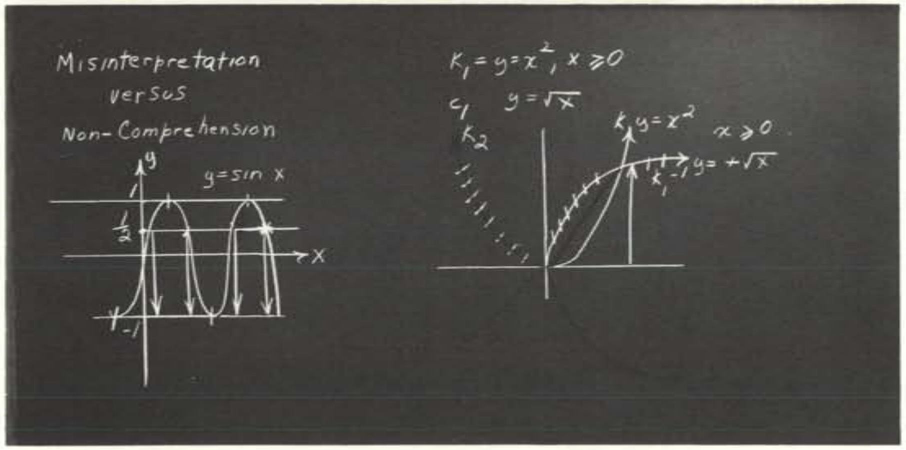




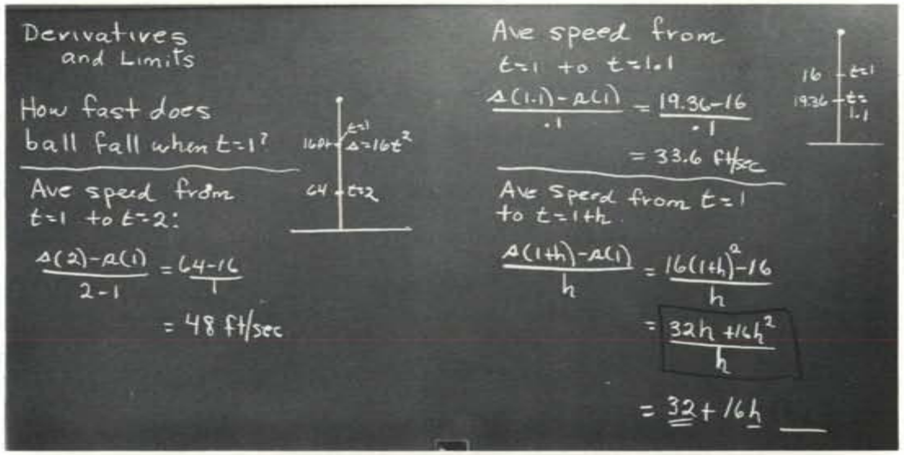

Aside
y = f(x)
y = 10^x
C = C1 ∪ C2 ∪ C3
flip wrt

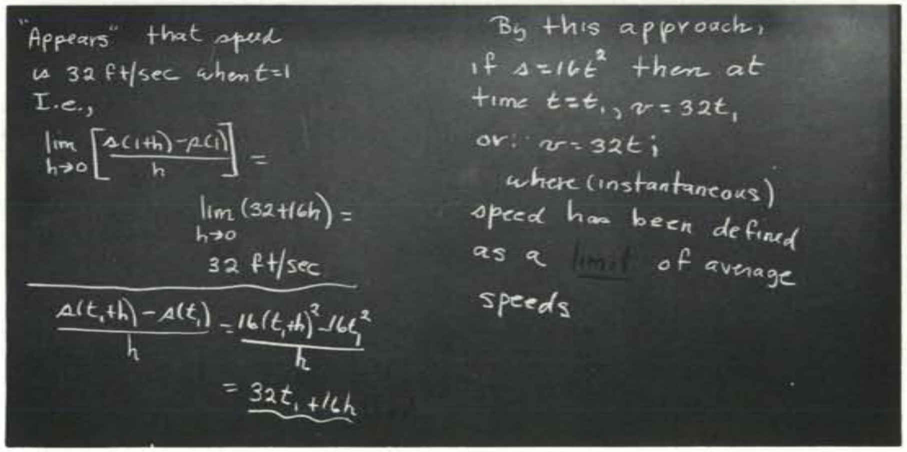

C1
C2
y = f(x)
y = f⁻¹(x)
y = √x = ±√x
y² = x
y = x

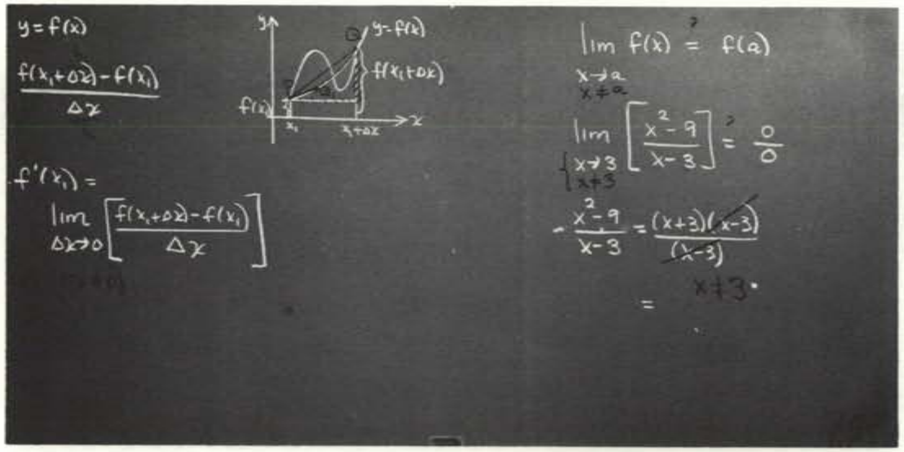

y = f(x)
y = 2x-7
x = (y+7)/2
x = f⁻¹(y)

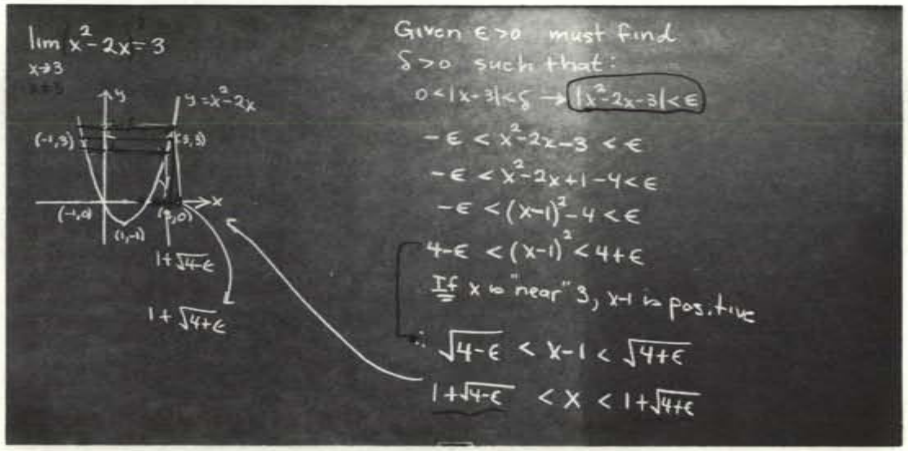

Misinterpretation
versus
Non-Comprehension
y = sin x
y = √x
y = x²
x ≥ 0

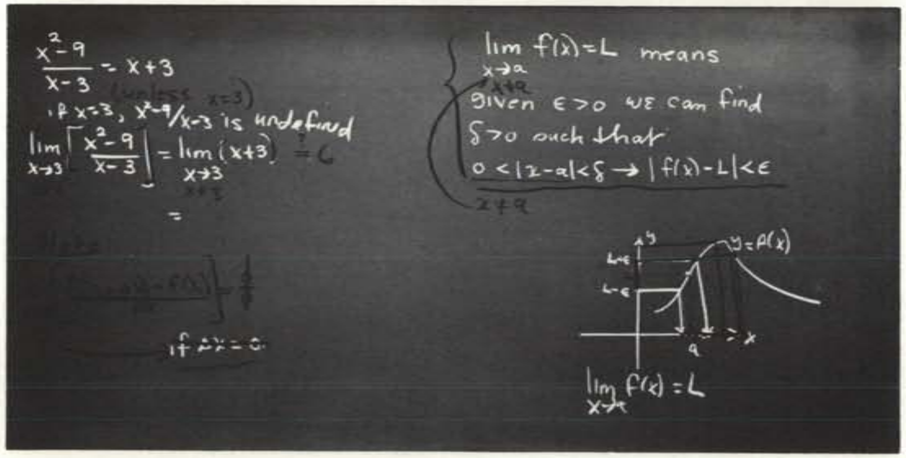
1.030 Derivatives and Limits **45 min.**

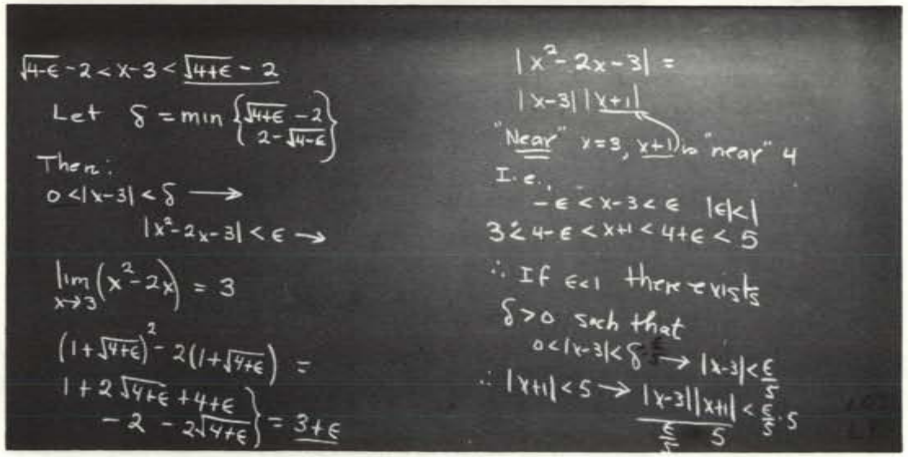




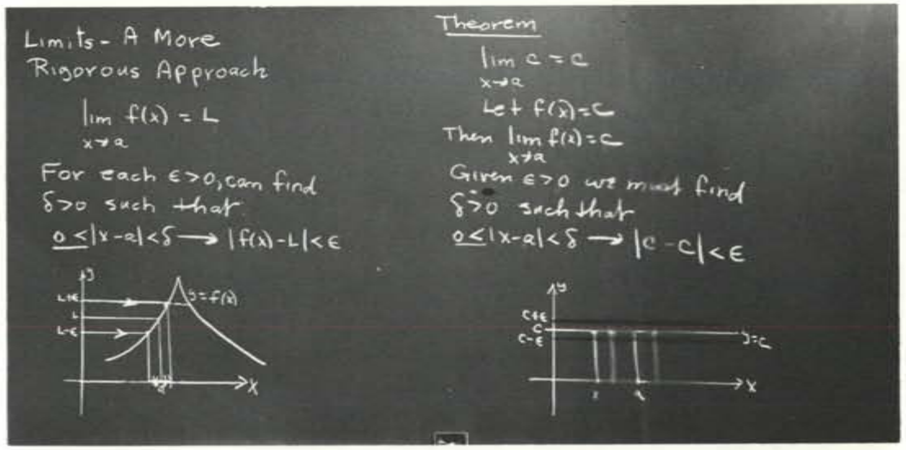

y = f(x)
f(x₁+Δx) − f(x₁)
Δx
f′(x₁) =
lim Δx→0 [f(x₁+Δx) − f(x₁) / Δx]
y = f(x)
f(x₁+Δx)
f(x₁)
lim x→a f(x) = f(a)
x ≠ a
lim x→3 [x² − 9 / x − 3] = 0/0
x ≠ 3
x² − 9 / x − 3 = (x+3)(x−3) / (x−3)
= x+3

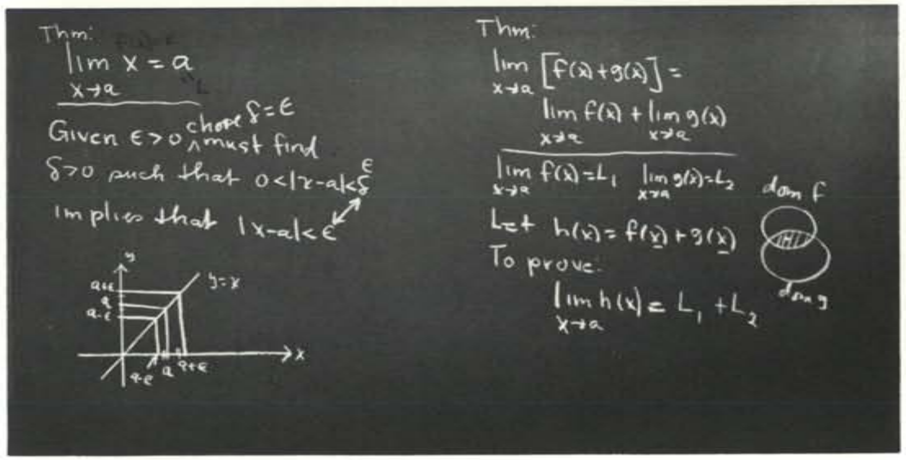

lim x→3 x² − 2x = 3
x ≠ 3
y = x² − 2x
(−1, 3)
(3, 3)
(−1, 0)
(1, −1)
(3, 0)
1 + √(4−ε)
1 + √(4+ε)
Given ε > 0 must find
δ > 0 such that:
0 < |x−3| < δ → |x² − 2x − 3| < ε
−ε < x² − 2x − 3 < ε
−ε < x² − 2x + 1 − 4 < ε
−ε < (x−1)² − 4 < ε
4 − ε < (x−1)² < 4 + ε
If x is "near" 3, x−1 is positive
∴ √(4−ε) < x − 1 < √(4+ε)
1 + √(4−ε) < x < 1 + √(4+ε)

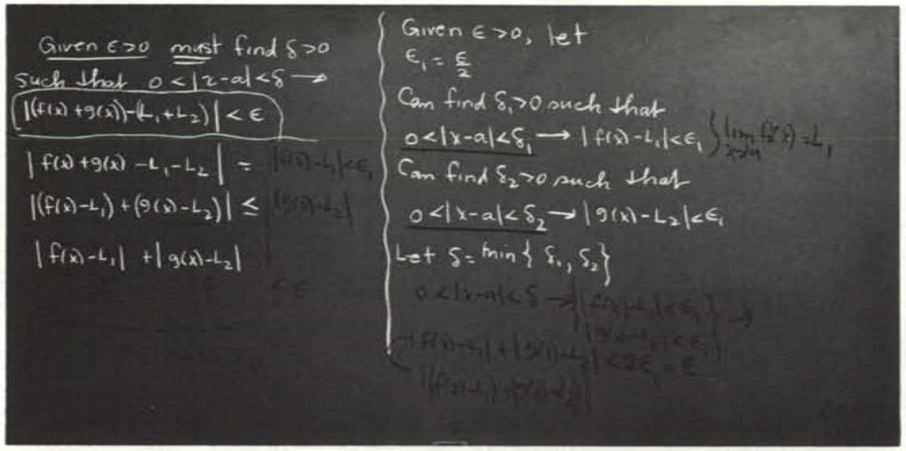

x² − 9 / x − 3 = x + 3 (unless x = 3)
if x = 3, x² − 9/x − 3 is undefined
lim x→3 [x² − 9 / x − 3] = lim x→3 (x+3) = 6
lim x→a f(x) = L means
x ≠ a
given ε > 0 we can find
δ > 0 such that
0 < |x − a| < δ → |f(x) − L| < ε
x ≠ a
y = f(x)
L + ε
L − ε
a
x
lim x→a f(x) = L

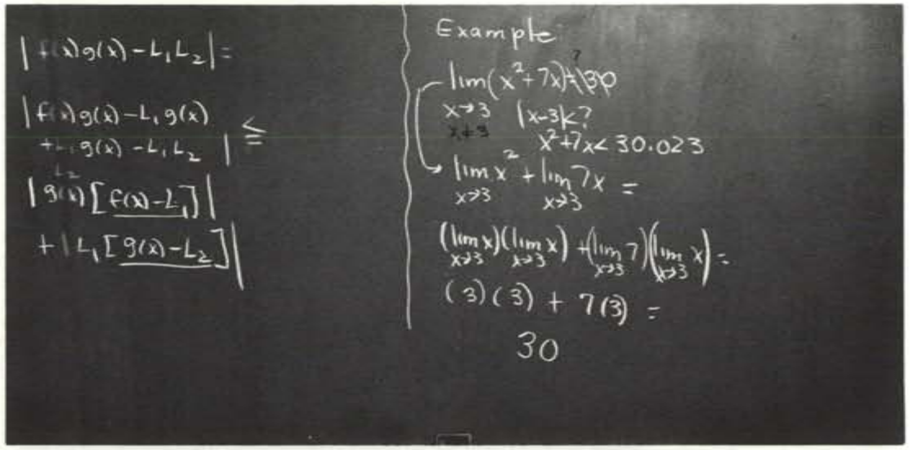

√(4−ε) − 2 < x − 3 < √(4+ε) − 2
Let δ = min { √(4+ε) − 2, 2 − √(4−ε) }
Then:
0 < |x − 3| < δ →
|x² − 2x − 3| < ε →
lim x→3 (x² − 2x) = 3
(1 + √(4+ε))² − 2(1 + √(4+ε)) =
1 + 2√(4+ε) + 4 + ε
− 2 − 2√(4+ε) = 3 + ε
|x² − 2x − 3| =
|x − 3| |x + 1|
"Near" x = 3, x + 1 is "near" 4
I.e.
−ε < x − 3 < ε |ε| < 1
3 < 4 − ε < x + 1 < 4 + ε < 5
∴ If ε < 1 there exists
δ > 0 such that
0 < |x − 3| < δ → |x − 3| < ε/5
∴ |x + 1| < 5 → |x − 3| |x + 1| < ε/5 · 5

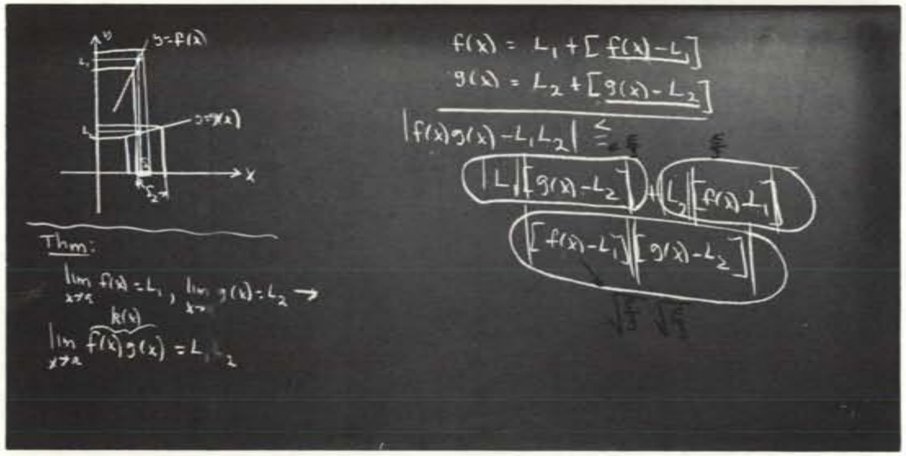
1.040 Limits: A More Rigorous Approach 46 min.

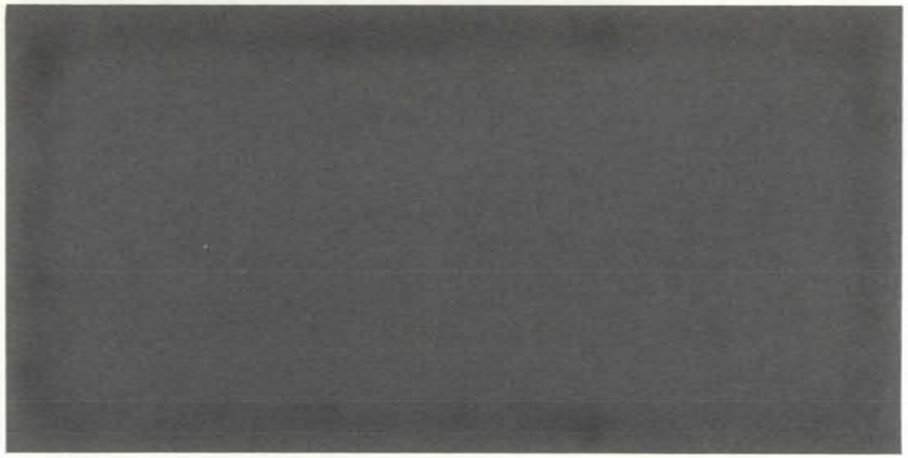




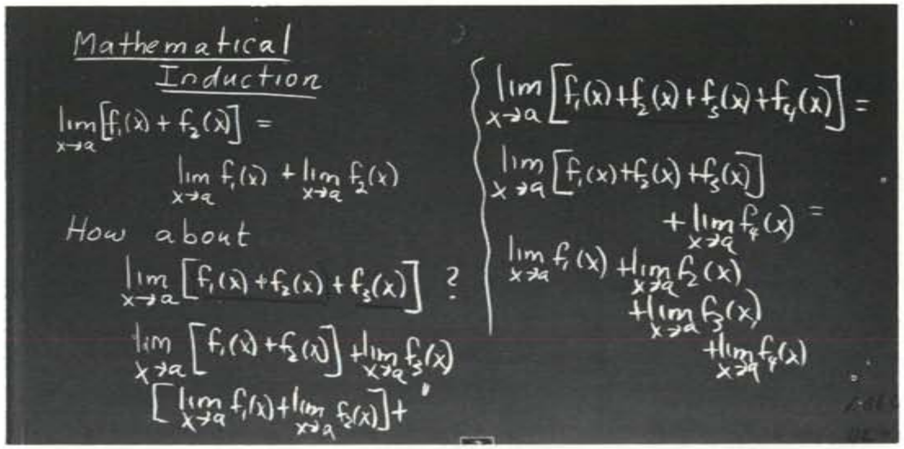

Given ε>0 must find δ>0
such that 0<|x-a|<δ →
|(f(x)+g(x))-(L1+L2)| < ε
|f(x)+g(x) - L1 - L2| =
|(f(x)-L1)+(g(x)-L2)| ≤
|f(x)-L1| + |g(x)-L2|
Given ε>0, let
ε1 = ε/2
Can find δ1>0 such that
0<|x-a|<δ1 → |f(x)-L1|<ε1
Can find δ2>0 such that
0<|x-a|<δ2 → |g(x)-L2|<ε1
Let δ = min{δ1, δ2}

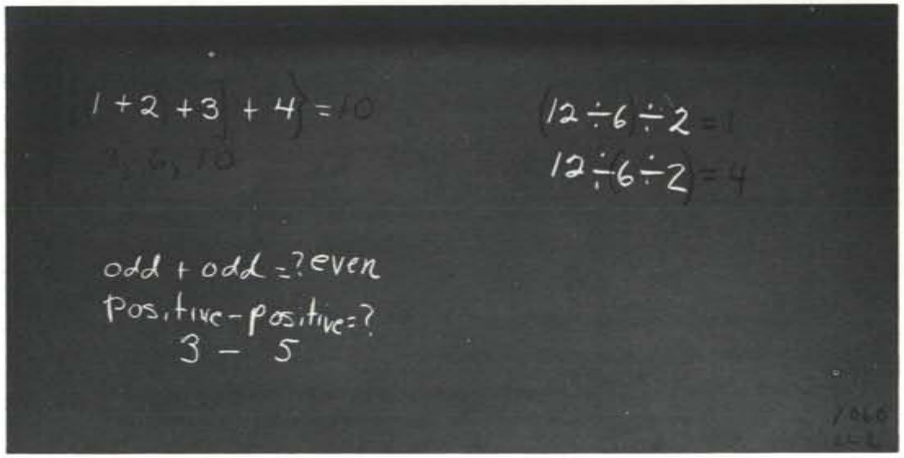

|f(x)g(x) - L1L2| =
|f(x)g(x) - L1g(x) + L1g(x) - L1L2| ≤
|g(x)[f(x)-L1]|
+ |L1[g(x)-L2]|
Example
lim (x²+7x) = 30
x→3
|x-3|<?
x²+7x < 30.023
lim x² + lim 7x =
(lim x)(lim x) + (lim 7)(lim x) =
(3)(3) + 7(3) =
30

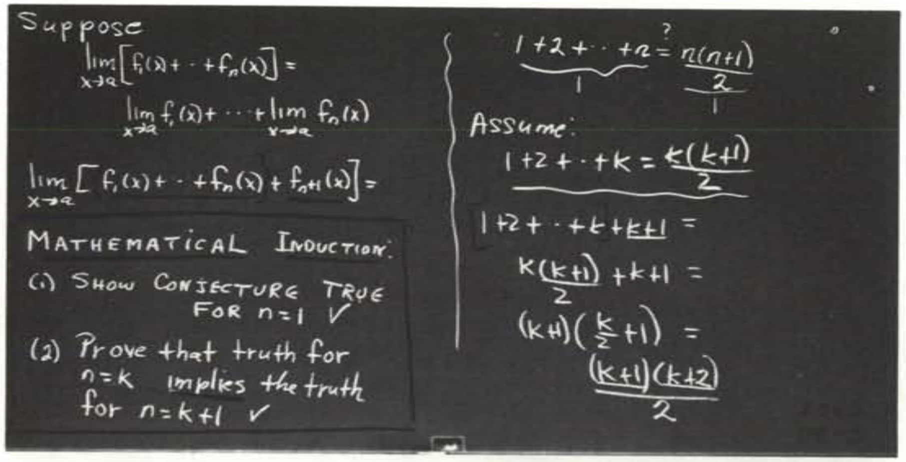

y=f(x)
y=g(x)
x
Thm:
lim f(x) = L1, lim g(x) = L2 →
lim f(x)g(x) = L1L2
f(x) = L1 + [f(x)-L1]
g(x) = L2 + [g(x)-L2]
|f(x)g(x)-L1L2| ≤
|L1|[g(x)-L2]| + |L2|[f(x)-L1]|
|[f(x)-L1]|[g(x)-L2]|

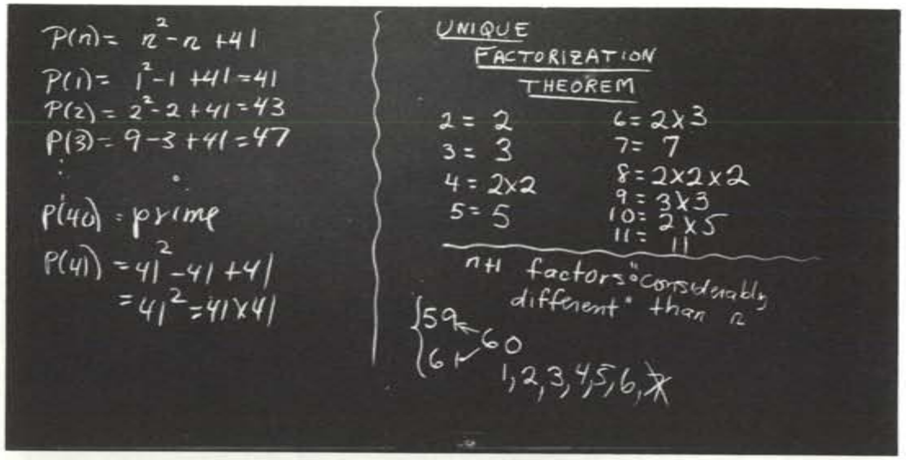


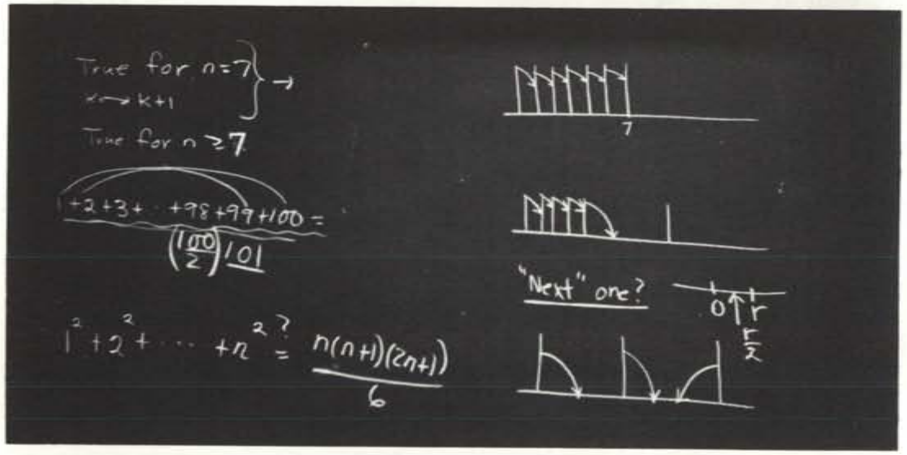
1.060 **Mathematical Induction** **29 min.**





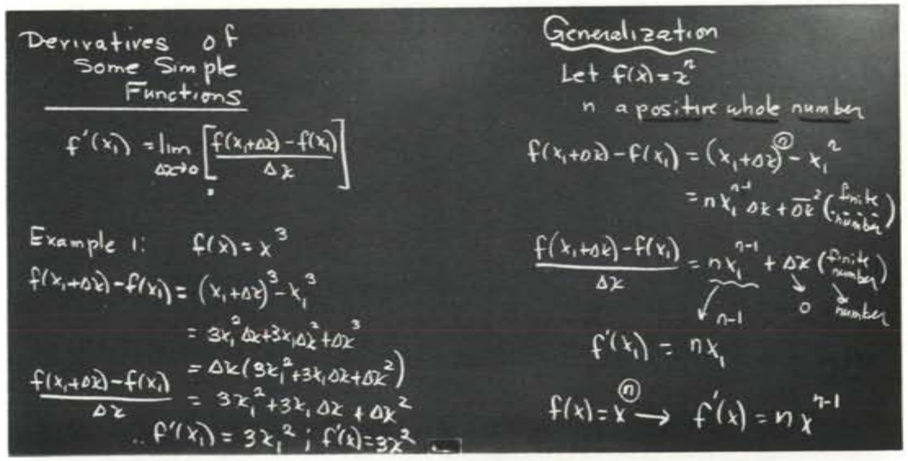

Suppose
lim [f1(x) + · + fn(x)] =
lim f1(x) + ··· + lim fn(x)
lim [f1(x) + · + fn(x) + fn+1(x)] =
MATHEMATICAL INDUCTION:
(1) SHOW CONJECTURE TRUE FOR n=1 ✓
(2) Prove that truth for n=k implies the truth for n=k+1 ✓
1+2+ · +n ≟ n(n+1)/2
Assume:
1+2+ · +k = k(k+1)/2
1+2+ · +k+k+1 =
k(k+1)/2 +k+1 =
(k+1)(k/2+1) =
(k+1)(k+2)/2

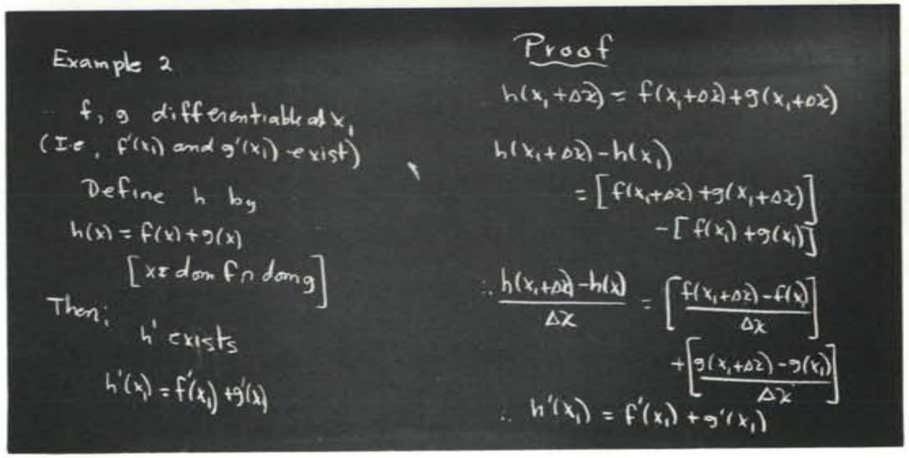

P(n) = n² − n + 41
P(1) = 1² − 1 + 41 = 41
P(2) = 2² − 2 + 41 = 43
P(3) = 9 − 3 + 41 = 47
P(40) = prime
P(41) = 41² − 41 + 41
= 41² = 41×41
UNIQUE FACTORIZATION THEOREM
2 = 2
3 = 3
4 = 2×2
5 = 5
6 = 2×3
7 = 7
8 = 2×2×2
9 = 3×3
10 = 2×5
11 = 11
n+1 factors "considerably different" than n
59
60
1,2,3,4,5,6


True for n=7
k→k+1
True for n≥7
1+2+3+ · +98+99+100 =
(100/2)101
1² + 2² + ··· + n² ≟ n(n+1)(2n+1)/6
7
"Next" one?
0
r
r/2

Block II: Differentiation

2.010 Derivatives of Some Simple Functions 28 min.





Beware of "Self-Evident"

$[f(x)g(x)]' \not\equiv f'(x)\,g'(x)$

$f(x) = x+1 \rightarrow f'(x) = 1$

$g(x) = x-1 \rightarrow g'(x) = 1$

$f(x)g(x) = x^2-1 \rightarrow [f(x)g(x)]' = \underline{2x}$

$\therefore [f(x)g(x)]' \not\equiv f'(x)g'(x)$

In fact, if

$h(x) = f(x)g(x)$

then:

$$h(x_1+\Delta x) - h(x_1) = f(x_1+\Delta x)\,g(x_1+\Delta x) - f(x_1)g(x_1)$$

$$= f(x_1+\Delta x)g(x_1+\Delta x) - f(x_1)g(x_1+\Delta x) + f(x_1)\,g(x_1+\Delta x) - f(x_1)g(x_1)$$

$$\therefore \frac{h(x_1+\Delta x)-h(x_1)}{\Delta x} = \left[\frac{f(x_1+\Delta x)-f(x_1)}{\Delta x}\right] g(x_1+\Delta x) + f(x_1)\left[\frac{g(x_1+\Delta x)-g(x_1)}{\Delta x}\right]$$

$$h'(x_1) = f'(x_1)\,g(x_1)^{(*)} + f(x_1)\,g'(x_1)$$

$(1(x_1-1) + (x_1+1)1 = 2x_1)$

Summary

The basic definition:

$$f'(x_1) = \lim_{\Delta x \to 0}\left[\frac{f(x_1+\Delta x)-f(x_1)}{\Delta x}\right]$$

never changes. But it can be manipulated to yield "convenient" "recipes".

(*) It is not true always that $\lim_{t \to a} g(t) = g(a)$

For example, let $g(t) = \frac{t^2-1}{t-1}$

$\lim_{t \to 1} g(t) = 2$, $g(1)$ doesn't exist! $(0/0)$

In our present case:

$$g(x_1+\Delta x) - g(x_1) = \left[\frac{g(x_1+\Delta x)-g(x_1)}{\Delta x}\right]\Delta x$$

$$\lim_{\Delta x \to 0}\left[g(x_1+\Delta x)-g(x_1)\right] = g'(x_1)\lim_{\Delta x \to 0}\Delta x = 0$$

$$\therefore \lim_{\Delta x \to 0} g(x_1+\Delta x) = g(x_1)$$

For a quotient we can show that if $h(x) = \frac{f(x)}{g(x)}$ then

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Example.

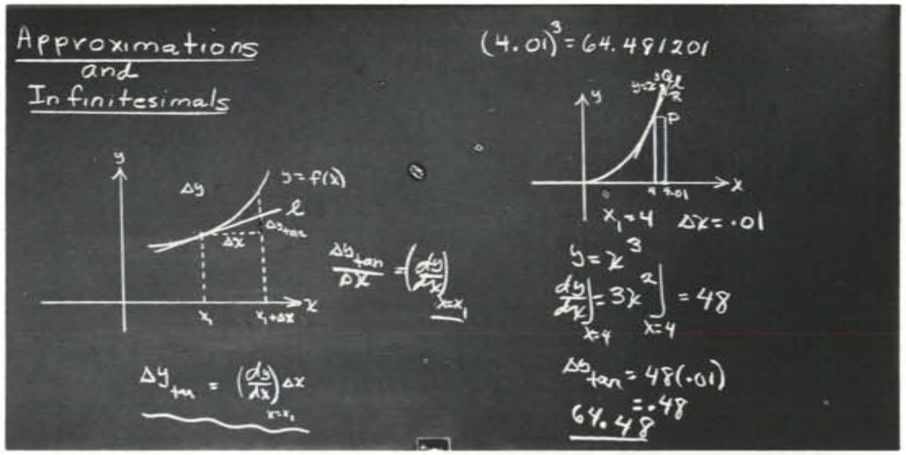
$f(x) = x^{-n}$ n, positive integer

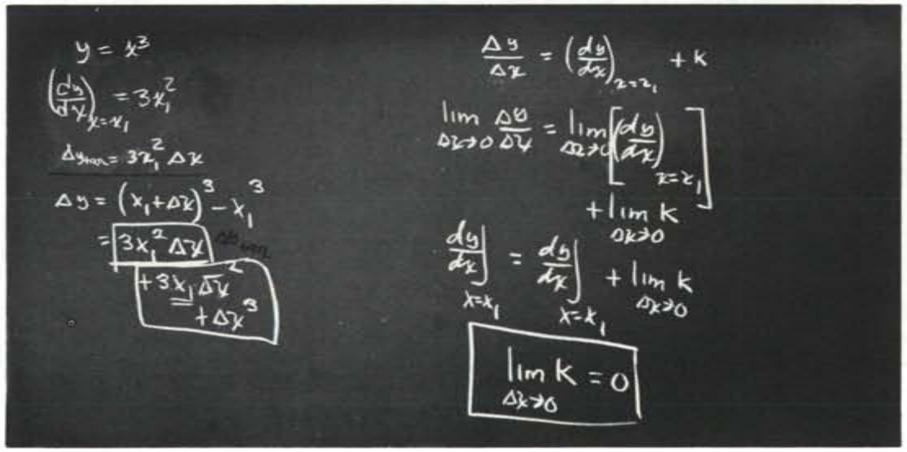
$= \frac{1}{x^n}$

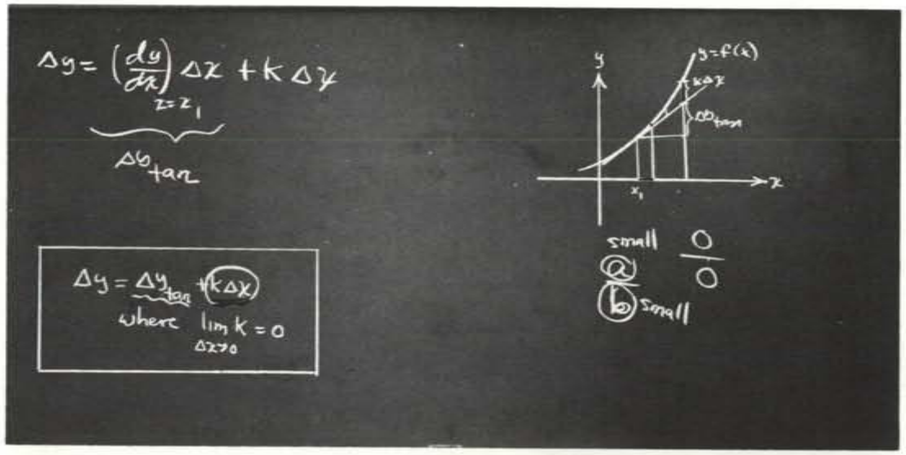
$$f'(x) = \frac{x^n(0) - 1(nx^{n-1})}{x^{2n}} = -nx^{n-1-2n}$$

$= \underline{-n\,x^{-n-1}}$

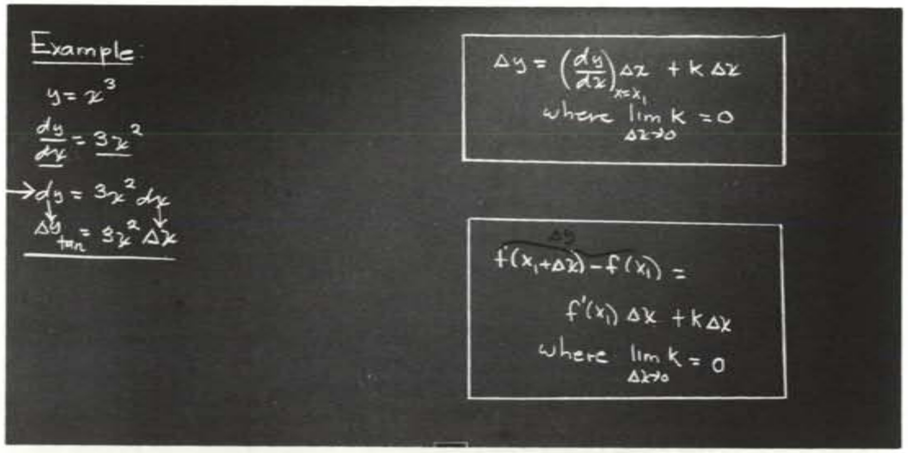
2.020 Approximations and Infinitesimals 34 min.



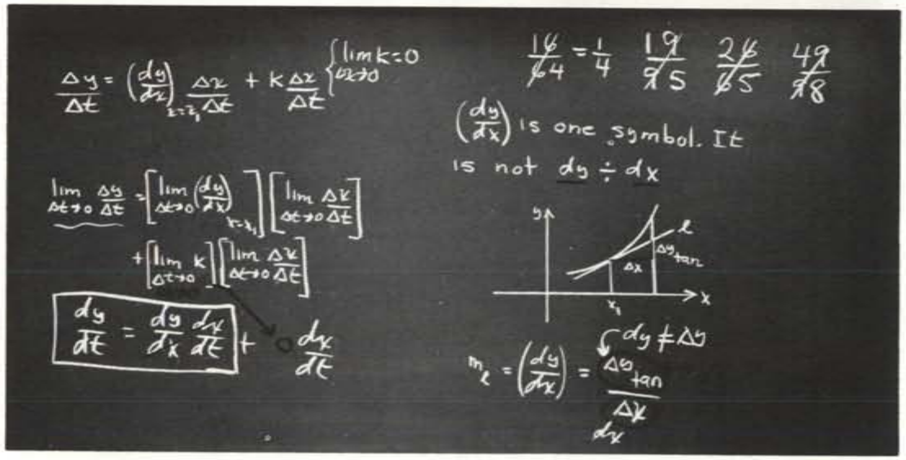




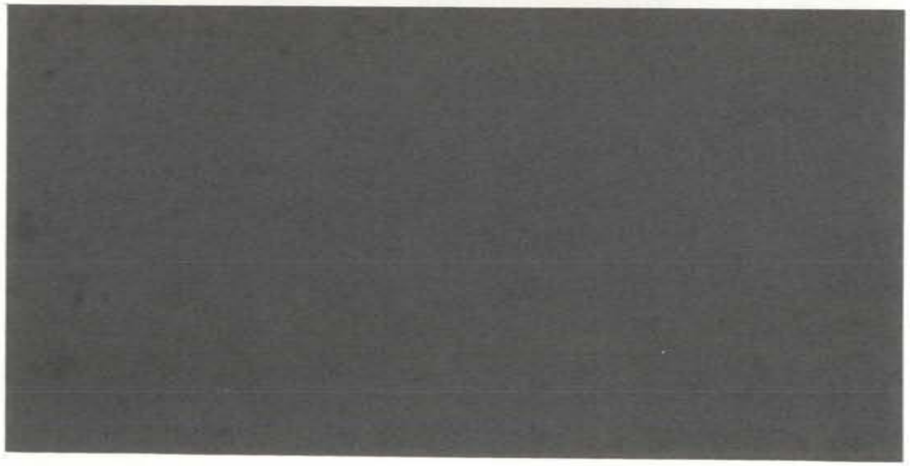
Δy = (dy/dx)_{x=x₁} Δx + k Δx
Δy_tan
y = f(x)
y
x
x₁
kΔx
Δy_tan
small
0/0
Δy = Δy_tan + kΔx
where lim k = 0
Δx→0



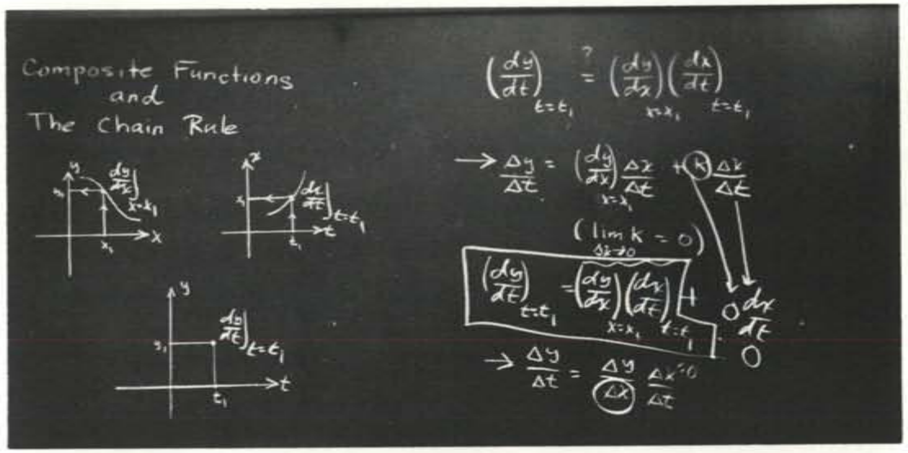
Example
y = x³
dy/dx = 3x²
dy = 3x² dx
Δy_tan = 3x² Δx
Δy = (dy/dx)_{x=x₁} Δx + k Δx
where lim k = 0
Δx→0
f(x₁+Δx) − f(x₁) =
f′(x₁) Δx + k Δx
where lim k = 0
Δx→0

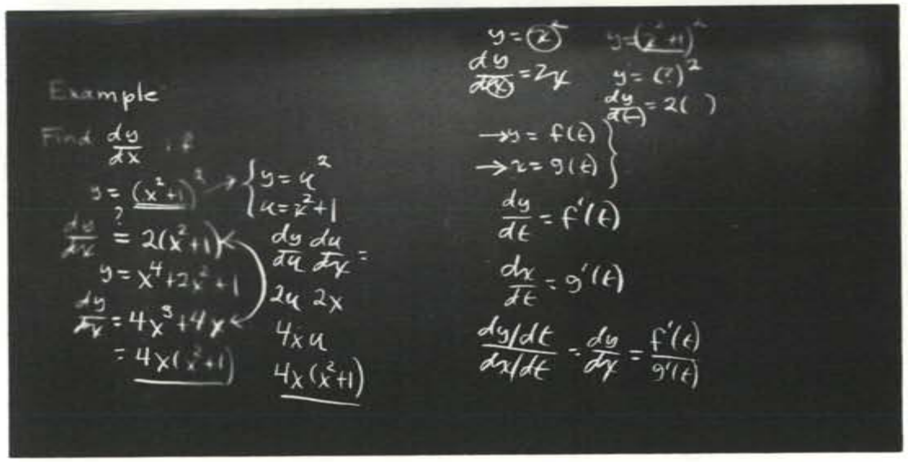


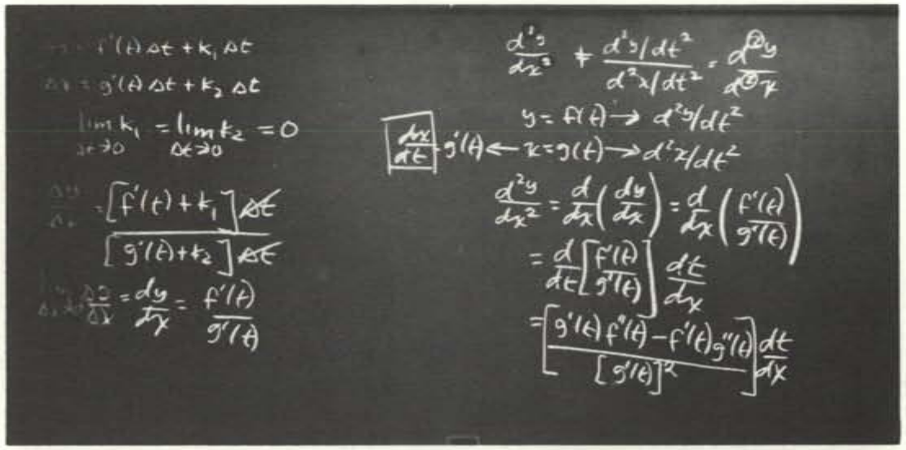
Δy/Δt = (dy/dx)_{x=x₁} Δx/Δt + k Δx/Δt
lim k = 0
Δx→0
16/64 = 1/4
19/95
26/65
49/98
(dy/dx) is one symbol. It
is not dy ÷ dx
lim Δy/Δt = [lim (dy/dx)] [lim Δx/Δt] + [lim k] [lim Δx/Δt]
Δt→0
dy/dt = dy/dx · dx/dt + 0 · dx/dt
y
x
x₁
Δx
Δy_tan
ℓ
dy ≠ Δy
m_ℓ = (dy/dx) = Δy_tan / Δx
dx



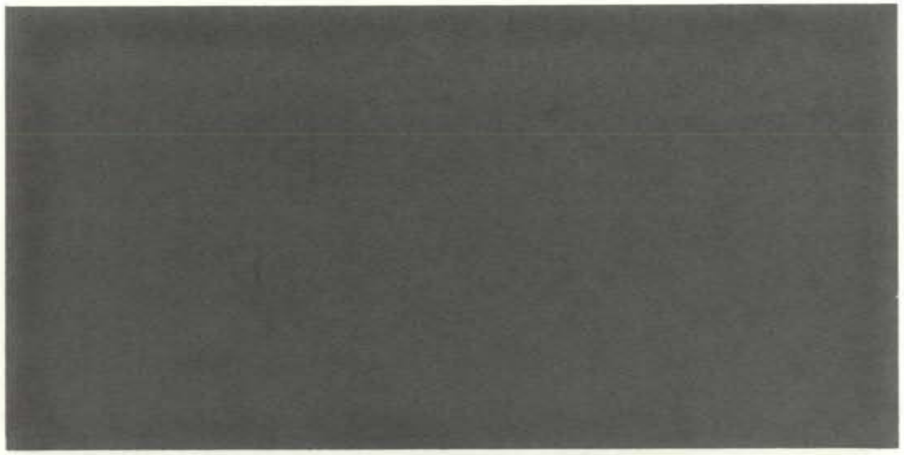
2.030 Composite Functions and the Chain Rule 39 min.

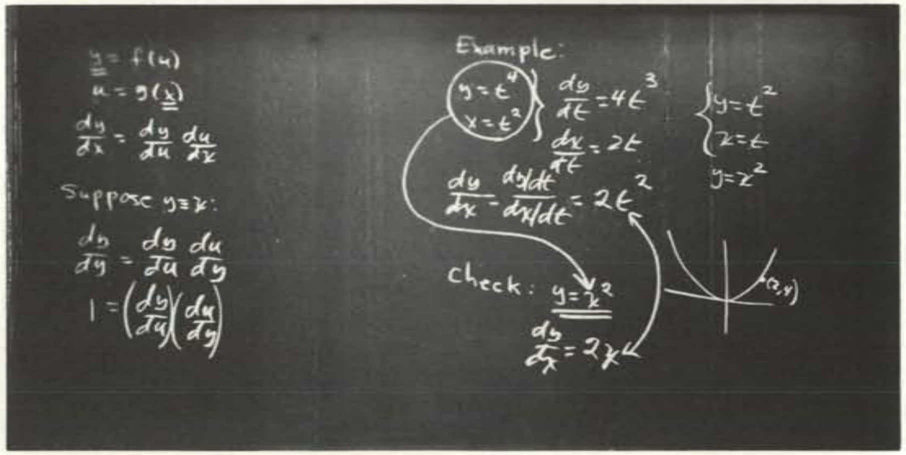




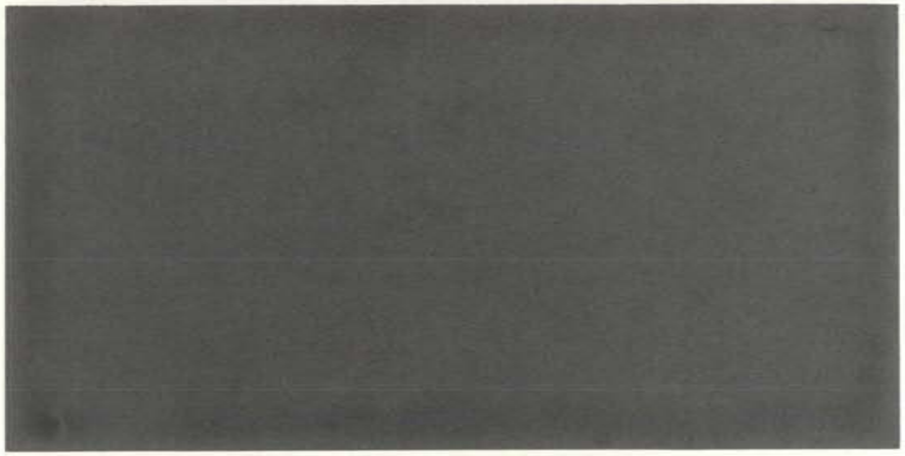


$\Delta x = g'(t)\Delta t + k_2 \Delta t$
$\lim_{\Delta t \to 0} k_1 = \lim_{\Delta t \to 0} k_2 = 0$
$\frac{[f'(t)+k_1]\Delta t}{[g'(t)+k_2]\Delta t}$
$= \frac{dy}{dx} = \frac{f'(t)}{g'(t)}$
$\frac{d^2y}{dx^2} \neq \frac{d^2y/dt^2}{d^2x/dt^2}$
$y = f(t) \to d^2y/dt^2$
$\boxed{\frac{dx}{dt}}\ g'(t) \leftarrow x = g(t) \to d^2x/dt^2$
$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{f'(t)}{g'(t)}\right)$
$= \frac{d}{dt}\left[\frac{f'(t)}{g'(t)}\right]\frac{dt}{dx}$
$= \left[\frac{g'(t)f''(t) - f'(t)g''(t)}{[g'(t)]^2}\right]\frac{dt}{dx}$

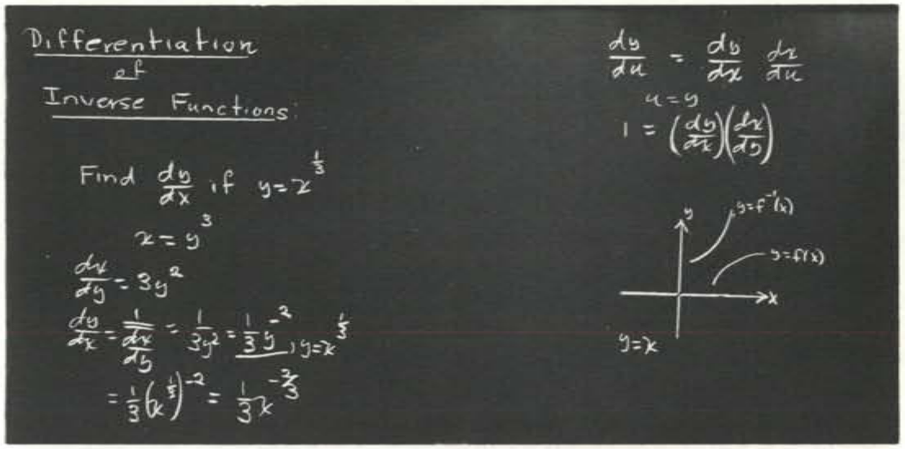


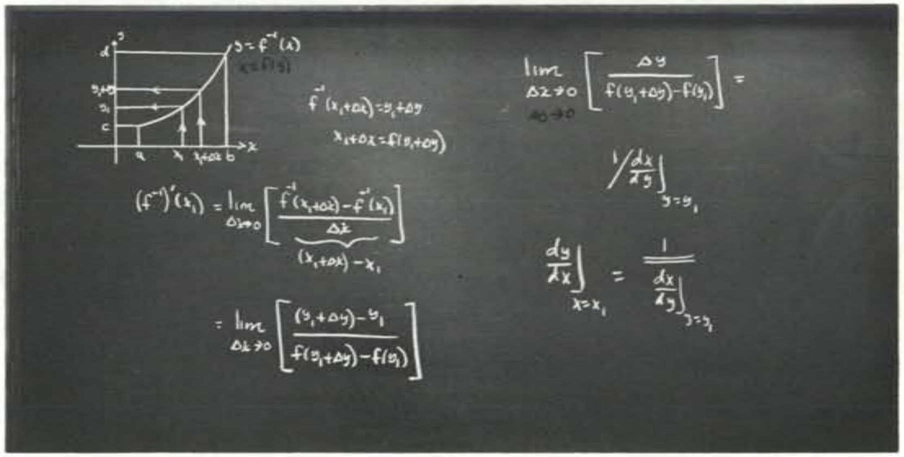


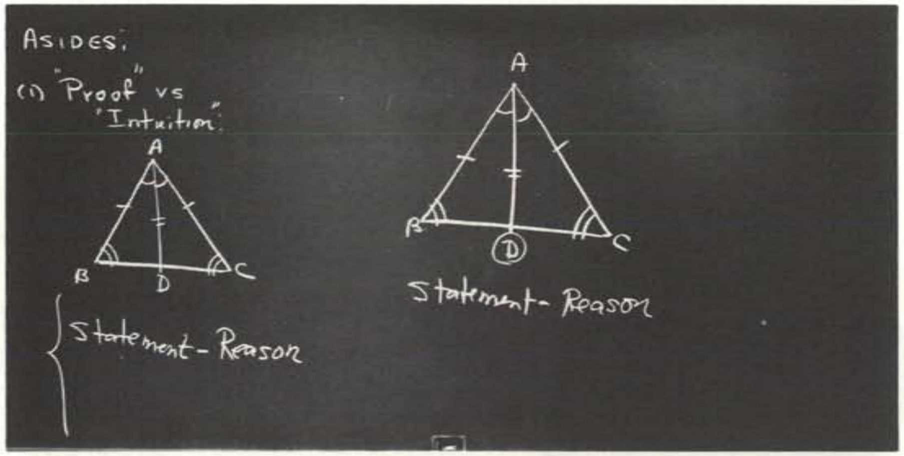
$y = f(u)$
$u = g(x)$
$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
Suppose $y \equiv x$:
$\frac{dy}{dy} = \frac{dy}{du}\frac{du}{dy}$
$1 = \left(\frac{dy}{du}\right)\left(\frac{du}{dy}\right)$
Example:
$y = t^4$
$x = t^2$
$\frac{dy}{dt} = 4t^3$
$\frac{dx}{dt} = 2t$
$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = 2t^2$
Check: $y = x^2$
$\frac{dy}{dx} = 2x$
$y = t^2$
$x = t$
$y = x^2$
$(2,4)$

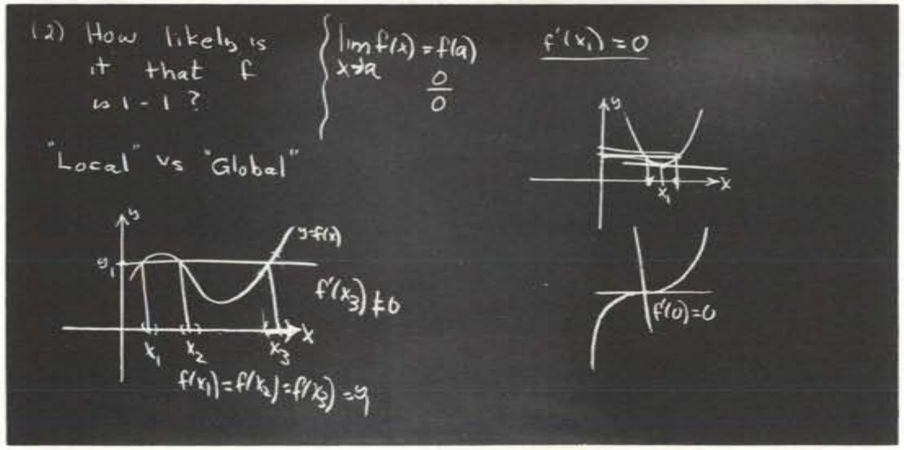


2.040 Differentiation of Inverse Functions 28 min.

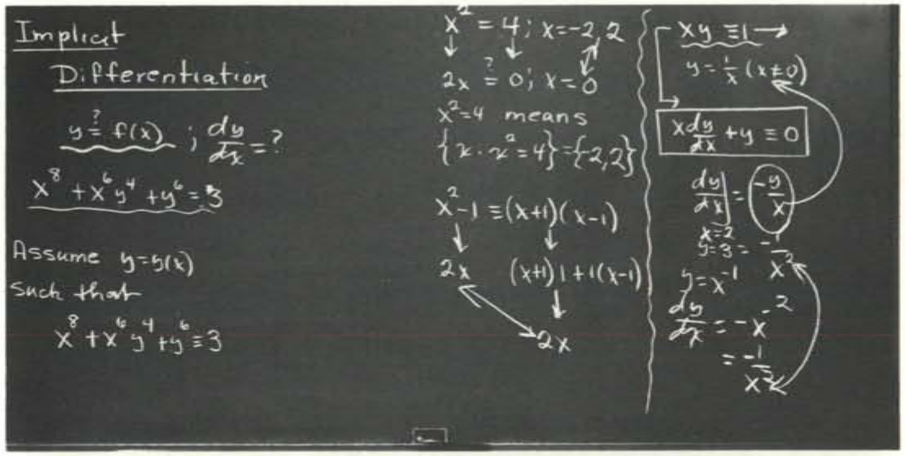


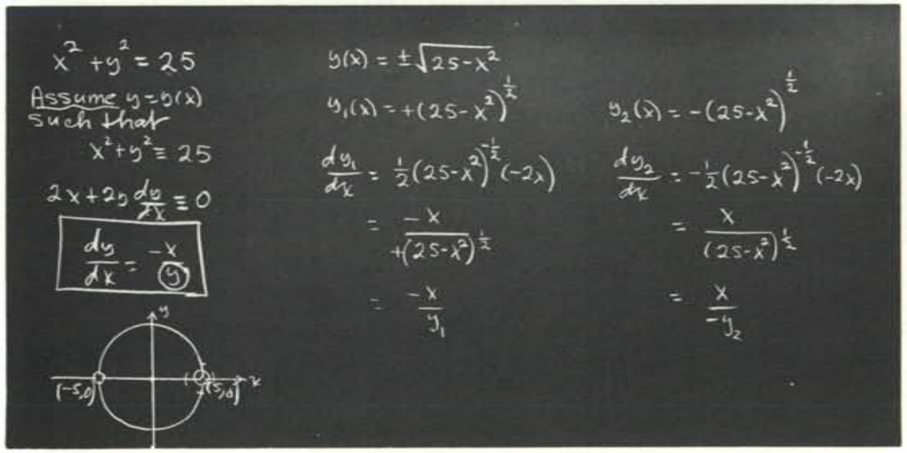



ASIDES:
(1) "Proof" vs
"Intuition"
A
B
D
C
Statement - Reason
A
B
D
C
Statement - Reason


(2) How likely is
it that f
is 1-1 ?
lim f(x) = f(a)
x→a
0/0
f'(x₁) = 0
"Local" vs "Global"
y
y₁
y=f(x)
f'(x₃) ≠ 0
x₁ x₂ x₃ x
f(x₁)=f(x₂)=f(x₃)=y
y
x₁
x
f'(0)=0

2.045 **Implicit Differentiation** 40 min.





ASIDE:

$y = x^{\frac{p}{q}}$, p, q integers

$y^q \equiv x^p$

$q y^{q-1} \frac{dy}{dx} = p x^{p-1}$

$\frac{dy}{dx} = \frac{p x^{p-1}}{q y^{q-1}}$

$= \frac{p x^{p-1}}{q (x^{p/q})^{q-1}}$

$= \frac{p}{q} x^{p-1-p+\frac{p}{q}} = \frac{p}{q} x^{\frac{p}{q}-1}$

Find the equation of the line tangent to $x^8 + x^6 y^4 + y^6 \equiv 3$ at the point $(1,1)$

$8x^7 + 6x^5 y^4 + 4x^6 y^3 \frac{dy}{dx} + 6y^5 \frac{dy}{dx} \equiv 0$

$\left.\frac{dy}{dx}\right|_{\substack{x=1\\y=1}} = \left.\frac{-(8x^7 + 6x^5 y^4)}{4x^6 y^3 + 6y^5}\right|_{\substack{x=1\\y=1}}$

$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{-14}{10} = -\frac{7}{5}$

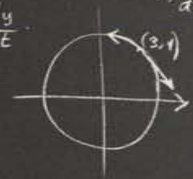
$\frac{y-1}{x-1} = -\frac{7}{5}$

$7x + 5y = 12$

Related Rates

A particle moves along the curve $x^2 + y^2 = 25$ (x, y in feet). At $(3,4)$, $\frac{dx}{dt} = 8$ ft/sec

Find $\frac{dy}{dt}$.



$-\frac{3}{4}(8)$

-6 ft/sec

Assume "near" $(3,4)$ x and y are differentiable functions of t. Then:

$x^2 + y^2 \equiv 25 \rightarrow$

$2x \frac{dx}{dt} + 2y \frac{dy}{dt} \equiv 0$

$\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}$

$\frac{dy}{dx} = 0 \longleftrightarrow x^5(8x^2 + 6y^4) = 0$

$\longleftrightarrow x = 0$

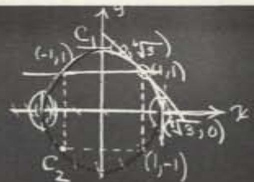
$\frac{dx}{dy} = 0 \longleftrightarrow y^3(4x^6 + 6y^2) = 0$

$\longleftrightarrow y = 0$

$x = 0 \longrightarrow y = \pm\sqrt[6]{3}$

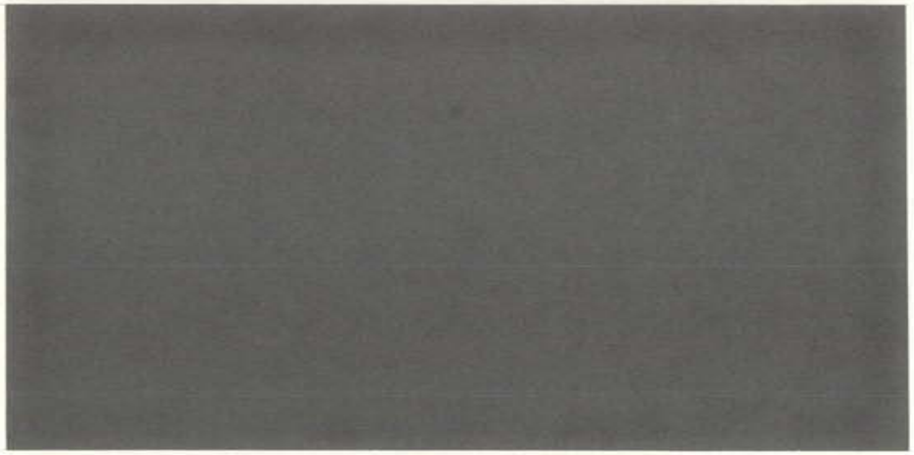
$y = 0 \longrightarrow x = \pm\sqrt[8]{3}$

$(x^8 + x^6 y^4 + y^6 \equiv 3)$

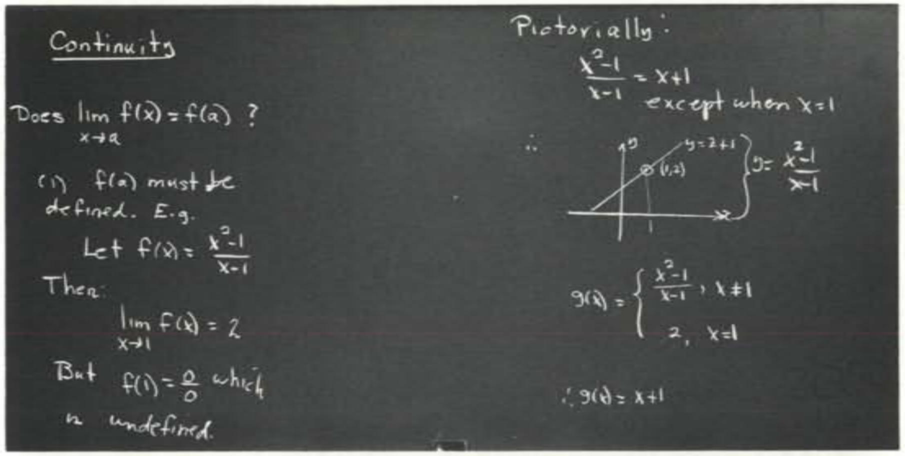


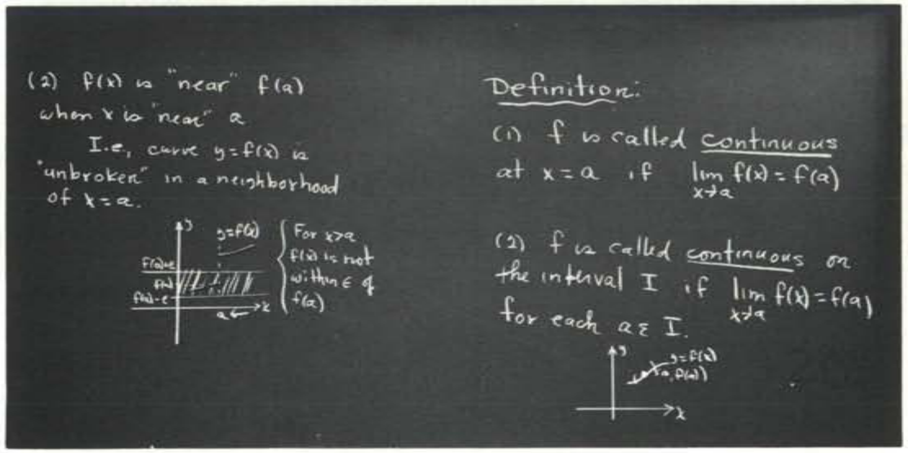
C_1: $x^8 + x^6 y^4 + y^6 = 3$ $y \geq 0$

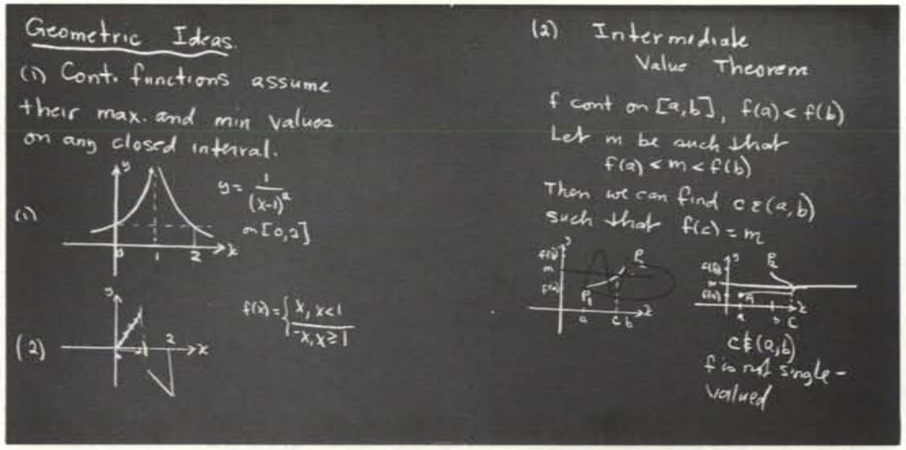
C_2: $x^8 + x^6 y^4 + y^6 = 3$ $y < 0$

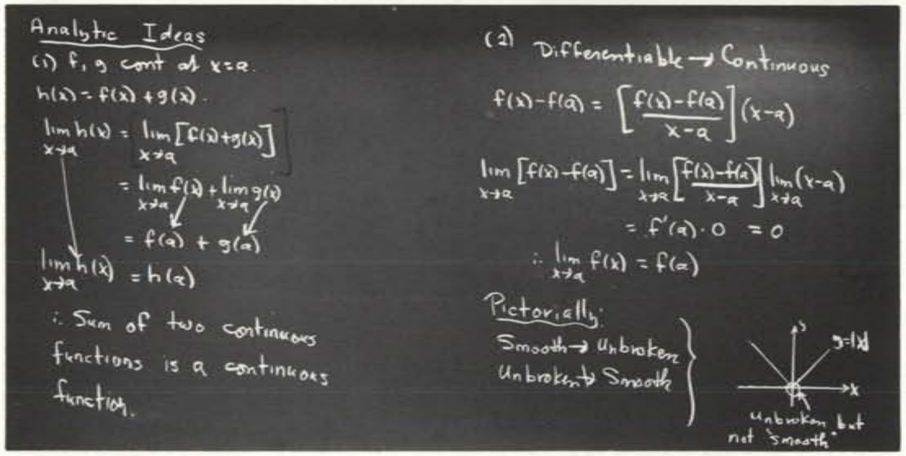


2.050 Continuity **22 min.**

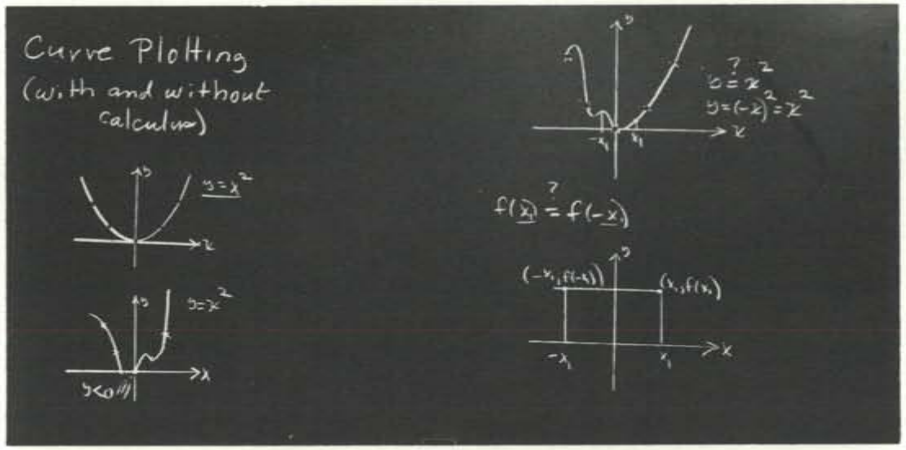


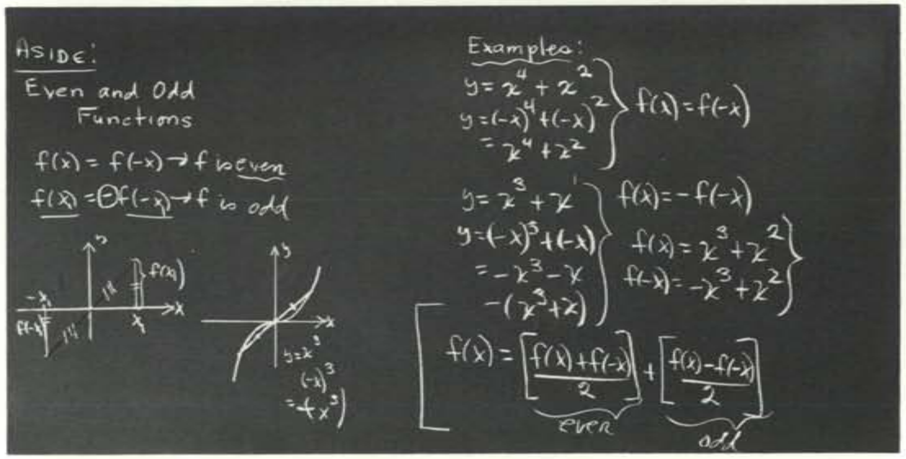


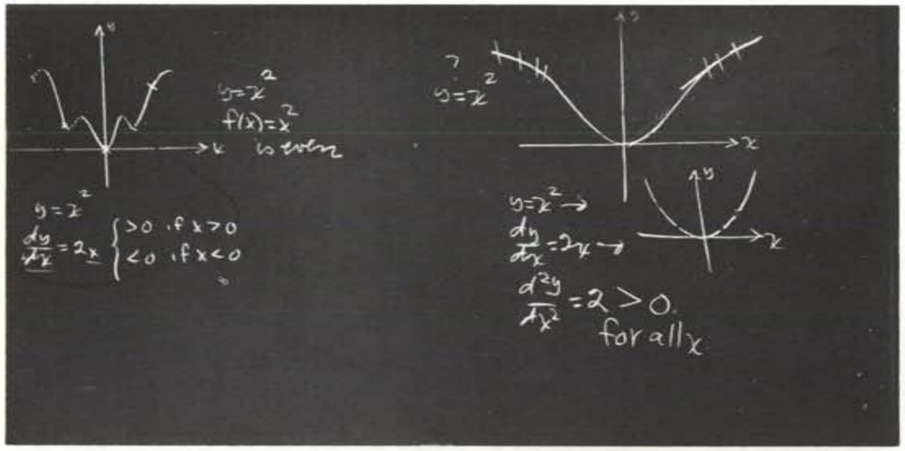

Geometric Ideas.
(1) Cont. functions assume their max. and min values on any closed interval.
(1)
y = 1/(x-1)^2 on [0,2]
(2)
f(x) = { x, x<1 ; -x, x≥1
(2) Intermediate Value Theorem
f cont on [a,b], f(a) < f(b)
Let m be such that f(a) < m < f(b)
Then we can find c ε (a,b) such that f(c) = m
c ∉ (a,b)
f is not single-valued

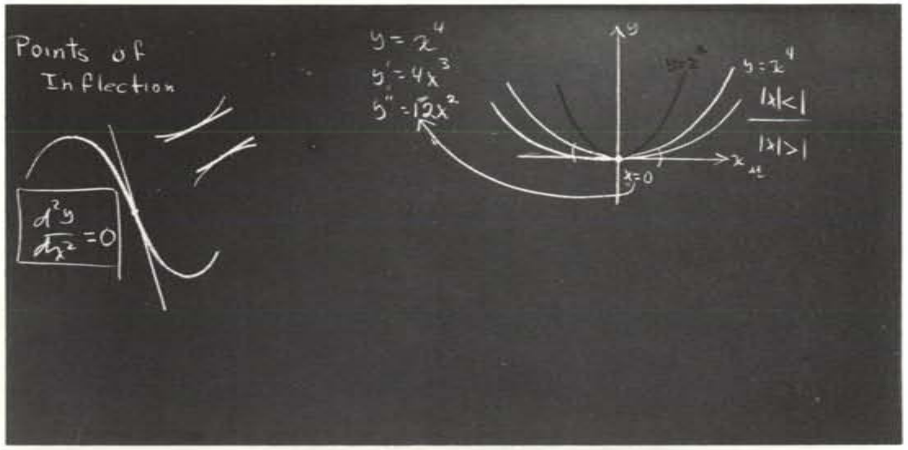

Analytic Ideas
(1) f, g cont at x=a.
h(x) = f(x) + g(x).
lim_{x→a} h(x) = lim_{x→a} [f(x)+g(x)]
= lim_{x→a} f(x) + lim_{x→a} g(x)
= f(a) + g(a)
lim_{x→a} h(x) = h(a)
∴ Sum of two continuous functions is a continuous function.
(2) Differentiable → Continuous
f(x) - f(a) = [(f(x)-f(a))/(x-a)] (x-a)
lim_{x→a} [f(x)-f(a)] = lim_{x→a} [(f(x)-f(a))/(x-a)] lim_{x→a} (x-a)
= f'(a)·0 = 0
∴ lim_{x→a} f(x) = f(a)
Pictorially:
Smooth → Unbroken
Unbroken ↛ Smooth
y=|x|
Unbroken but not "smooth"

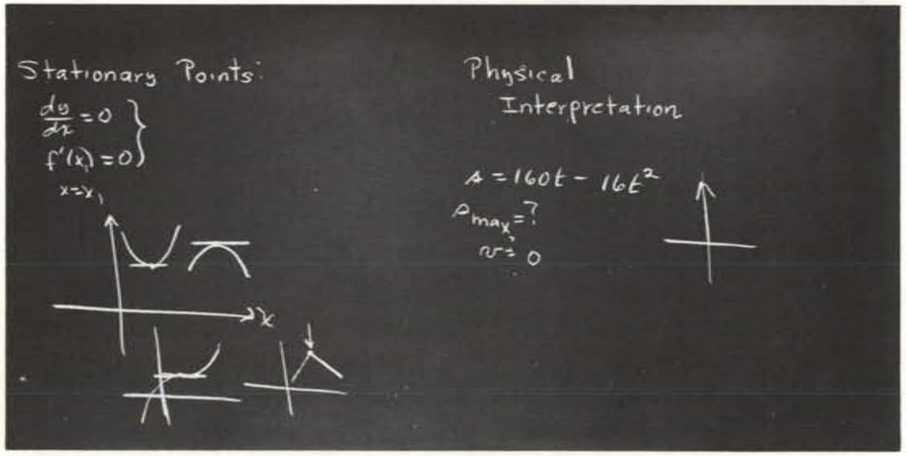
2.060 Curve Plotting **31 min.**

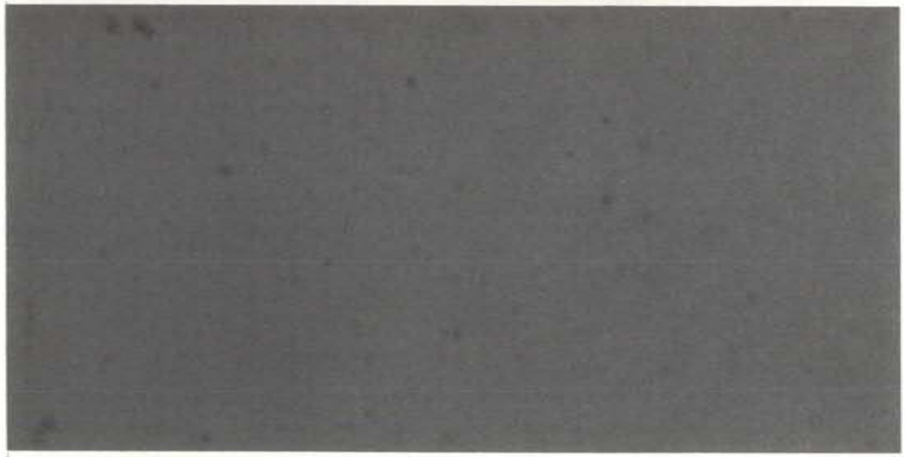




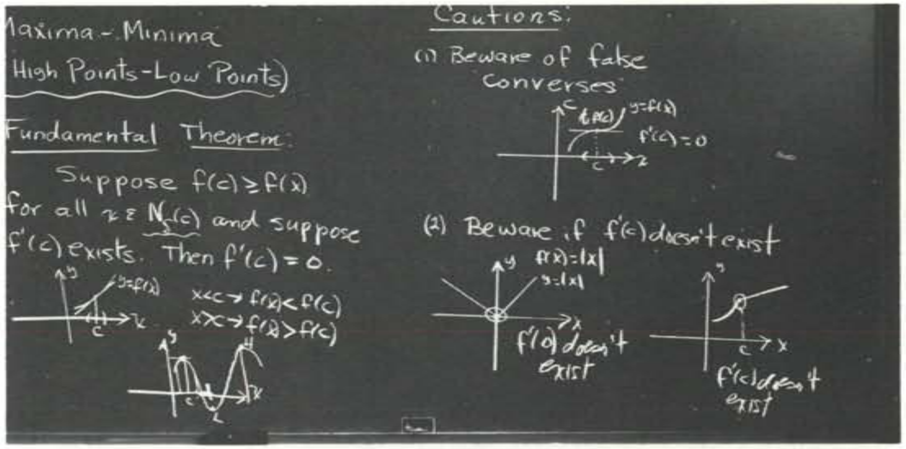

y=x²
f(x)=x²
is even
dy/dx = 2x
>0 if x>0
<0 if x<0
d²y/dx² = 2 > 0
for all x

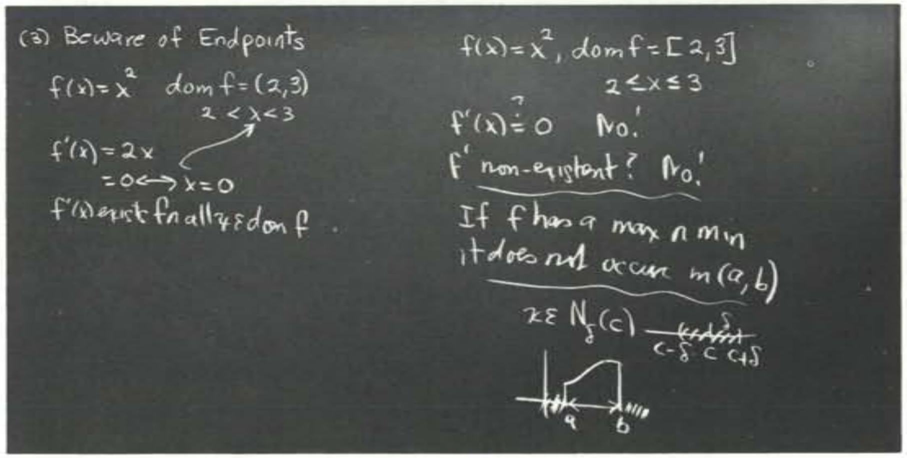

Points of
Inflection
y = x⁴
y' = 4x³
y'' = 12x²
d²y/dx² = 0
x=0

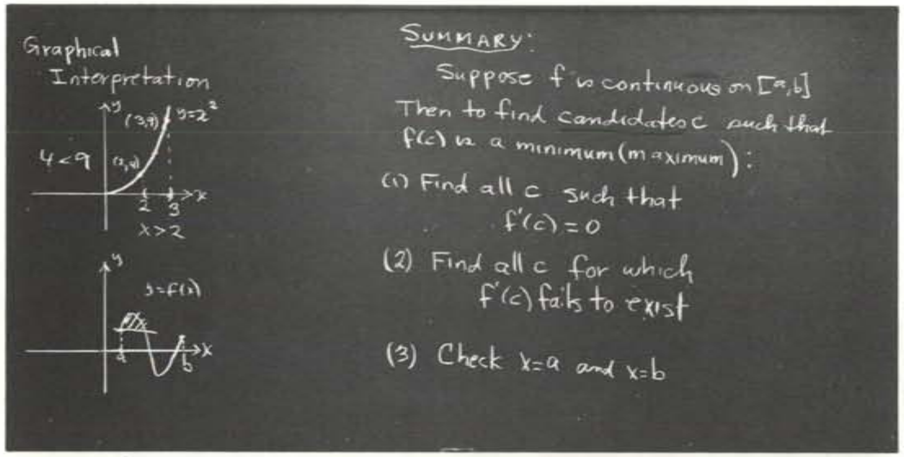

Stationary Points:
dy/dx = 0
f'(x) = 0
Physical
Interpretation
s = 160t − 16t²
s_max = ?
v = 0

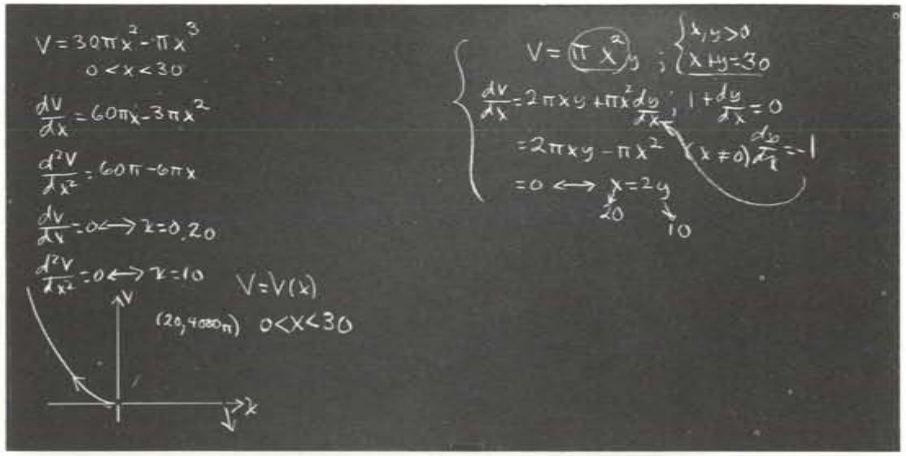


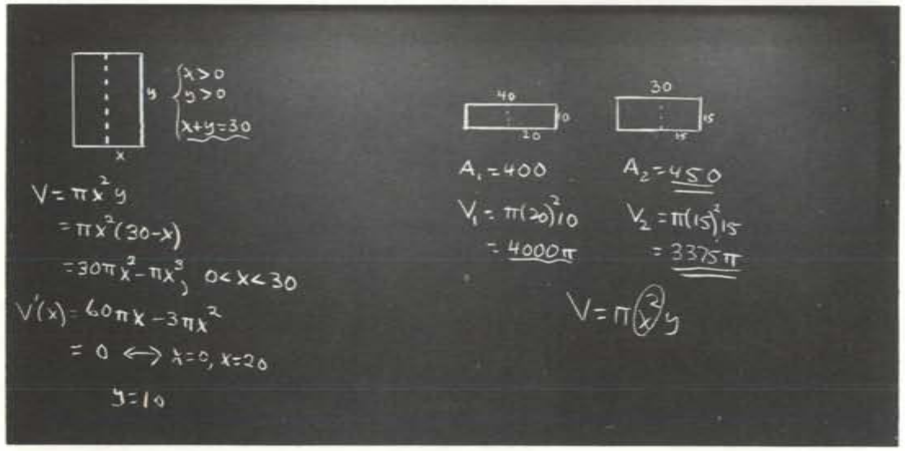
2.070 **Maxima-Minima** **34 min.**

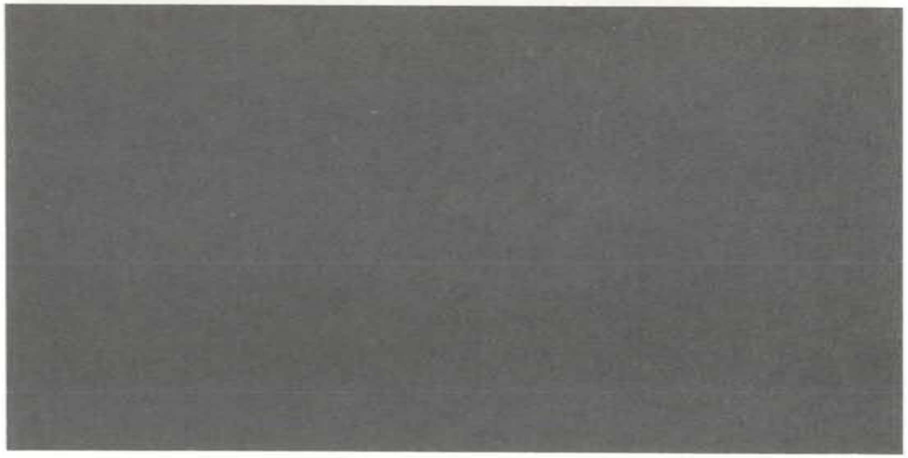




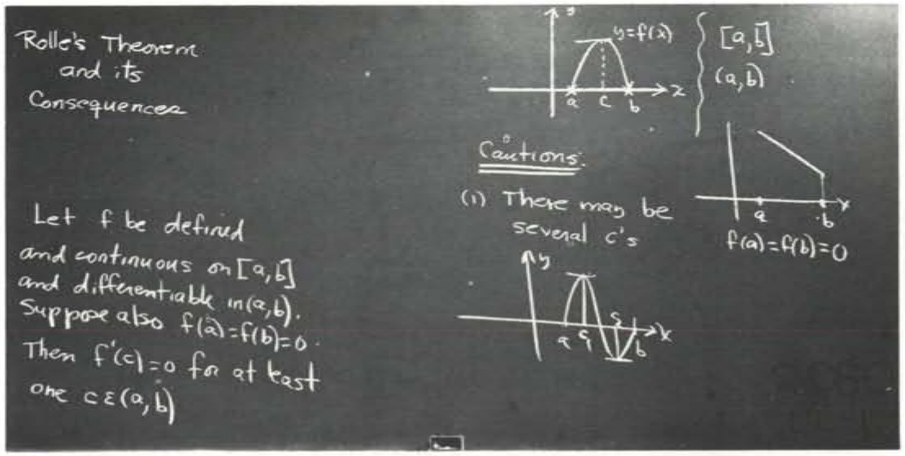

Graphical Interpretation
y=x^2
4<9
x>2
y=f(x)
SUMMARY:
Suppose f is continuous on [a,b]
Then to find candidates c such that f(c) is a minimum (maximum):
(1) Find all c such that f'(c) = 0
(2) Find all c for which f'(c) fails to exist
(3) Check x=a and x=b

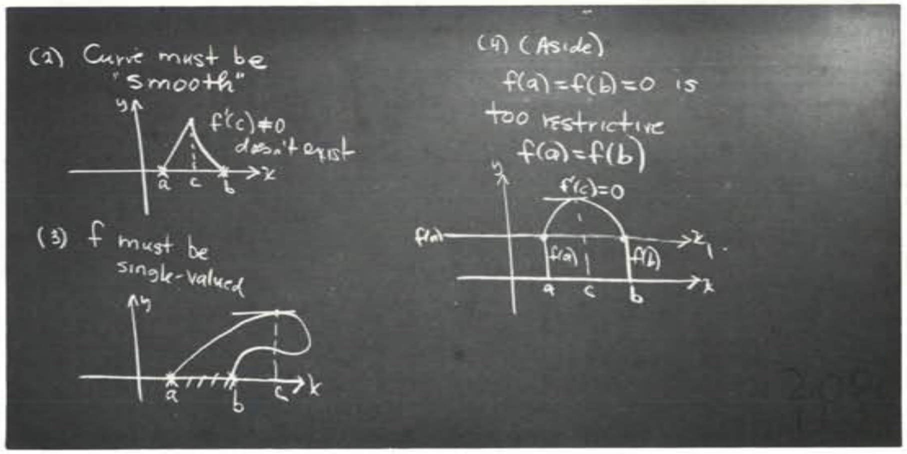

V = 30πx² - πx³
0 < x < 30
dV/dx = 60πx - 3πx²
d²V/dx² = 60π - 6πx
dV/dx = 0 ⟷ x = 0, 20
d²V/dx² = 0 ⟷ x = 10
V = V(x)
(20, 4000π) 0<x<30
V = πx²y ; x, y > 0 ; x+y=30
dV/dx = 2πxy + πx² dy/dx ; 1 + dy/dx = 0
= 2πxy - πx² (x ≠ 0) dy/dx = -1
= 0 ⟷ x = 2y
20 10

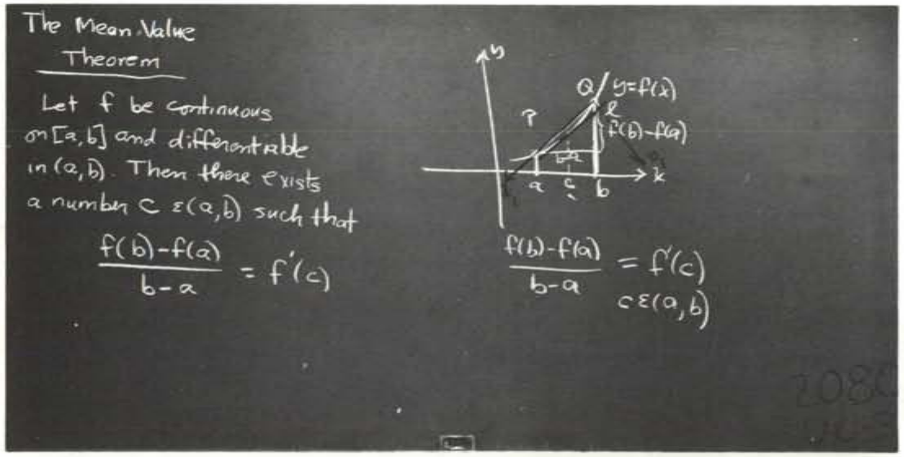

x>0
y>0
x+y=30
V = πx²y
= πx²(30-x)
= 30πx² - πx³, 0<x<30
V'(x) = 60πx - 3πx²
= 0 ⟷ x=0, x=20
y=10
A₁ = 400
A₂ = 450
V₁ = π(20)²10
= 4000π
V₂ = π(15)²15
= 3375π
V = πx²y

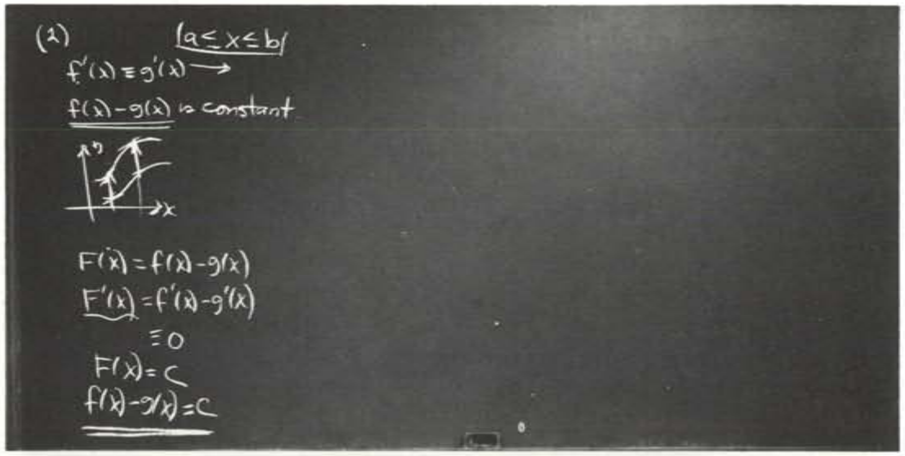


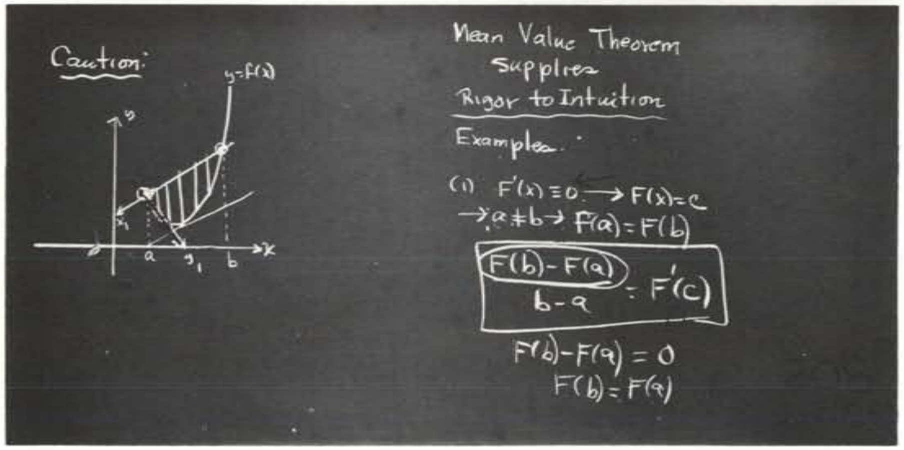
2.080 **Rolle's Theorem and its Consequences** **30 min.**

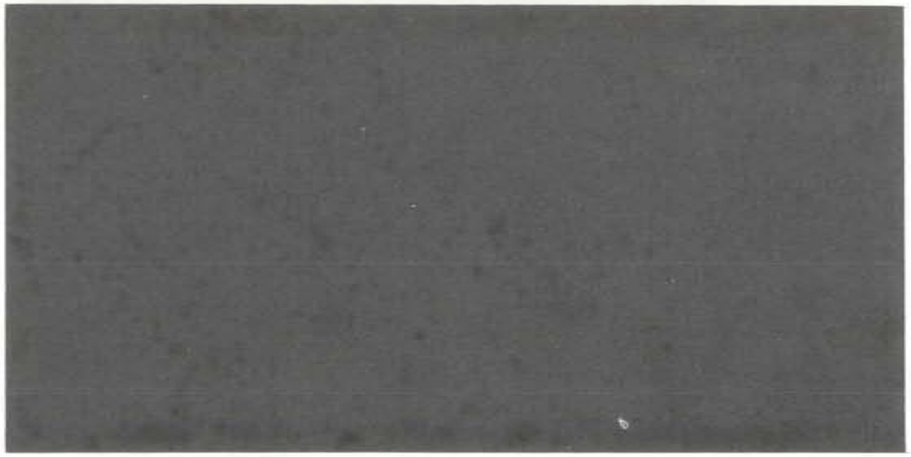




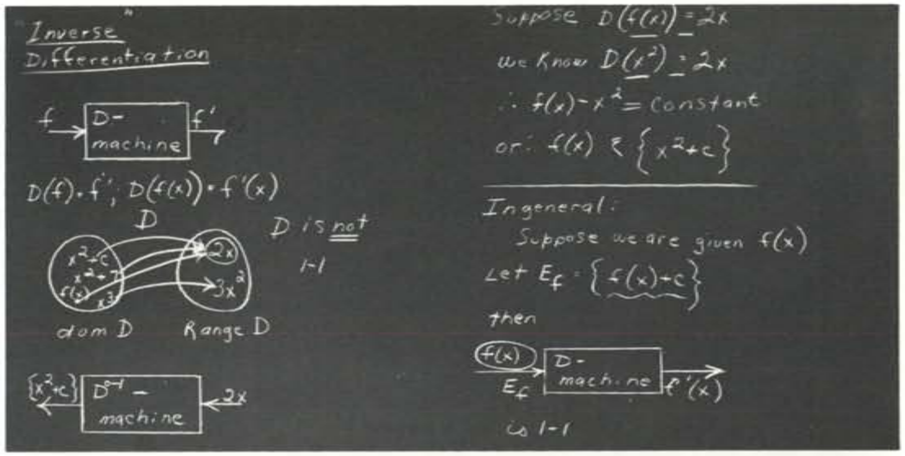

The Mean Value Theorem
Let f be continuous on [a,b] and differentiable in (a,b). Then there exists a number c ε(a,b) such that
f(b)-f(a) / b-a = f'(c)
y
Q
y=f(x)
P
ℓ
f(b)-f(a)
b-a
a
c
b
x
f(b)-f(a) / b-a = f'(c)
c ε(a,b)

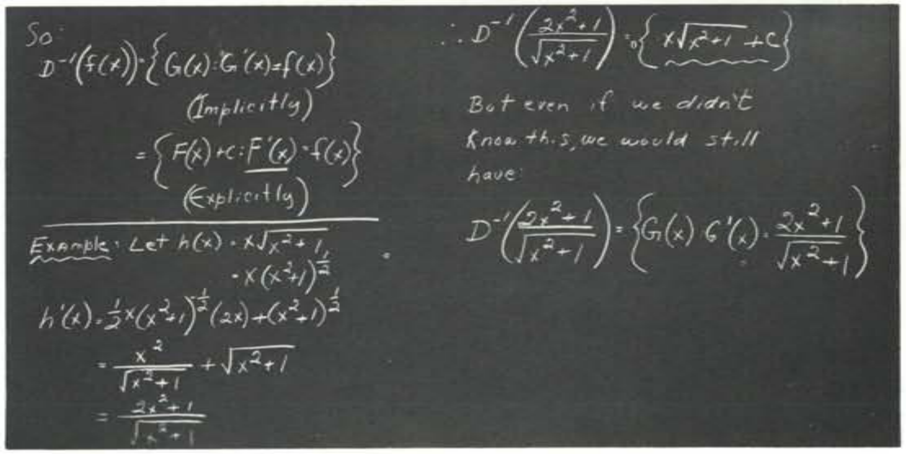

(2)
a ≤ x ≤ b
f'(x) ≡ g'(x) →
f(x)-g(x) is constant
y
x
F(x) = f(x)-g(x)
F'(x) = f'(x)-g'(x)
≡ 0
F(x) = C
f(x)-g(x) = C

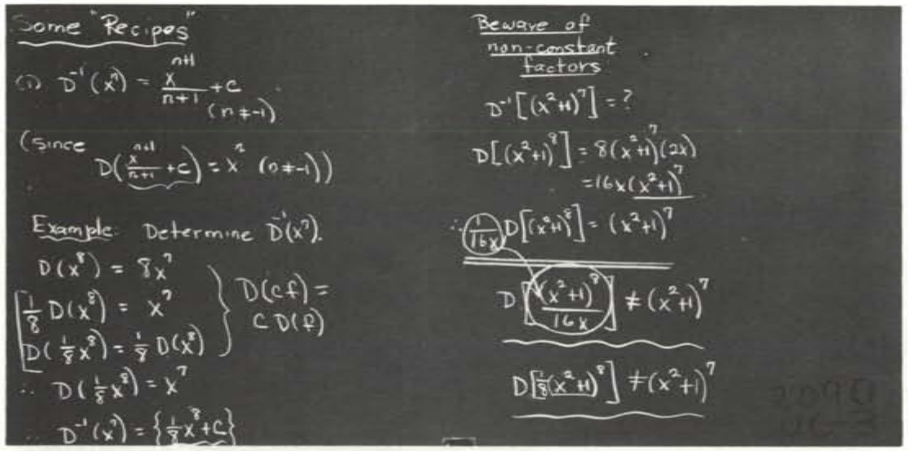

Caution:
y=f(x)
y
a
c1
b
x
Mean Value Theorem
Supplies
Rigor to Intuition
Examples:
(1) F'(x) ≡ 0 → F(x)=c
→ a≠b → F(a)=F(b)
F(b)-F(a) / b-a = F'(c)
F(b)-F(a) = 0
F(b)=F(a)

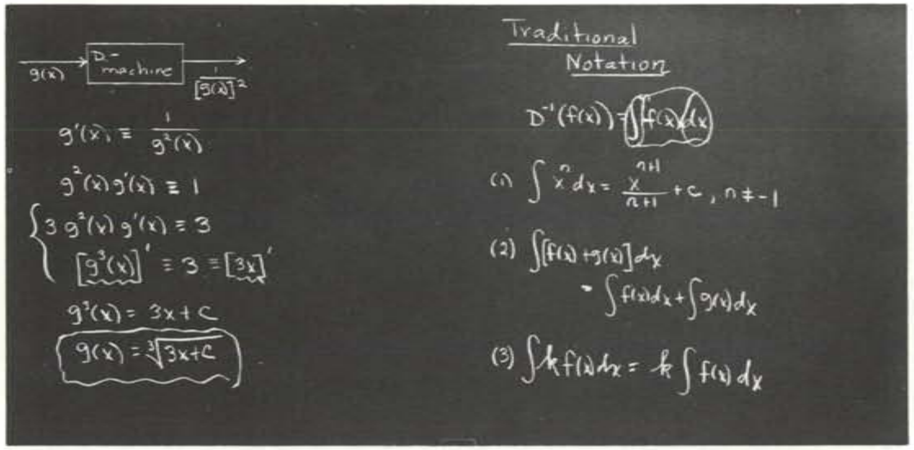


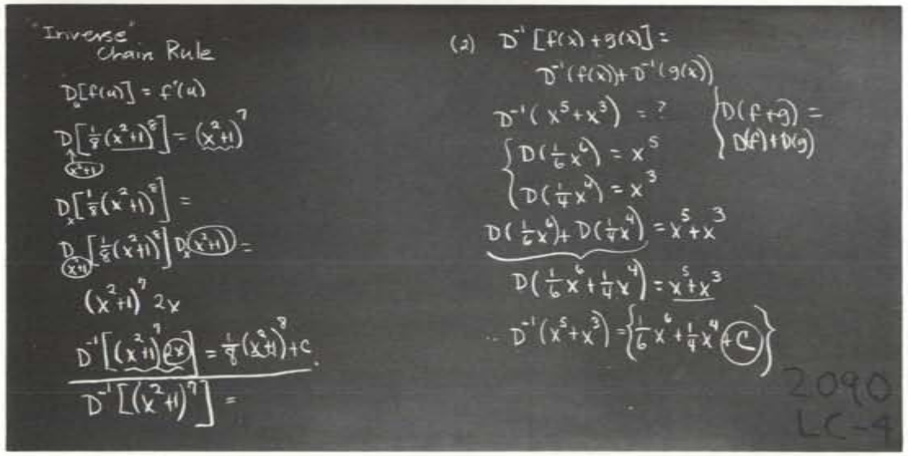
2.090 Inverse Differentiation **42 min.**

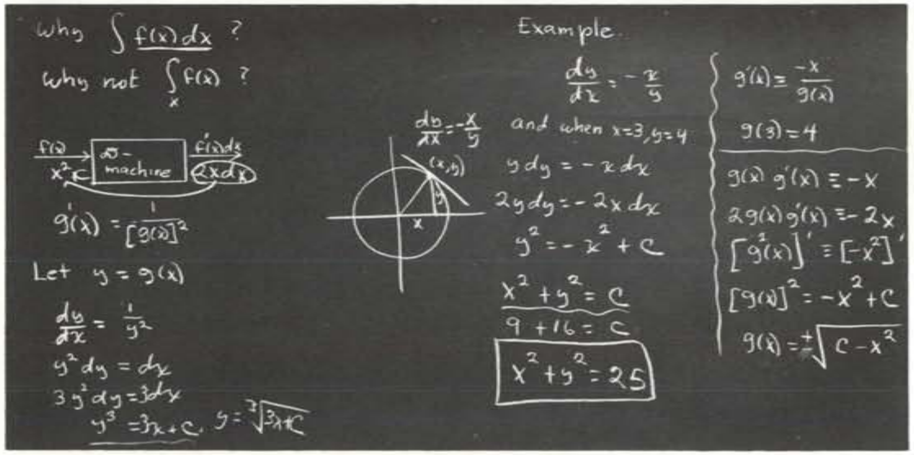




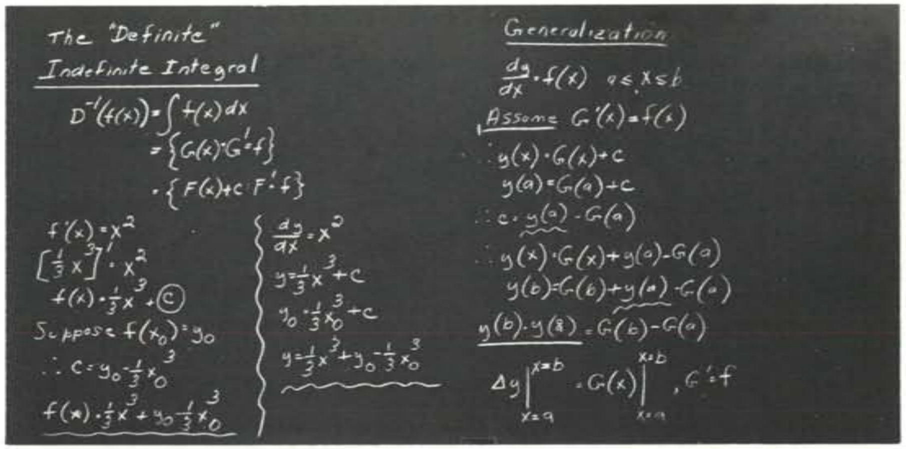

Some "Recipes"
(1) $D^{-1}(x^n) = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
(since $D\left(\frac{x^{n+1}}{n+1} + C\right) = x^n \quad (n \neq -1)$)
Example: Determine $D^{-1}(x^7)$.
$D(x^8) = 8x^7$
$\frac{1}{8} D(x^8) = x^7$
$D(\frac{1}{8}x^8) = \frac{1}{8} D(x^8)$
$D(cf) = c\,D(f)$
$\therefore D(\frac{1}{8}x^8) = x^7$
$\therefore D^{-1}(x^7) = \{\frac{1}{8}x^8 + C\}$
Beware of non-constant factors
$D^{-1}[(x^2+1)^7] = ?$
$D[(x^2+1)^8] = 8(x^2+1)^7(2x) = 16x(x^2+1)^7$
$\frac{1}{16x} D[(x^2+1)^8] = (x^2+1)^7$
$D\left[\frac{(x^2+1)^8}{16x}\right] \neq (x^2+1)^7$
$D[\frac{1}{8}(x^2+1)^8] \neq (x^2+1)^7$

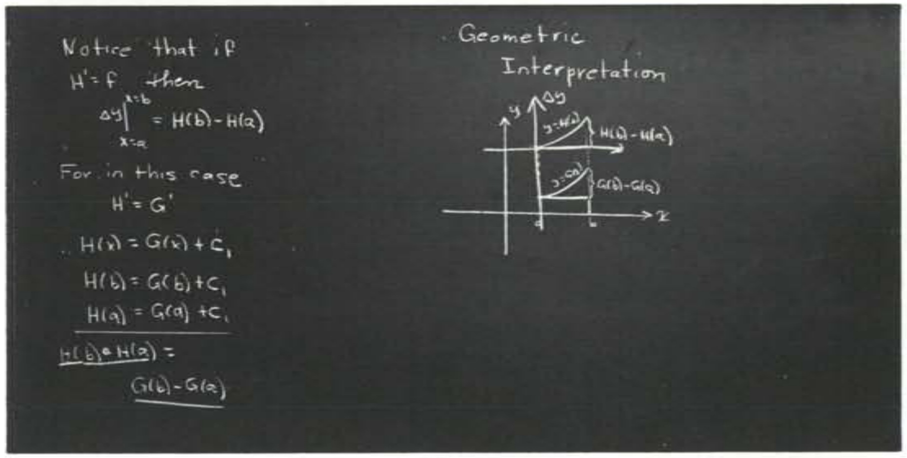

$g(x) \rightarrow$ D-machine $\rightarrow \frac{1}{[g(x)]^2}$
$g'(x) \equiv \frac{1}{g^2(x)}$
$g^2(x) g'(x) \equiv 1$
$3g^2(x) g'(x) \equiv 3$
$[g^3(x)]' \equiv 3 \equiv [3x]'$
$g^3(x) = 3x + C$
$g(x) = \sqrt[3]{3x+C}$
Traditional Notation
$D^{-1}(f(x)) = \int f(x)\,dx$
(1) $\int x^n dx = \frac{x^{n+1}}{n+1} + C, \; n \neq -1$
(2) $\int [f(x) + g(x)]\,dx = \int f(x)dx + \int g(x)dx$
(3) $\int k f(x)\,dx = k \int f(x)\,dx$

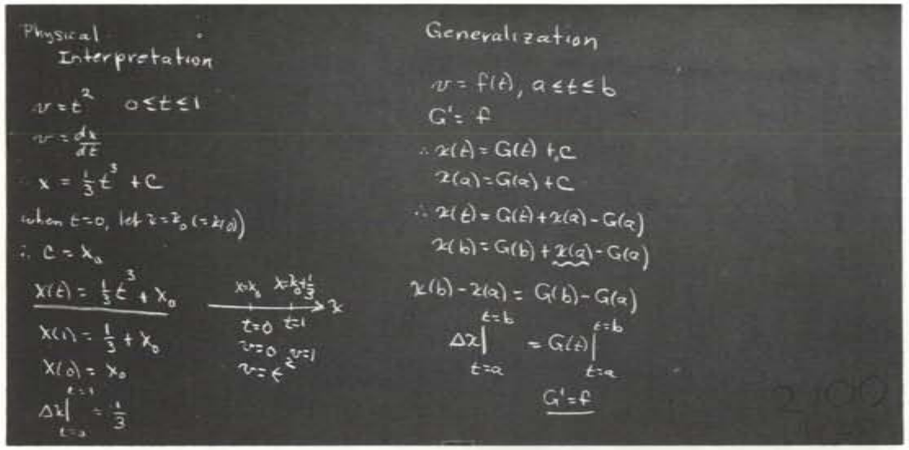

"Inverse" Chain Rule
$D_x[f(u)] = f'(u)$
$D_x[\frac{1}{8}(x^2+1)^8] = (x^2+1)^7$
$D_x[\frac{1}{8}(x^2+1)^8] =$
$D_{x^2+1}[\frac{1}{8}(x^2+1)^8] D_x(x^2+1) =$
$(x^2+1)^7\, 2x$
$D^{-1}[(x^2+1)^7 (2x)] = \frac{1}{8}(x^2+1)^8 + C$
$D^{-1}[(x^2+1)^7] =$
(2) $D^{-1}[f(x) + g(x)] = D^{-1}(f(x)) + D^{-1}(g(x))$
$D^{-1}(x^5 + x^3) = ?$
$D(\frac{1}{6}x^6) = x^5$
$D(\frac{1}{4}x^4) = x^3$
$D(f+g) = D(f) + D(g)$
$D(\frac{1}{6}x^6) + D(\frac{1}{4}x^4) = x^5 + x^3$
$D(\frac{1}{6}x^6 + \frac{1}{4}x^4) = x^5 + x^3$
$\therefore D^{-1}(x^5 + x^3) = \{\frac{1}{6}x^6 + \frac{1}{4}x^4 + C\}$

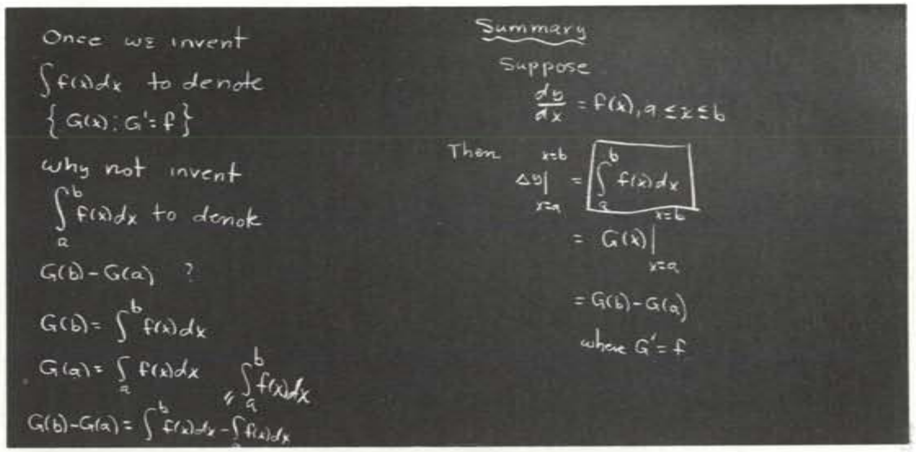

Why $\int f(x)\,dx$?
Why not $\int_x f(x)$?
$f(x)$, x^2 → D-machine → $f(x)dx$, $2x\,dx$
$g'(x) = \frac{1}{[g(x)]^2}$
Let $y = g(x)$
$\frac{dy}{dx} = \frac{1}{y^2}$
$y^2 dy = dx$
$3y^2 dy = 3dx$
$y^3 = 3x + C, \; y = \sqrt[3]{3x+C}$
Example
$\frac{dy}{dx} = -\frac{x}{y}$
and when $x=3, y=4$
$\frac{dy}{dx} = -\frac{x}{y}$
(x,y)
$y\,dy = -x\,dx$
$2y\,dy = -2x\,dx$
$y^2 = -x^2 + C$
$x^2 + y^2 = C$
$9 + 16 = C$
$x^2 + y^2 = 25$
$g'(x) \equiv \frac{-x}{g(x)}$
$g(3) = 4$
$g(x) g'(x) \equiv -x$
$2g(x) g'(x) \equiv -2x$
$[g^2(x)]' \equiv [-x^2]'$
$[g(x)]^2 = -x^2 + C$
$g(x) = \pm\sqrt{C - x^2}$

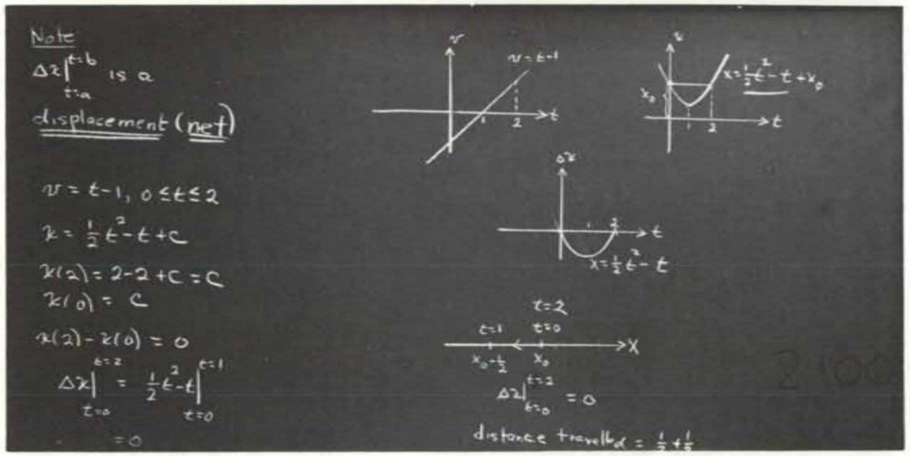
2.100 The "Definite" Indefinite Integral 29 min.

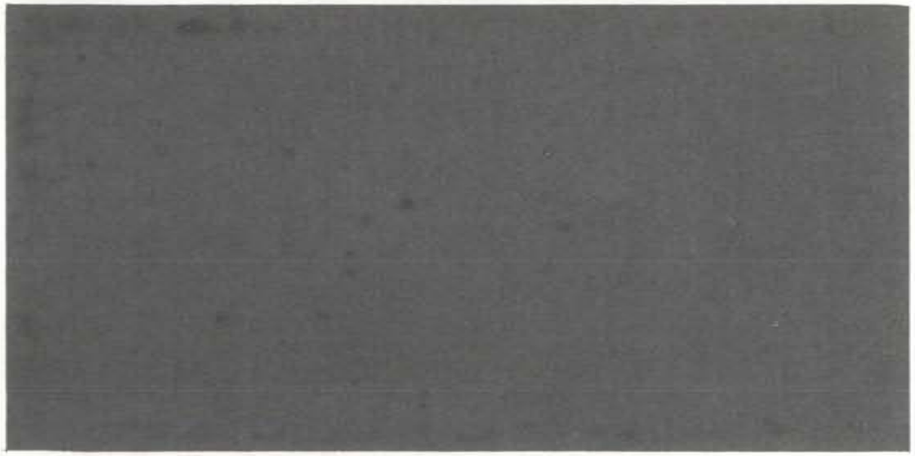





Physical Interpretation
v = t² 0≤t≤1
v = dx/dt
x = ⅓t³ + C
when t=0, let x=x₀ (=x(0))
∴ C = x₀
x(t) = ⅓t³ + x₀
x(1) = ⅓ + x₀
x(0) = x₀
Δx|t=0 to t=1 = ⅓
x=x₀ x=x₀+⅓
t=0 t=1
v=0 v=1
v=t²
Generalization
v = f(t), a≤t≤b
G' = f
∴ x(t) = G(t) + C
x(a) = G(a) + C
∴ x(t) = G(t) + x(a) - G(a)
x(b) = G(b) + x(a) - G(a)
x(b) - x(a) = G(b) - G(a)
Δx|t=a to t=b = G(t)|t=a to t=b
G' = f

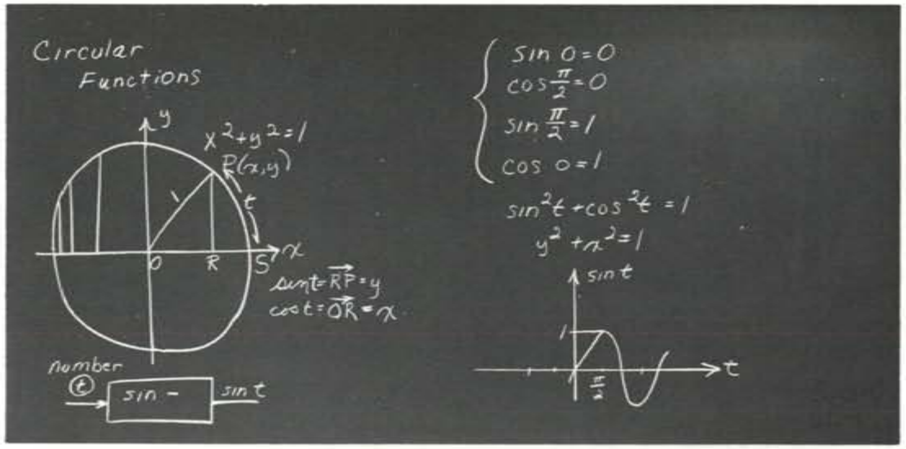

Once we invent
∫f(x)dx to denote
{G(x): G' = f}
why not invent
∫ₐᵇ f(x)dx to denote
G(b) - G(a) ?
G(b) = ∫ᵇ f(x)dx
G(a) = ∫ₐ f(x)dx
G(b) - G(a) = ∫ᵇ f(x)dx - ∫ₐ f(x)dx
≠ ∫ₐᵇ f(x)dx
Summary
Suppose
dy/dx = f(x), a ≤ x ≤ b
Then Δy|x=a to x=b = ∫ₐᵇ f(x)dx
= G(x)|x=a to x=b
= G(b) - G(a)
where G' = f

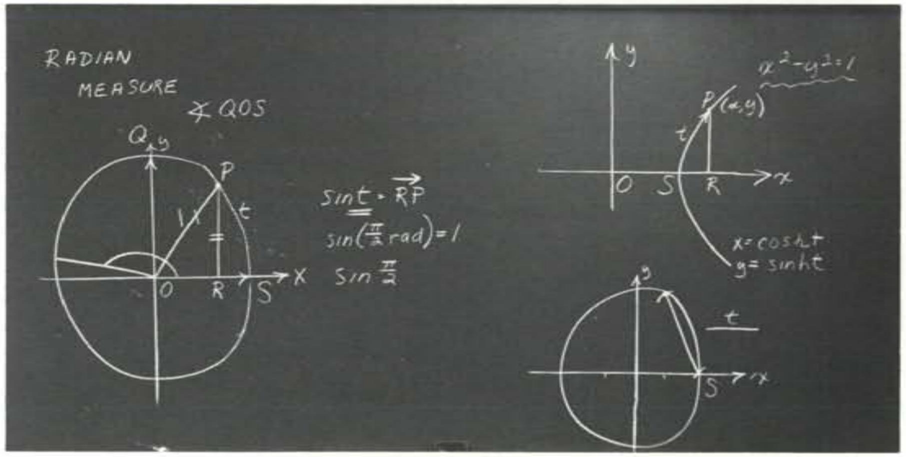

Note
Δx|t=a to t=b is a
displacement (net)
v = t-1, 0≤t≤2
x = ½t² - t + C
x(2) = 2-2+C = C
x(0) = C
x(2) - x(0) = 0
Δx|t=0 to t=2 = ½t² - t|t=0 to t=1
= 0
v
v = t-1
1
2
t
x
x = ½t² - t + x₀
x₀
1
2
t
x
1
2
t
x = ½t² - t
t=2
t=1 t=0
X
x₀-½ x₀
Δx|t=0 to t=2 = 0
distance travelled = ½ + ½

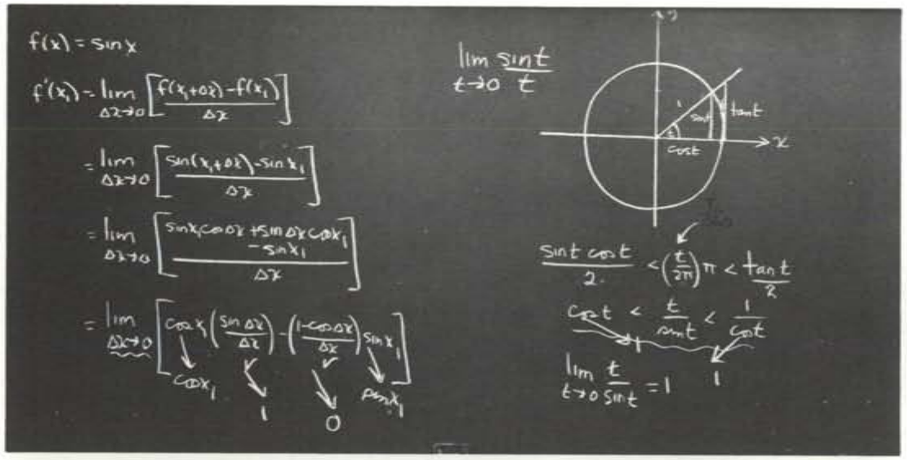


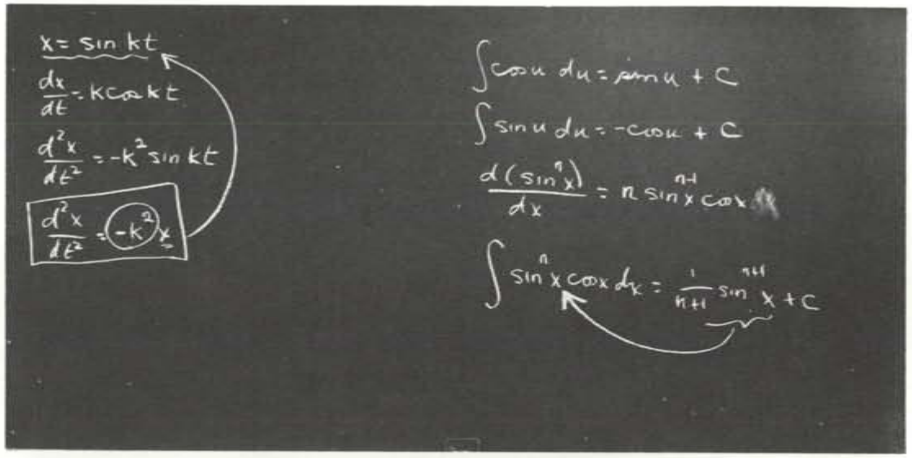
Block III: The Circular Functions

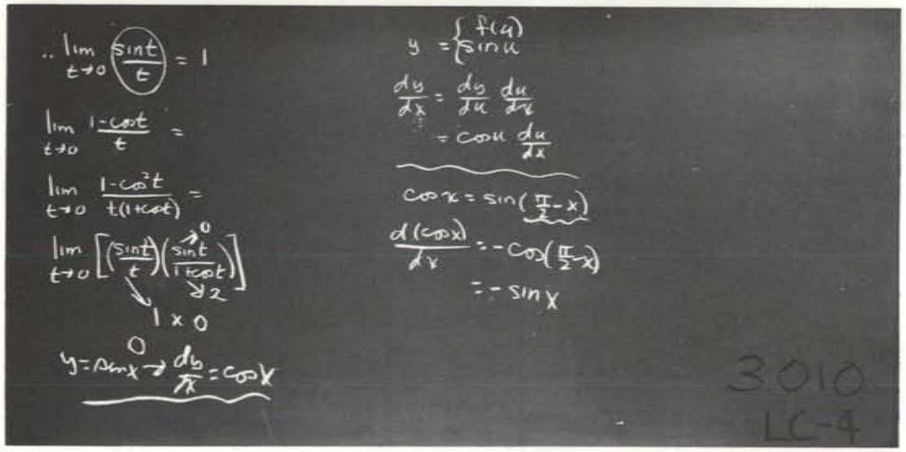
3.010 Circular Functions **35 min.**

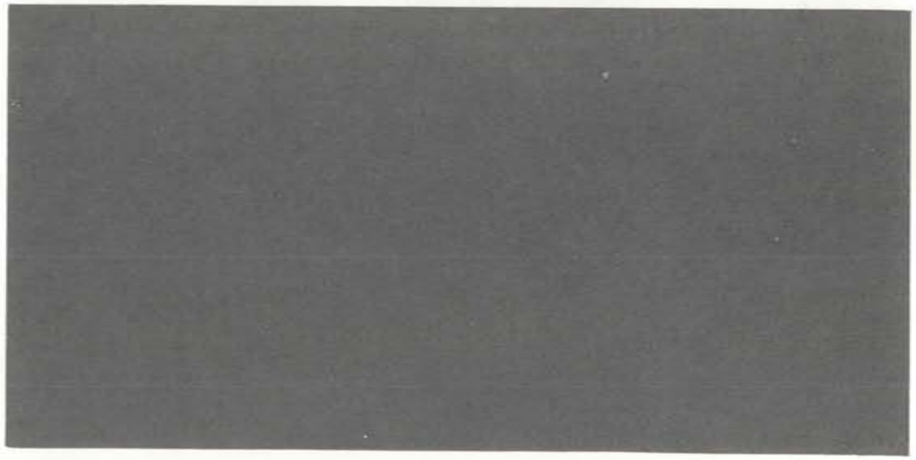




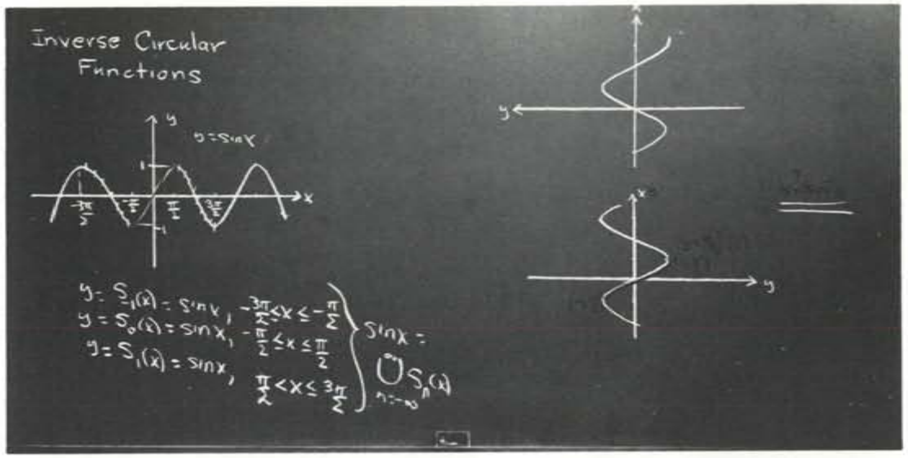

f(x) = sin x
f'(x₁) = lim Δx→0 [(f(x₁+Δx) − f(x₁)) / Δx]
= lim Δx→0 [(sin(x₁+Δx) − sin x₁) / Δx]
= lim Δx→0 [(sin x₁ cos Δx + sin Δx cos x₁ − sin x₁) / Δx]
= lim Δx→0 [cos x₁ (sin Δx / Δx) − ((1 − cos Δx) / Δx) sin x₁]
cos x₁
1
0
sin x₁
lim t→0 sin t / t
sin t
tan t
cos t
x
sin t cos t / 2 < (t / 2π) π < tan t / 2
cos t < t / sin t < 1 / cos t
lim t→0 t / sin t = 1
1
1

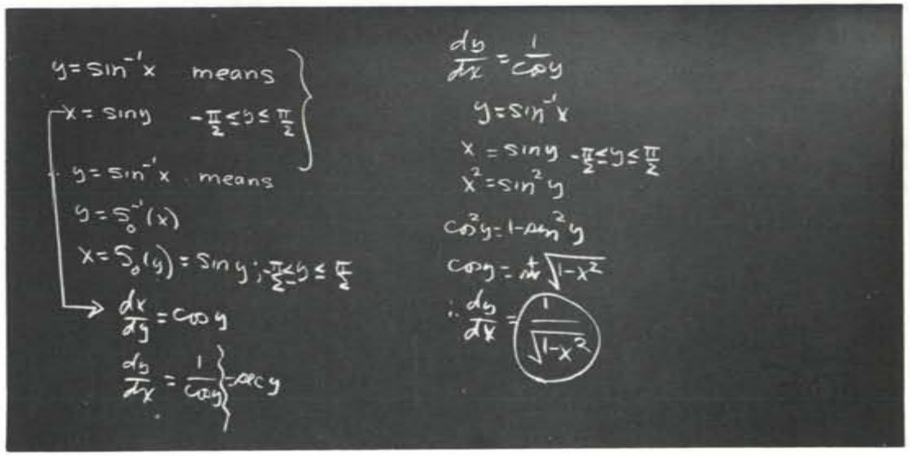

x = sin kt
dx/dt = k cos kt
d²x/dt² = −k² sin kt
d²x/dt² = −k² x
∫ cos u du = sin u + C
∫ sin u du = −cos u + C
d(sinⁿ x)/dx = n sinⁿ⁻¹ x cos x
∫ sinⁿ x cos x dx = 1/(n+1) sinⁿ⁺¹ x + C

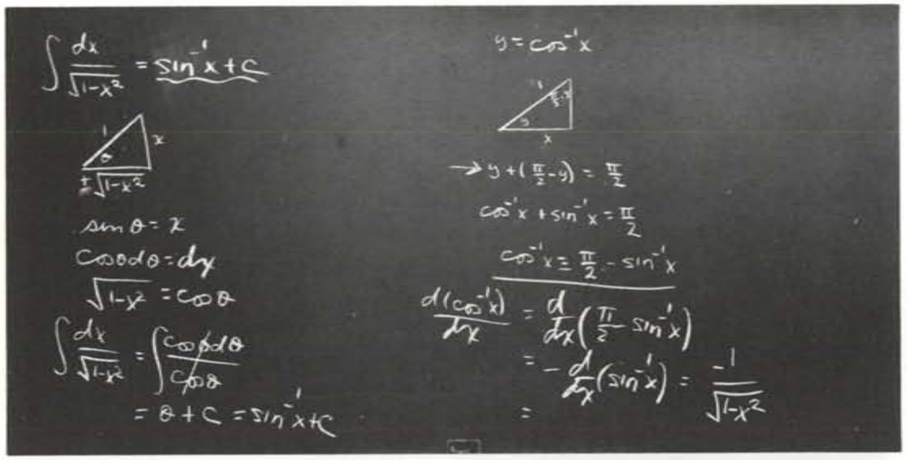

∴ lim t→0 sin t / t = 1
lim t→0 (1 − cos t) / t =
lim t→0 (1 − cos² t) / (t(1 + cos t)) =
lim t→0 [(sin t / t) (sin t / (1 + cos t))]
0
2
1 × 0
0
y = sin x → dy/dx = cos x
y = f(u) = sin u
dy/dx = dy/du · du/dx
= cos u du/dx
cos x = sin(π/2 − x)
d(cos x)/dx = −cos(π/2 − x)
= −sin x
3.010
LC-4



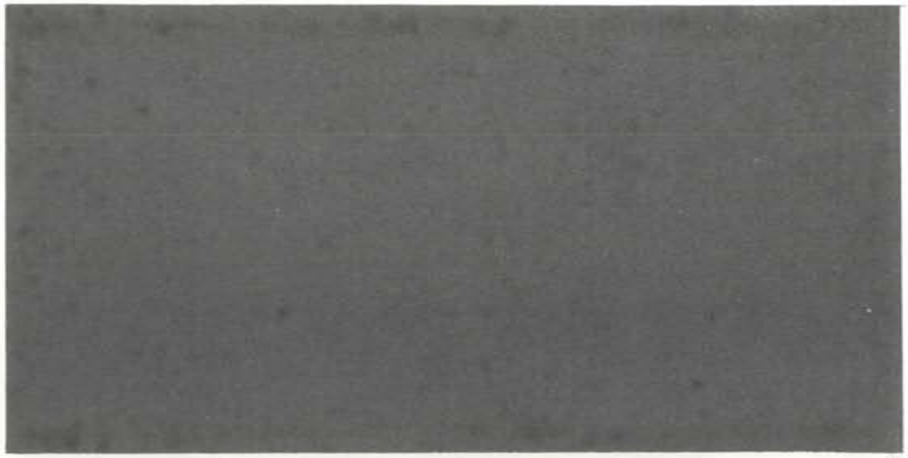
3.020 Inverse Circular Functions **26 min.**

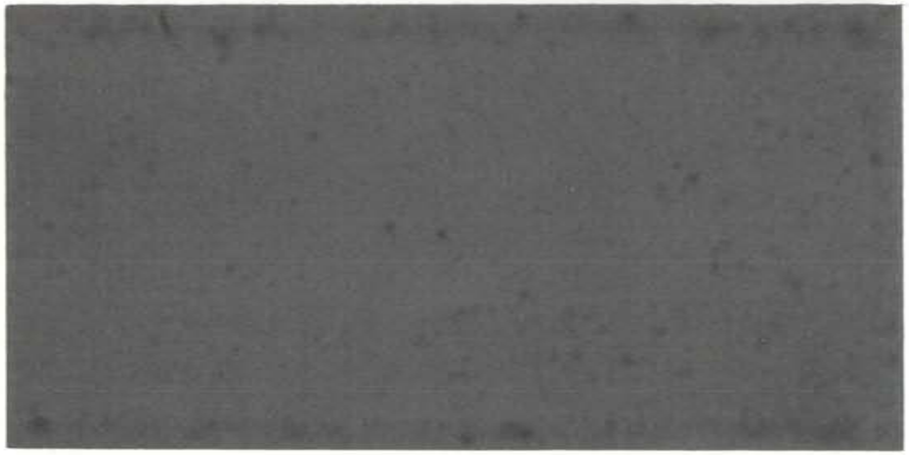


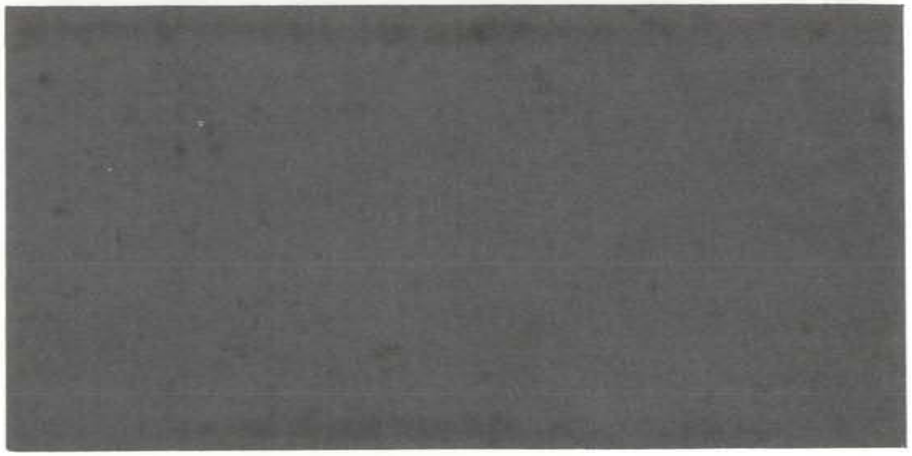




$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C$
$\sin\theta = x$
$\cos\theta d\theta = dx$
$\sqrt{1-x^2} = \cos\theta$
$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos\theta d\theta}{\cos\theta}$
$= \theta + C = \sin^{-1}x + C$
$y = \cos^{-1}x$
$\rightarrow y + (\frac{\pi}{2} - y) = \frac{\pi}{2}$
$\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$
$\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x$
$\frac{d(\cos^{-1}x)}{dx} = \frac{d}{dx}(\frac{\pi}{2} - \sin^{-1}x)$
$= -\frac{d}{dx}(\sin^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$

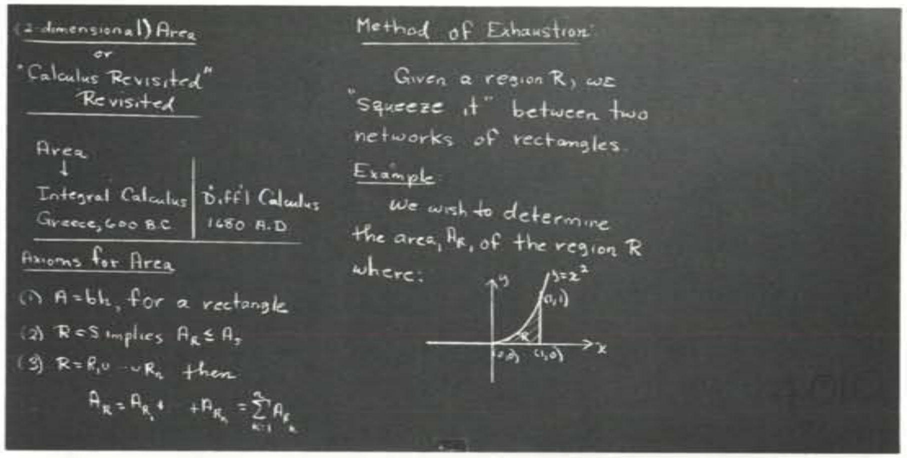


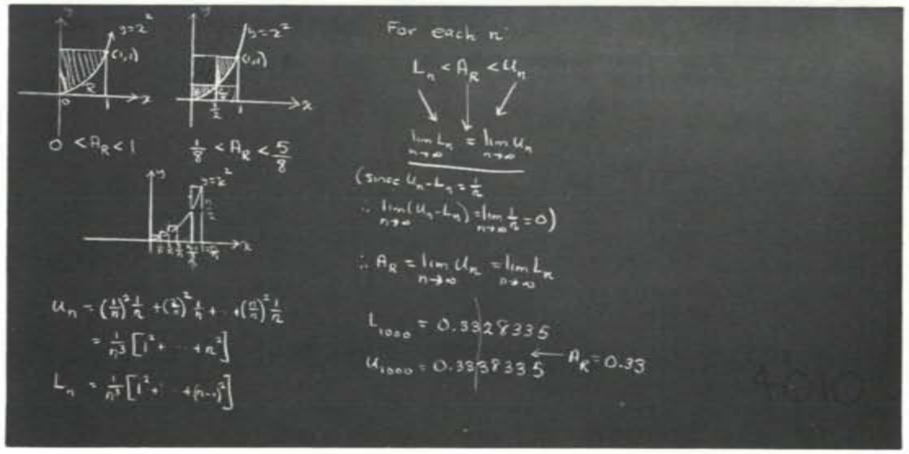


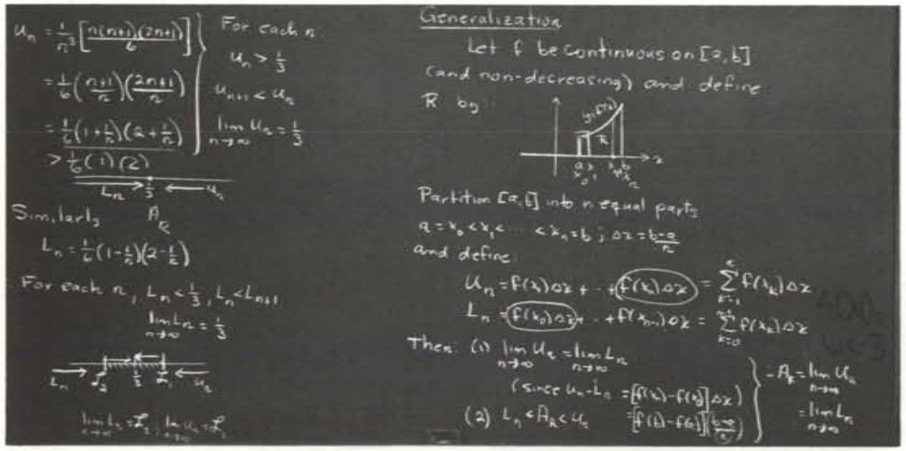


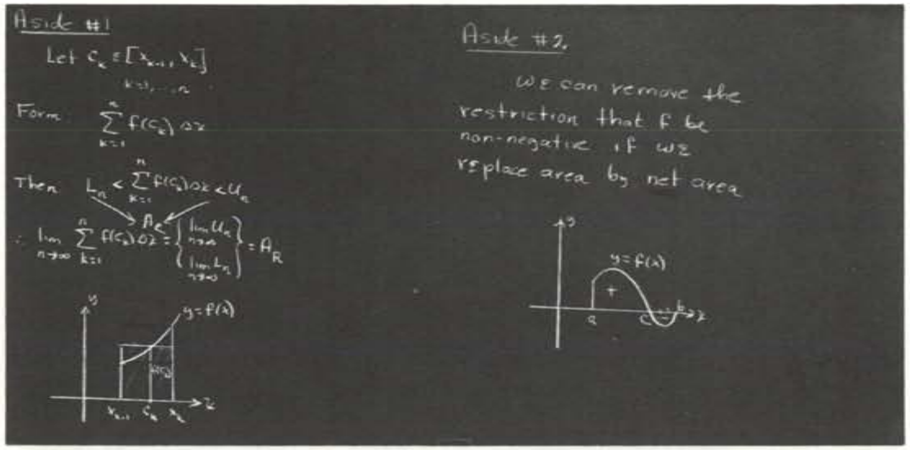
Block IV: The Definite Integral

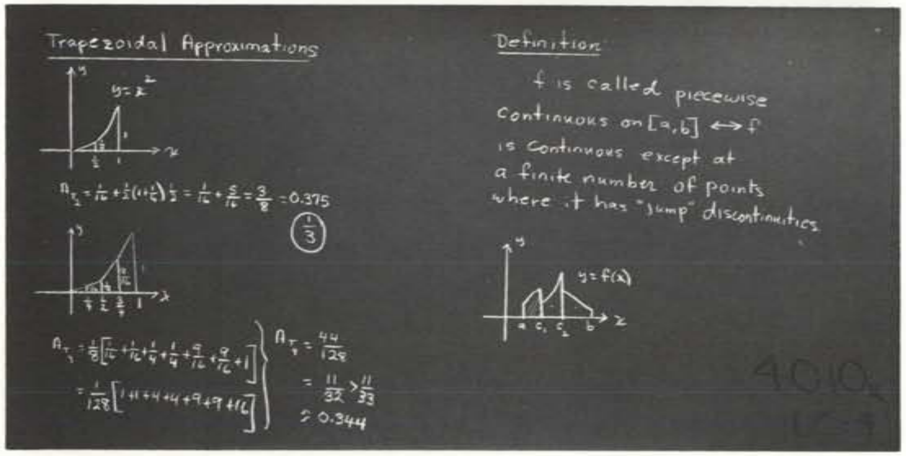
4.010 2-dimensional Area **36 min.**

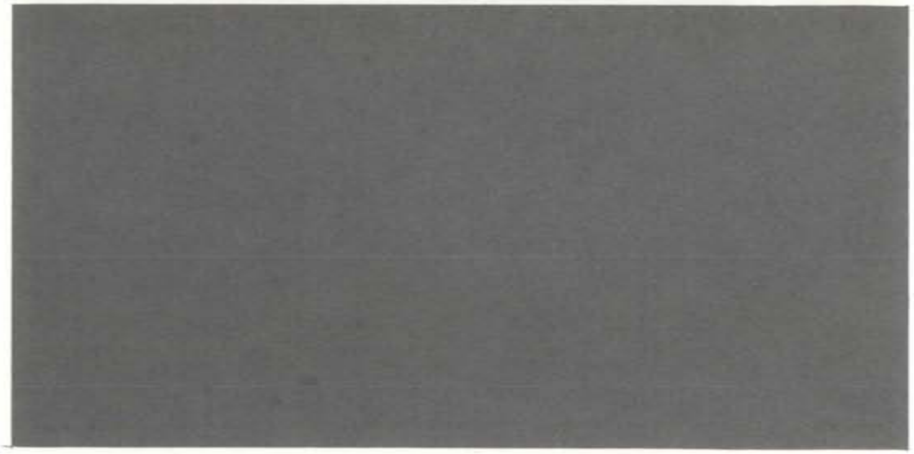




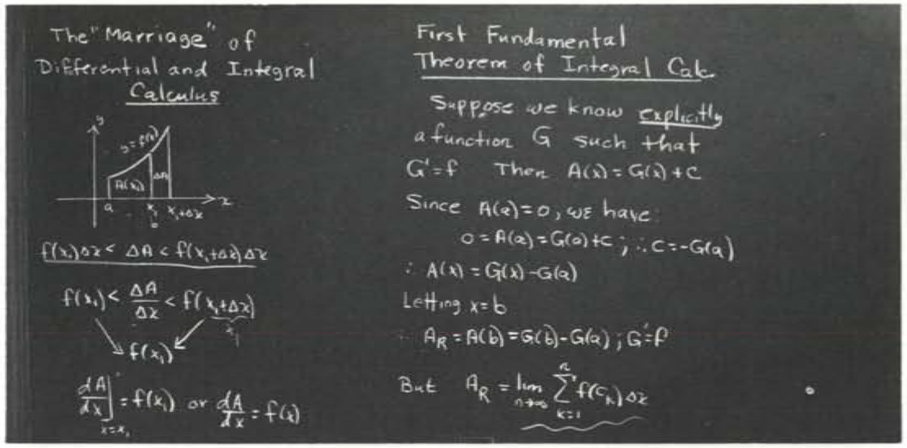

For each n
Similarly
For each n, $L_n < \frac{1}{3}$, $L_n < L_{n+1}$
Generalization
Let f be continuous on [a,b]
(and non-decreasing) and define
R by:
Partition [a,b] into n equal parts
and define
Then: (1) $\lim_{n\to\infty} U_n = \lim_{n\to\infty} L_n$
(2) $L_n < A_R < U_n$

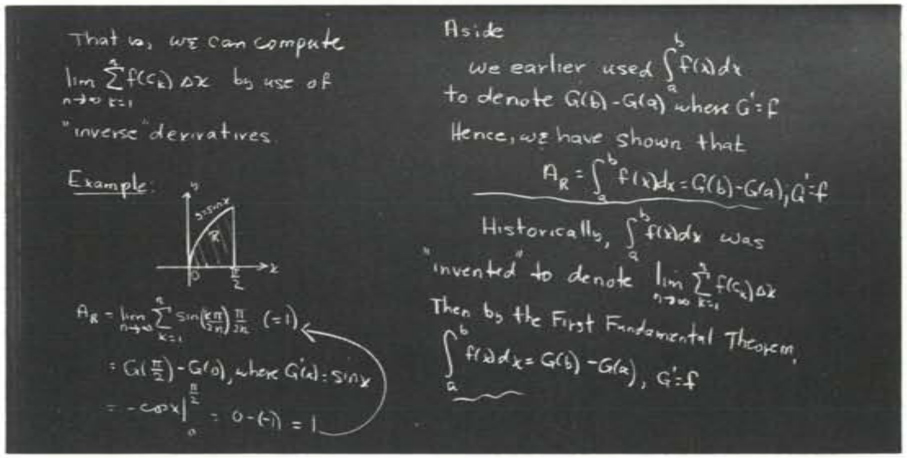

Aside #1
Let $c_k \in [x_{k-1}, x_k]$
Form $\sum_{k=1}^{n} f(c_k)\Delta x$
Then $L_n < \sum_{k=1}^{n} f(c_k)\Delta x < U_n$
y = f(x)
Aside #2.
We can remove the
restriction that f be
non-negative if we
replace area by net area
y = f(x)

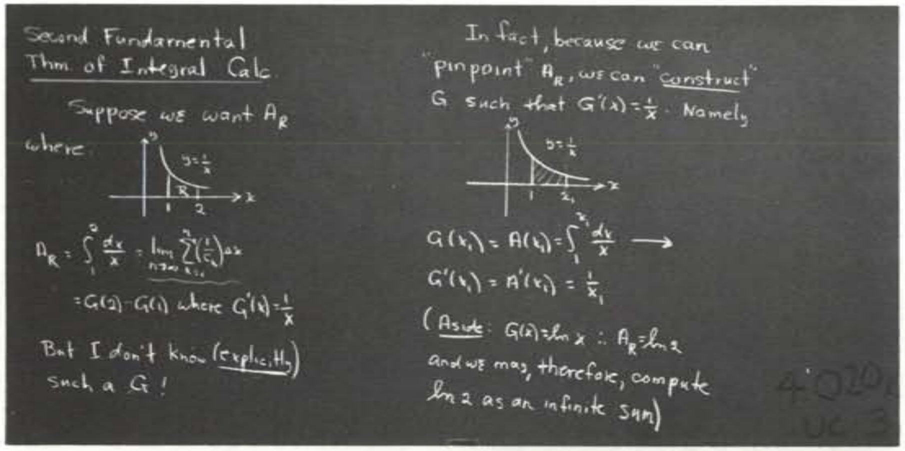

Trapezoidal Approximations
$y = x^2$
$= 0.375$
$A_{T_4} = \frac{44}{128}$
$= \frac{11}{32} > \frac{11}{33}$
$\doteq 0.344$
Definition
f is called piecewise
continuous on [a,b] ⟷ f
is continuous except at
a finite number of points
where it has "jump" discontinuities
y = f(x)



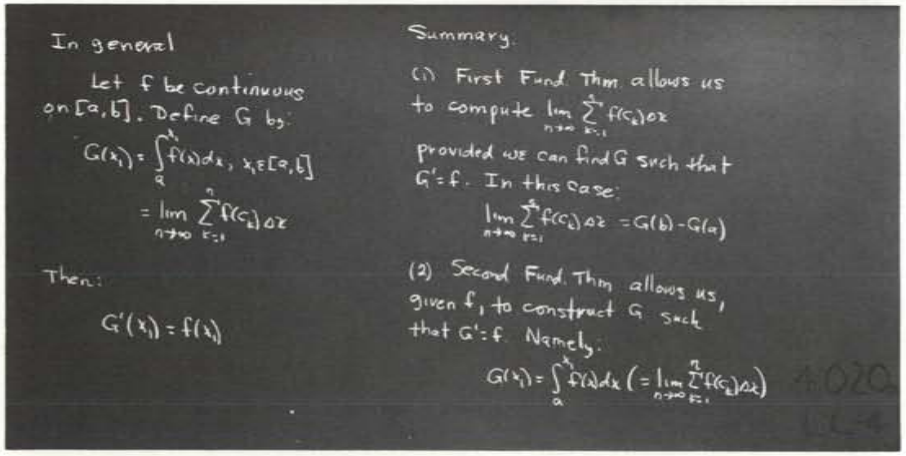
4.020 **Marriage of Differential & Integral Calculus** **30 min.**





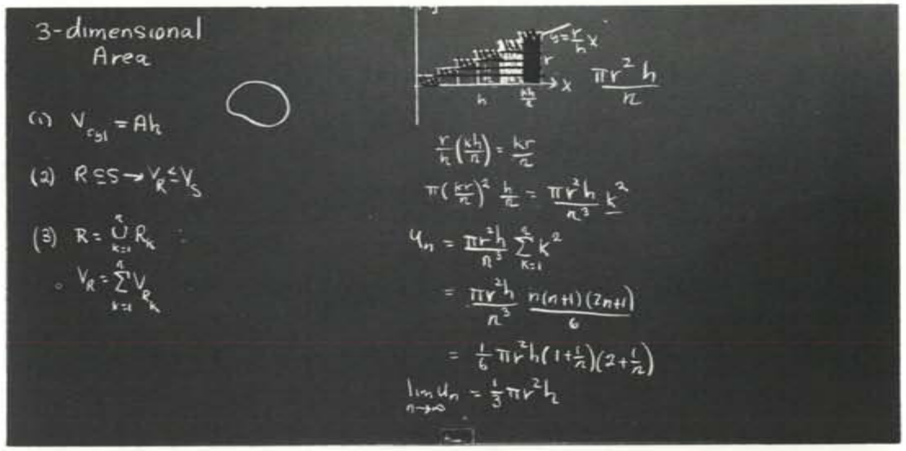


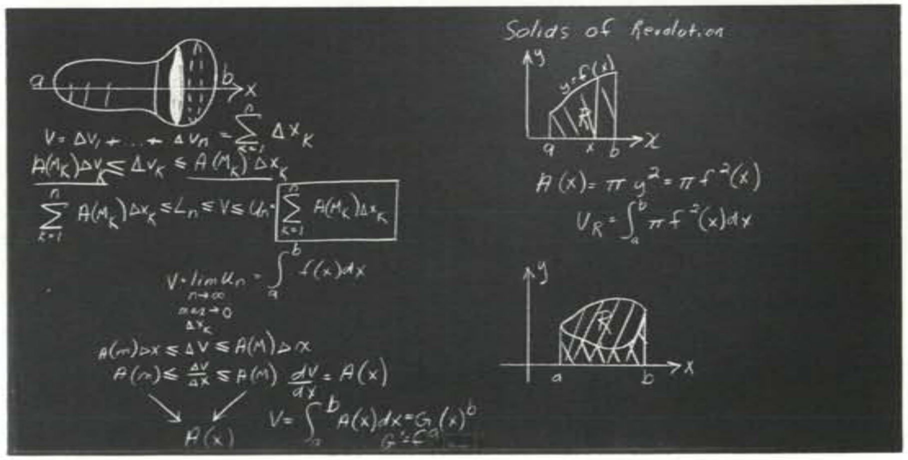
Second Fundamental Thm. of Integral Calc.
Suppose we want A_R where
$y=\frac{1}{x}$
$A_R = \int_1^2 \frac{dx}{x} = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{c_k}\right)\Delta x$
$= G(2) - G(1)$ where $G'(x) = \frac{1}{x}$
But I don't know (explicitly) such a G!
In fact, because we can "pinpoint" A_R, we can "construct" G such that $G'(x) = \frac{1}{x}$. Namely
$y=\frac{1}{x}$
$G(x_1) = A(x_1) = \int_1^{x_1} \frac{dx}{x} \longrightarrow$
$G'(x_1) = A'(x_1) = \frac{1}{x_1}$
(Aside: $G(x) = \ln x \therefore A_R = \ln 2$ and we may, therefore, compute $\ln 2$ as an infinite sum)
4020 UC 3

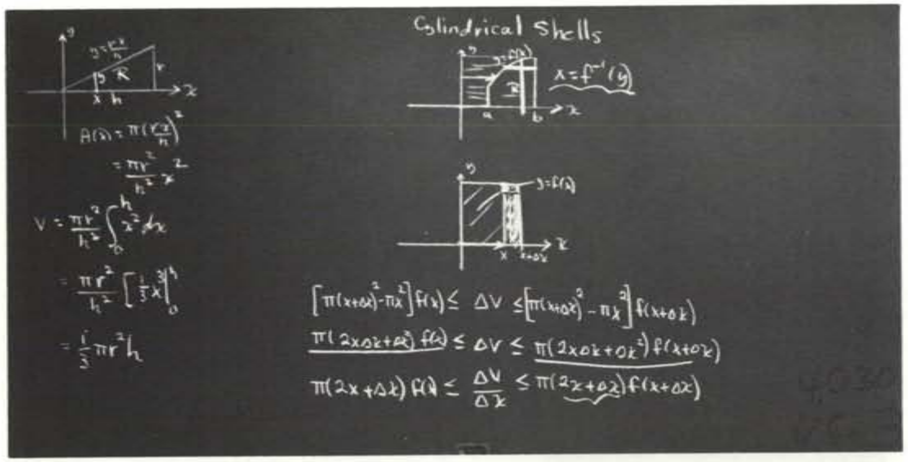


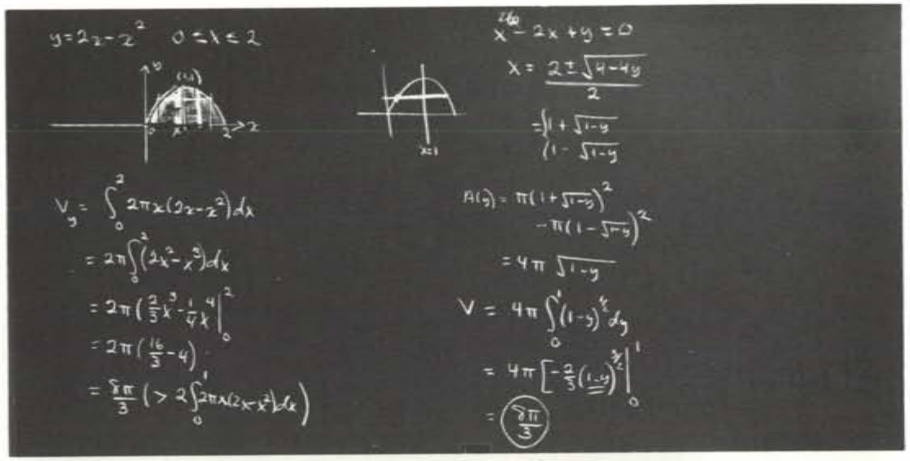
In general
Let f be continuous on [a,b]. Define G by:
$G(x_1) = \int_a^{x_1} f(x)\,dx,\ x_1 \in [a,b]$
$= \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x$
Then:
$G'(x_1) = f(x_1)$
Summary
(1) First Fund. Thm. allows us to compute $\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x$ provided we can find G such that $G' = f$. In this case:
$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x = G(b) - G(a)$
(2) Second Fund. Thm. allows us, given f, to construct G such that $G' = f$. Namely:
$G(x_1) = \int_a^{x_1} f(x)\,dx \left(= \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x\right)$
4020

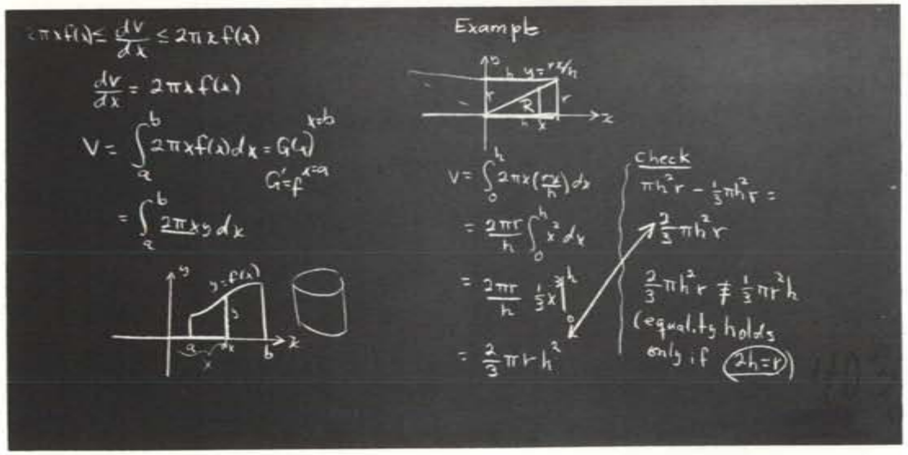
4.030 3-dimensional Area (Volume) 42 min.

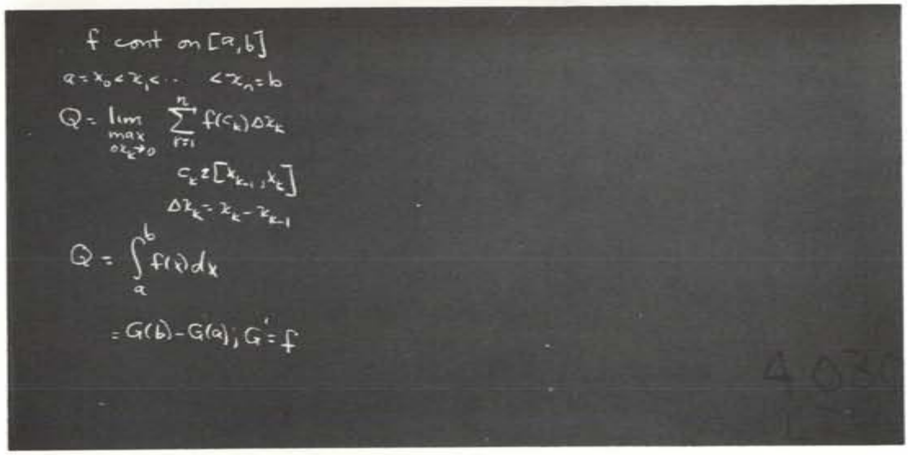




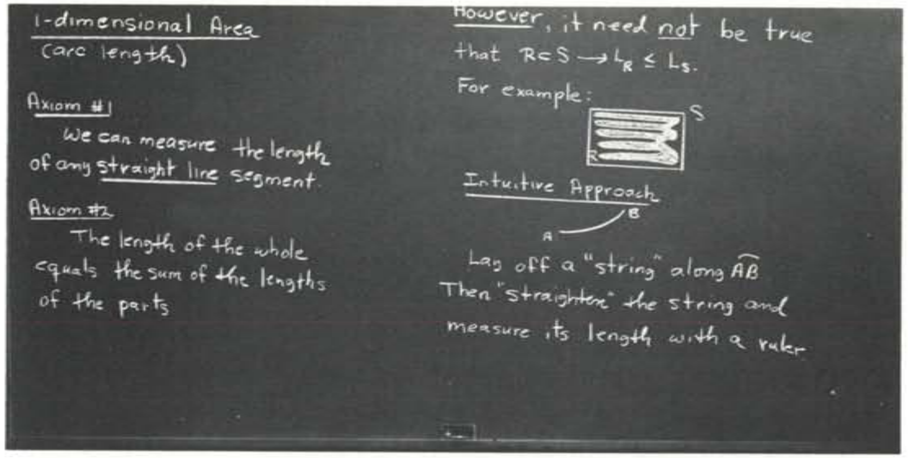

Cylindrical Shells
V = πr²/h² ∫₀ʰ x² dx
= (1/3)πr²h
ΔV

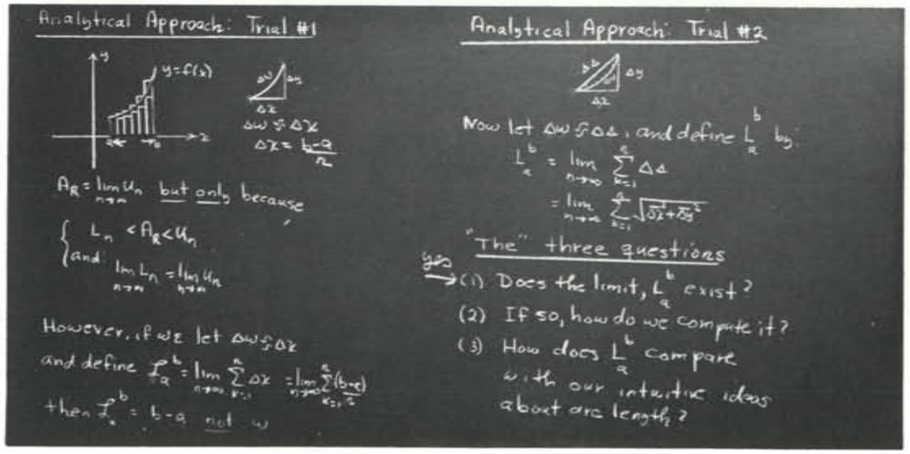

y = 2x − x²   0 ≤ x ≤ 2
= 2π(16/3 − 4)
= 8π/3

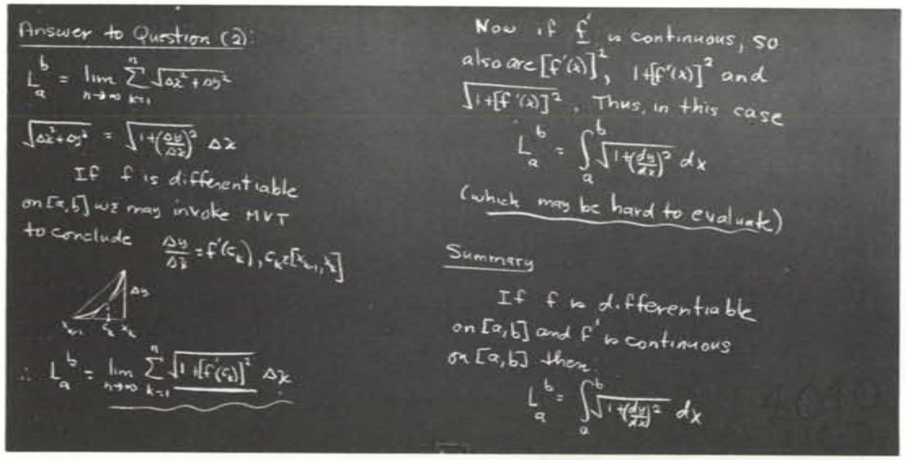

Example
dV/dx = 2πx f(x)
Check
(equality holds
only if 2h = r)

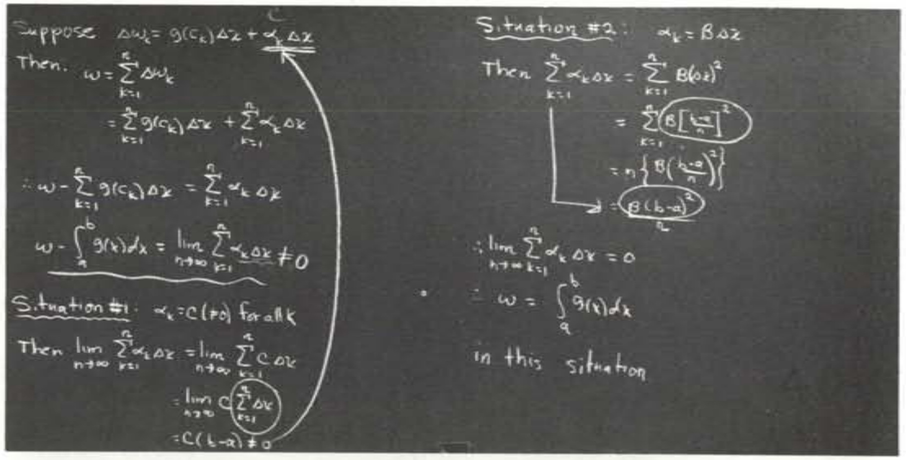

f cont on [a,b]
Q = ∫ₐᵇ f(x) dx
= G(b) − G(a); G′ = f

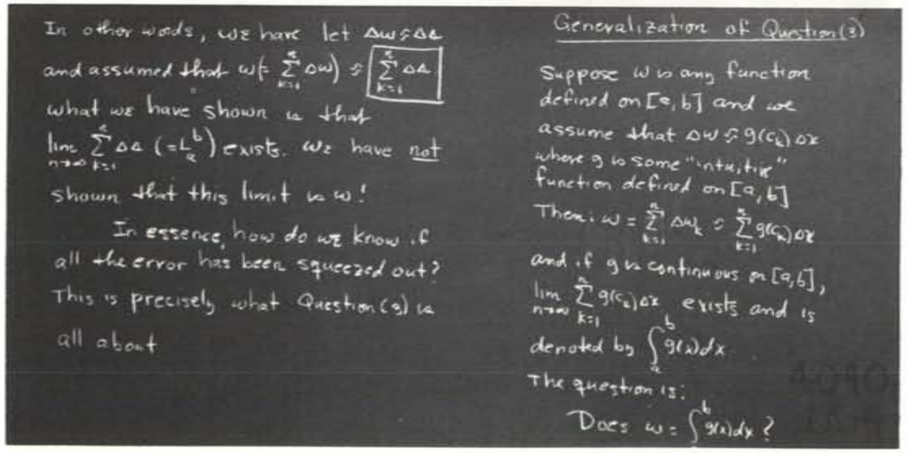
4.040 1-dimensional Area (Arc Length) 36 min.

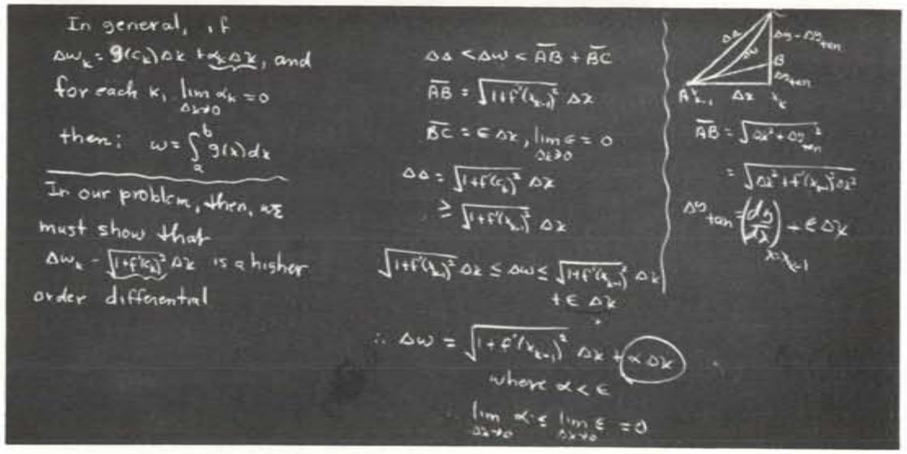





Answer to Question (2):
$L_a^b = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{\Delta x^2 + \Delta y^2}$
$\sqrt{\Delta x^2+\Delta y^2} = \sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x$
If f is differentiable on [a,b] we may invoke MVT to conclude $\frac{\Delta y}{\Delta x} = f'(c_k)$, $c_k \in [x_{k-1}, x_k]$
Δy
x_{k-1} c_k x_k
$\therefore L_a^b = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1+[f'(c_k)]^2}\,\Delta x$
Now if f' is continuous, so also are $[f'(x)]^2$, $1+[f'(x)]^2$ and $\sqrt{1+[f'(x)]^2}$. Thus, in this case
$L_a^b = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$
(which may be hard to evaluate)
Summary
If f is differentiable on [a,b] and f' is continuous on [a,b] then
$L_a^b = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

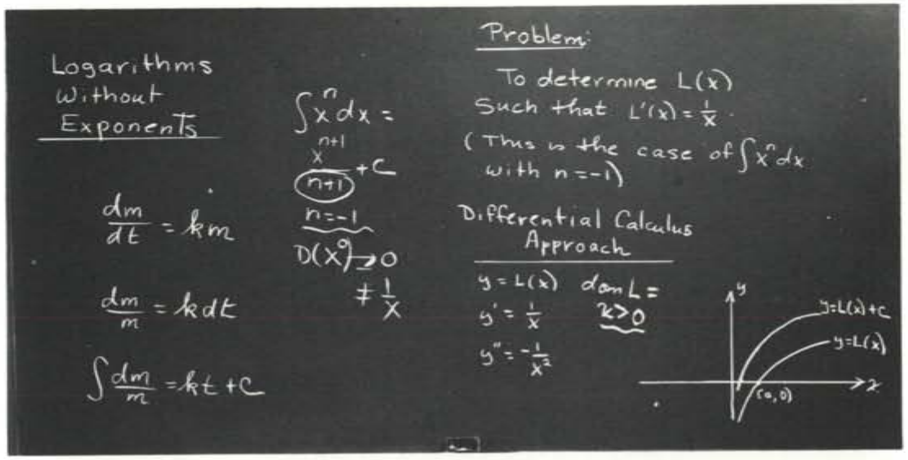

Suppose $\Delta w_k = g(c_k)\Delta x + \alpha_k \Delta x$
Then. $w = \sum_{k=1}^{n} \Delta w_k$
$= \sum_{k=1}^{n} g(c_k)\Delta x + \sum_{k=1}^{n} \alpha_k \Delta x$
$\therefore w - \sum_{k=1}^{n} g(c_k)\Delta x = \sum_{k=1}^{n} \alpha_k \Delta x$
$w - \int_a^b g(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} \alpha_k \Delta x \neq 0$
Situation #1: $\alpha_k = c\ (\neq 0)$ for all k
Then $\lim_{n\to\infty} \sum_{k=1}^{n} \alpha_k \Delta x = \lim_{n\to\infty} \sum_{k=1}^{n} c\,\Delta x$
$= \lim_{n\to\infty} c \sum_{k=1}^{n} \Delta x$
$= c(b-a) \neq 0$
Situation #2: $\alpha_k = B\Delta x$
Then $\sum_{k=1}^{n} \alpha_k \Delta x = \sum_{k=1}^{n} B(\Delta x)^2$
$= \sum_{k=1}^{n} B\left[\frac{b-a}{n}\right]^2$
$= n\left\{B\left(\frac{b-a}{n}\right)^2\right\}$
$= \frac{B(b-a)^2}{n}$
$\therefore \lim_{n\to\infty} \sum_{k=1}^{n} \alpha_k \Delta x = 0$
$\therefore w = \int_a^b g(x)dx$
in this situation

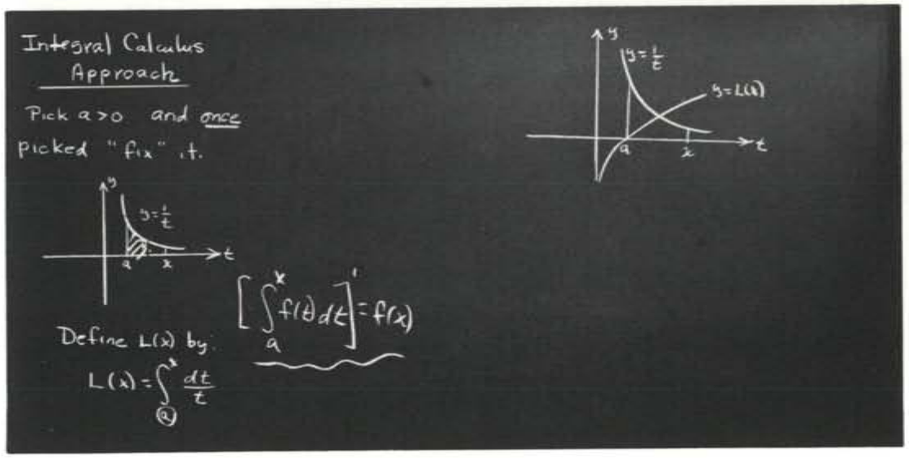

In other words, we have let $\Delta w \doteq \Delta s$ and assumed that $w \left(= \sum_{k=1}^{n} \Delta w\right) \doteq \boxed{\sum_{k=1}^{n} \Delta s}$
what we have shown is that $\lim_{n\to\infty} \sum_{k=1}^{n} \Delta s \ (= L_a^b)$ exists. We have not shown that this limit is w!
In essence, how do we know if all the error has been squeezed out? This is precisely what Question (3) is all about
Generalization of Question (3)
Suppose w is any function defined on [a,b] and we assume that $\Delta w \doteq g(c_k)\Delta x$ where g is some "intuitive" function defined on [a,b]
Then: $w = \sum_{k=1}^{n} \Delta w_k \doteq \sum_{k=1}^{n} g(c_k)\Delta x$
and if g is continuous on [a,b], $\lim_{n\to\infty} \sum_{k=1}^{n} g(c_k)\Delta x$ exists and is denoted by $\int_a^b g(x)dx$
The question is:
Does $w = \int_a^b g(x)dx$?

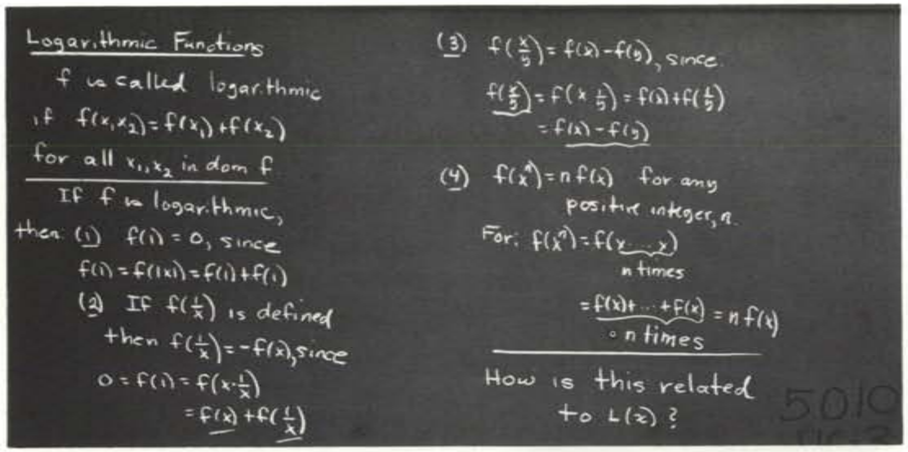
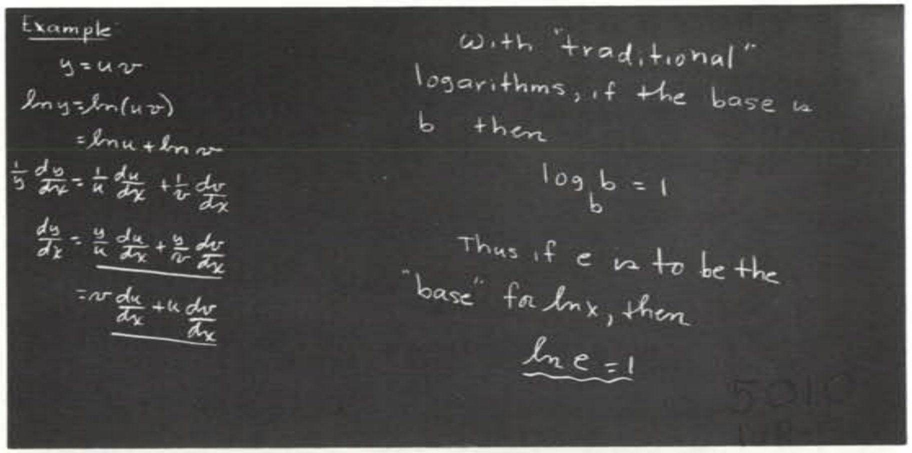
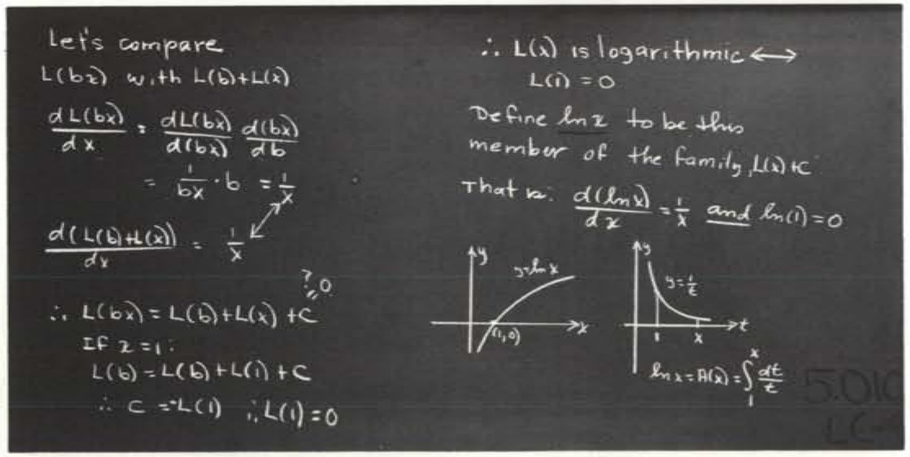
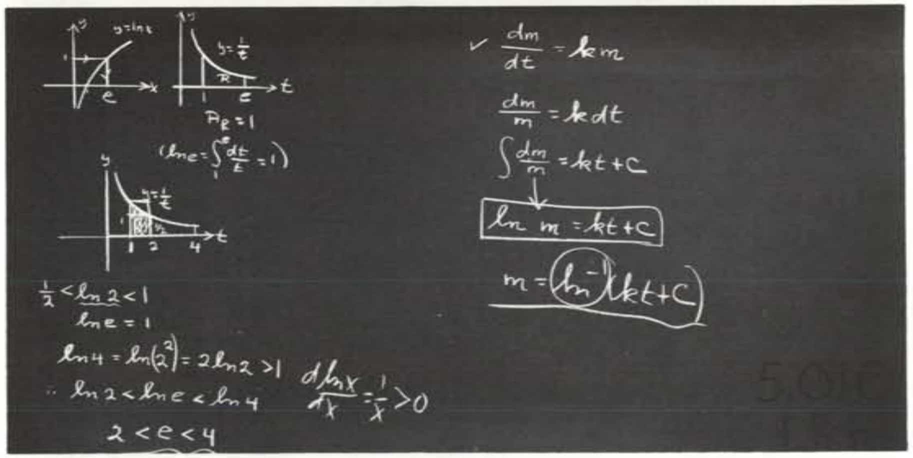

In general, if
$\Delta w_k = g(c_k)\Delta x + \alpha_k \Delta x$, and for each k, $\lim_{\Delta x\to 0} \alpha_k = 0$
then: $w = \int_a^b g(x)dx$
In our problem, then, we must show that
$\Delta w_k - \sqrt{1+f'(c_k)^2}\,\Delta x$ is a higher order differential
$\Delta s < \Delta w < \overline{AB} + \overline{BC}$
$\overline{AB} = \sqrt{1+f'(x_{k-1})^2}\,\Delta x$
$\overline{BC} = \epsilon\Delta x$, $\lim_{\Delta x\to 0} \epsilon = 0$
$\Delta s = \sqrt{1+f'(c_k)^2}\,\Delta x$
$\geq \sqrt{1+f'(x_{k-1})^2}\,\Delta x$
$\sqrt{1+f'(x_{k-1})^2}\,\Delta x \leq \Delta w \leq \sqrt{1+f'(x_{k-1})^2}\,\Delta x + \epsilon\Delta x$
$\therefore \Delta w = \sqrt{1+f'(x_{k-1})^2}\,\Delta x + \alpha\Delta x$
where $\alpha < \epsilon$
$\lim_{\Delta x\to 0} \alpha \leq \lim_{\Delta x\to 0} \epsilon = 0$
Δs Δw $\Delta y - \Delta y_{tan}$ B Δy_{tan}
A x_{k-1} Δx x_k
$\overline{AB} = \sqrt{\Delta x^2 + \Delta y_{tan}^2}$
$= \sqrt{\Delta x^2 + f'(x_{k-1})^2 \Delta x^2}$
$\Delta y_{tan} = \left(\frac{dy}{dx}\right)_{x=x_{k-1}} + \epsilon\Delta x$

Block V: Transcendental Functions

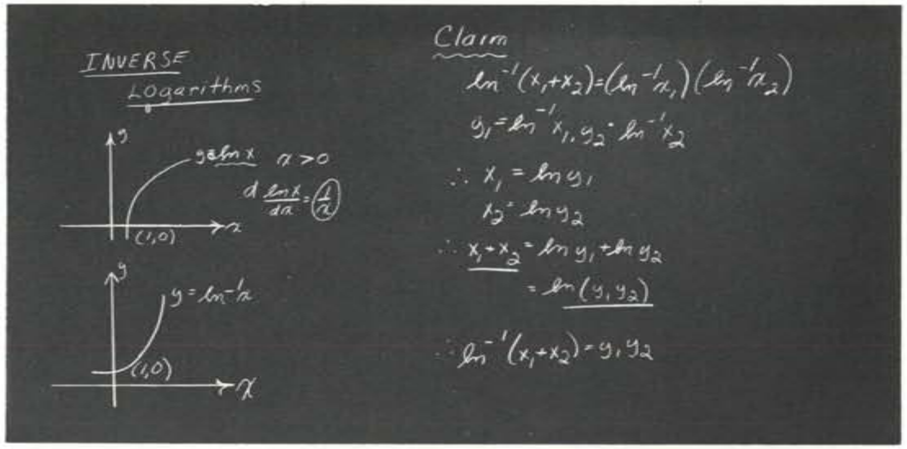
5.010 **Logarithms without Exponents** **34 min.**

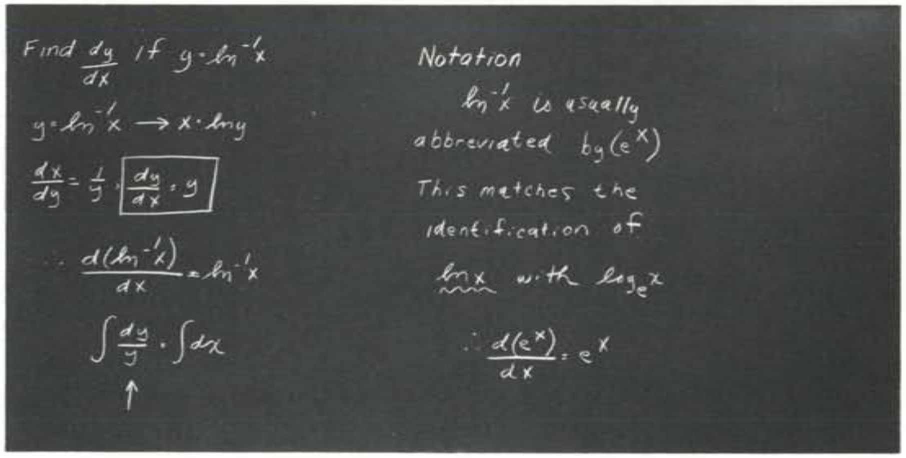





Logarithmic Functions
f is called logarithmic
if $f(x_1 x_2) = f(x_1) + f(x_2)$
for all x_1, x_2 in dom f
If f is logarithmic,
then: (1) $f(1) = 0$, since
$f(1) = f(1 \times 1) = f(1) + f(1)$
(2) If $f(\frac{1}{x})$ is defined
then $f(\frac{1}{x}) = -f(x)$, since
$0 = f(1) = f(x \cdot \frac{1}{x})$
$= f(x) + f(\frac{1}{x})$
(3) $f(\frac{x}{y}) = f(x) - f(y)$, since
$f(\frac{x}{y}) = f(x \cdot \frac{1}{y}) = f(x) + f(\frac{1}{y})$
$= f(x) - f(y)$
(4) $f(x^n) = n f(x)$ for any positive integer, n.
For: $f(x^n) = f(\underbrace{x \cdots x}_{n \text{ times}})$
$= \underbrace{f(x) + \cdots + f(x)}_{n \text{ times}} = n f(x)$
How is this related to L(x)?
5010

Example
$y = uv$
$\ln y = \ln(uv)$
$= \ln u + \ln v$
$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx}$
$\frac{dy}{dx} = \frac{y}{u}\frac{du}{dx} + \frac{y}{v}\frac{dv}{dx}$
$= v\frac{du}{dx} + u\frac{dv}{dx}$
With "traditional" logarithms, if the base is b then
$\log_b b = 1$
Thus if e is to be the "base" for ln x, then
$\ln e = 1$
5010

Let's compare
L(bx) with L(b)+L(x)
$\frac{dL(bx)}{dx} = \frac{dL(bx)}{d(bx)} \frac{d(bx)}{db}$
$= \frac{1}{bx} \cdot b = \frac{1}{x}$
$\frac{d(L(b)+L(x))}{dx} = \frac{1}{x}$
$\therefore L(bx) = L(b) + L(x) + C \stackrel{?}{=} 0$
If x = 1:
$L(b) = L(b) + L(1) + C$
$\therefore C = -L(1)$ $\therefore L(1) = 0$
$\therefore$ L(x) is logarithmic $\longleftrightarrow$ L(1) = 0
Define ln x to be this member of the family, L(x)+C
That is: $\frac{d(\ln x)}{dx} = \frac{1}{x}$ and $\ln(1) = 0$
y = ln x
(1,0)
$y = \frac{1}{t}$
$\ln x = A(x) = \int_1^x \frac{dt}{t}$
5010

y = ln x
e
$y = \frac{1}{t}$
R
1
e
$A_R = 1$
$(\ln e = \int_1^e \frac{dt}{t} = 1)$
$y = \frac{1}{t}$
1
2
4
$\frac{1}{2} < \ln 2 < 1$
$\ln e = 1$
$\ln 4 = \ln(2^2) = 2 \ln 2 > 1$
$\therefore \ln 2 < \ln e < \ln 4$
$\frac{d \ln x}{dx} = \frac{1}{x} > 0$
$2 < e < 4$
$\checkmark$ $\frac{dm}{dt} = km$
$\frac{dm}{m} = k\,dt$
$\int \frac{dm}{m} = kt + C$
$\boxed{\ln m = kt + C}$
$m = \ln^{-1}(kt + C)$
5010

5.020 Inverse Logarithms 21 min.





$$\frac{de^u}{dx} = \frac{de^u}{du}\,\frac{du}{dx} = e^u\,\frac{du}{dx}$$

$$\int e^{-x^2}\,dx = ?$$

$$\int 2xe^{-x^2}\,dx = ?$$

$$u = -x^2$$

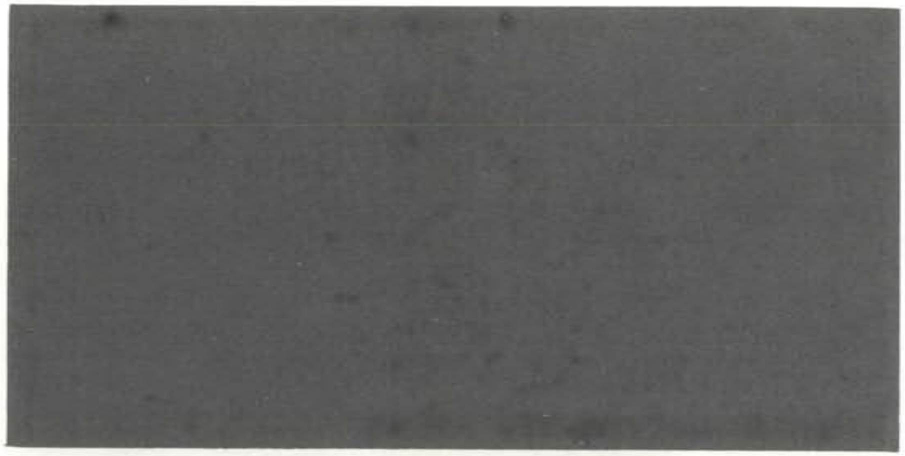
$$du = -2x\,dx$$

$$\int 2xe^{-x^2}\,dx = \int -e^u\,du = -e^u + C = -e^{-x^2} + C$$

$$\int e^{-x^2}\,d(-x^2) = e^{-x^2} + C$$

$$\int -2xe^{-x^2}\,dx = e^{-x^2} + C$$

5020 LC-3



$$y'' - 5y' + 6y = 0$$

Try $y_T = e^{rx}$

$$y_T' = re^{rx}$$

$$y_T'' = r^2e^{rx}$$

$$r^2e^{rx} - 5re^{rx} + 6e^{rx} = 0$$

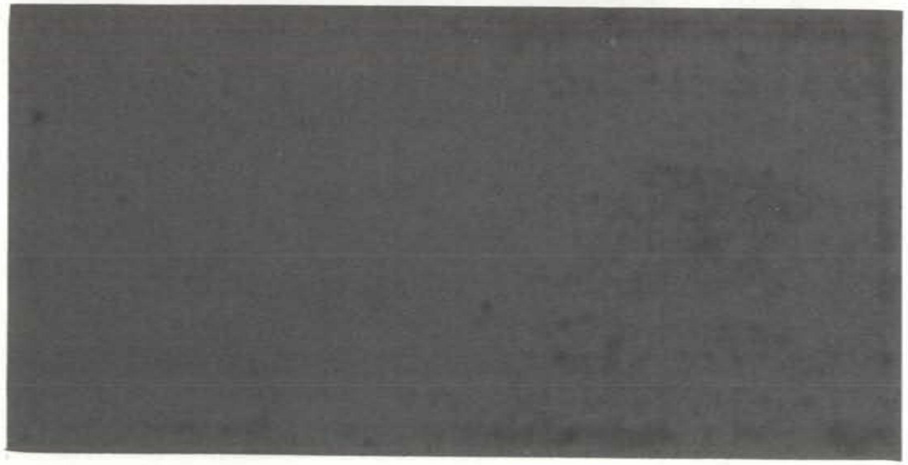
$$e^{rx}(r^2 - 5r + 6) = 0$$

$\therefore r = 2$ or $r = 3$

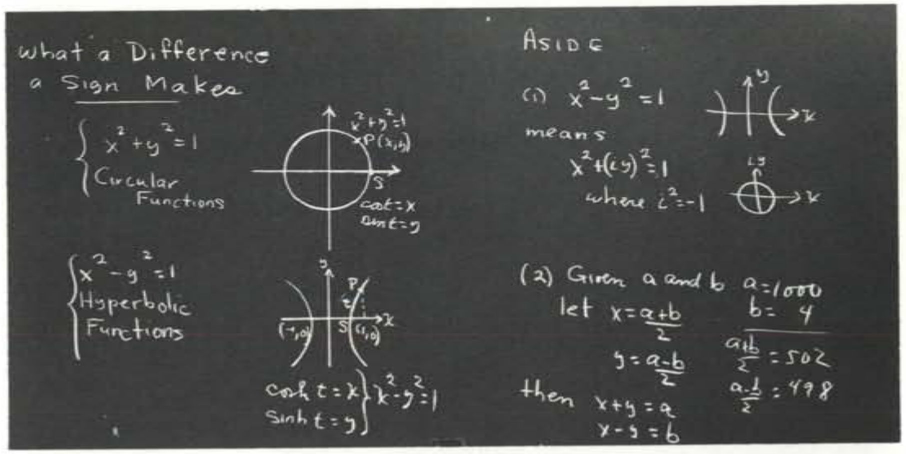
$y = e^{2x}$	$y = e^{3x}$
$y' = 2e^{2x}$	$y' = 3e^{3x}$
$y'' = 4e^{2x}$	$y'' = 9e^{3x}$
$y'' - 5y' + 6y \equiv 0$	$y'' - 5y' + 6y \equiv 0$

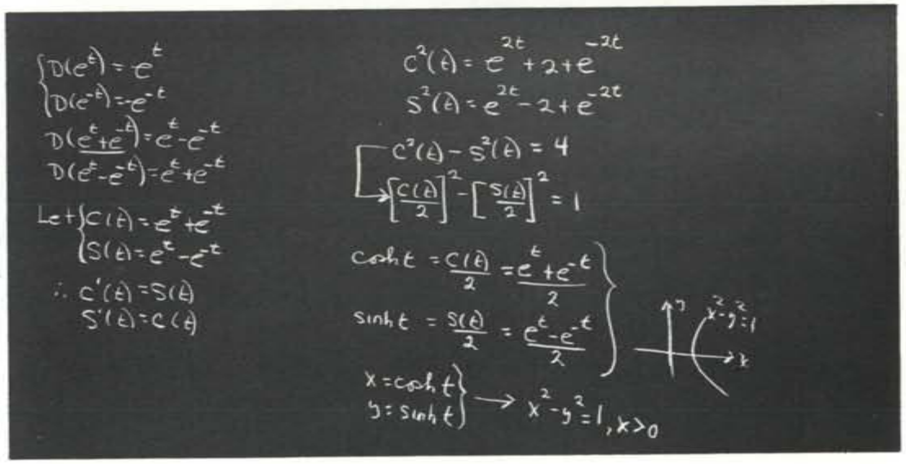
In general if a and b are constants $y_T = e^{rx}$ "transforms" $y'' + ay' + by = 0$ into $r^2 + ar + b = 0$

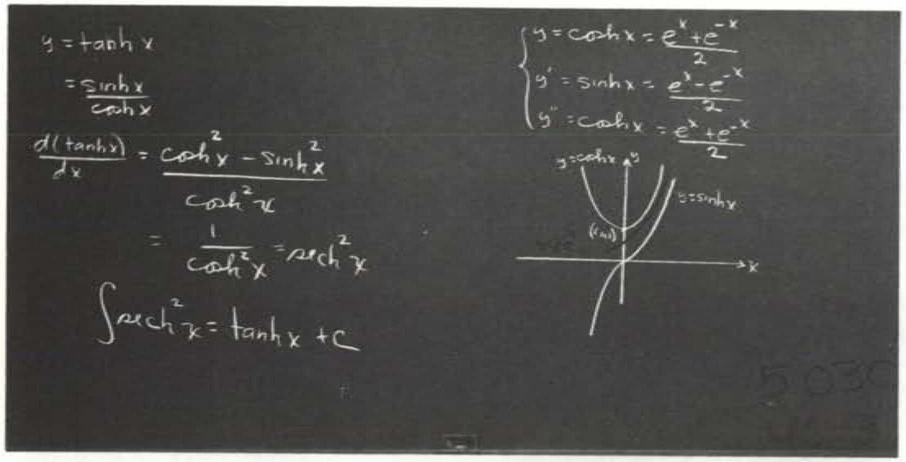
5020 LC-4



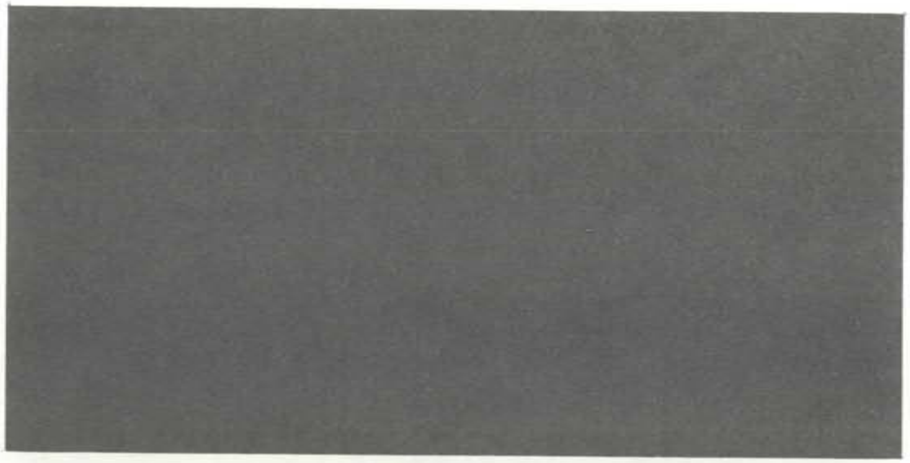
5.030 **What a Difference a Sign Makes** **27 min.**

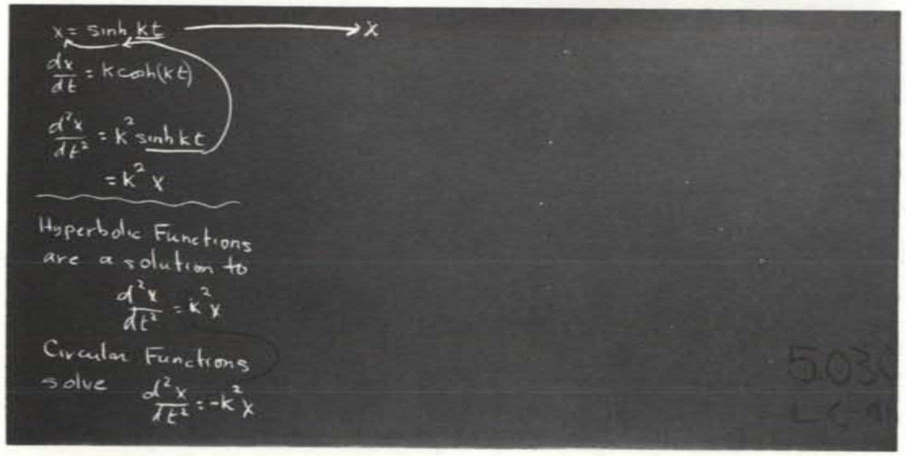




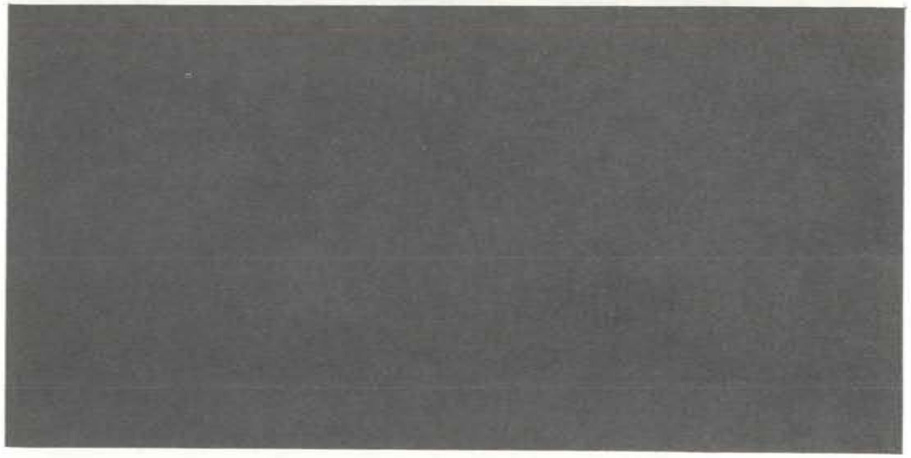


$y = \tanh x$
$= \frac{\sinh x}{\cosh x}$
$\frac{d(\tanh x)}{dx} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$
$= \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$
$\int \operatorname{sech}^2 x = \tanh x + C$
$y = \cosh x = \frac{e^x + e^{-x}}{2}$
$y' = \sinh x = \frac{e^x - e^{-x}}{2}$
$y'' = \cosh x = \frac{e^x + e^{-x}}{2}$
$y = \cosh x$
y
$y = \sinh x$
$(0,1)$
x

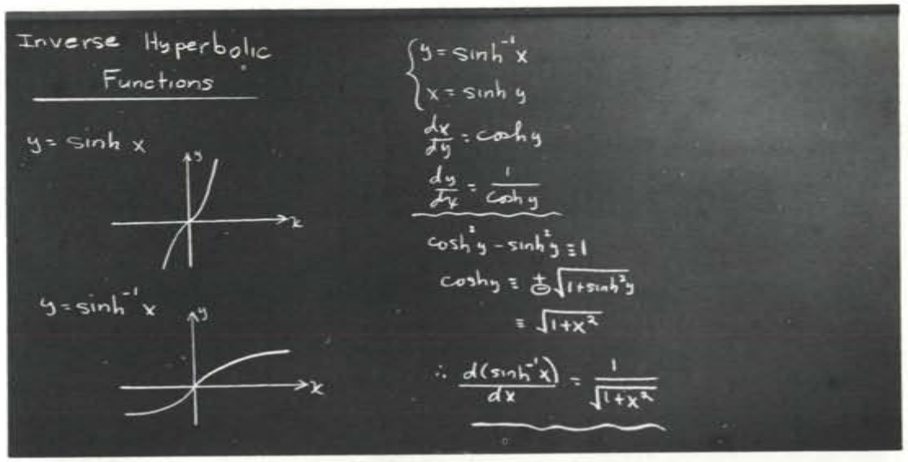


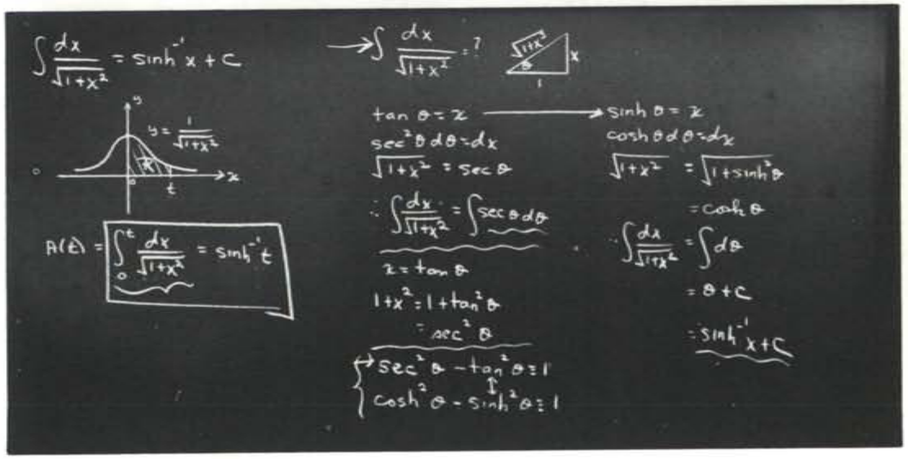


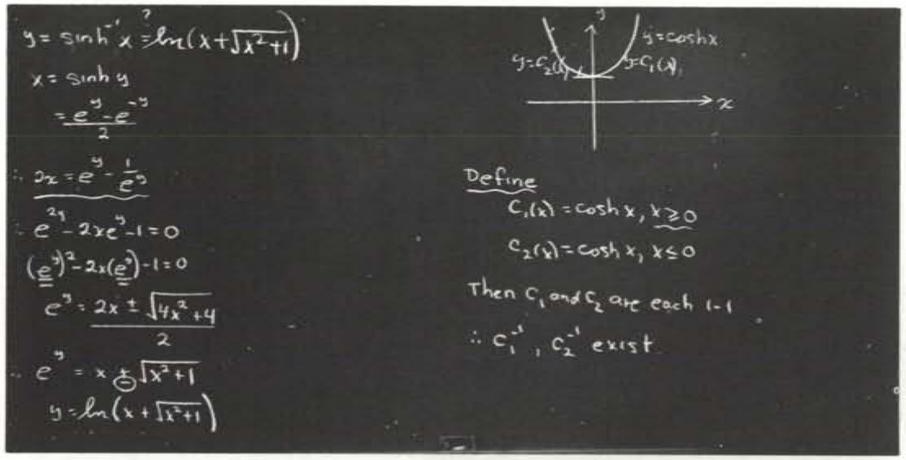
$x = \sinh kt \longrightarrow x$
$\frac{dx}{dt} = k\cosh(kt)$
$\frac{d^2x}{dt^2} = k^2 \sinh kt$
$= k^2 x$
Hyperbolic Functions
are a solution to
$\frac{d^2x}{dt^2} = k^2 x$
Circular Functions
solve $\frac{d^2x}{dt^2} = -k^2 x$

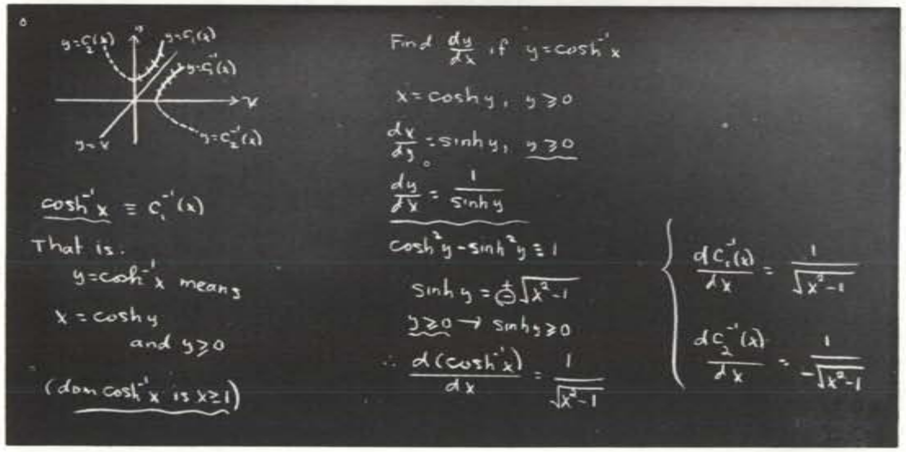


5.040 Inverse Hyperbolic Functions **30 min.**





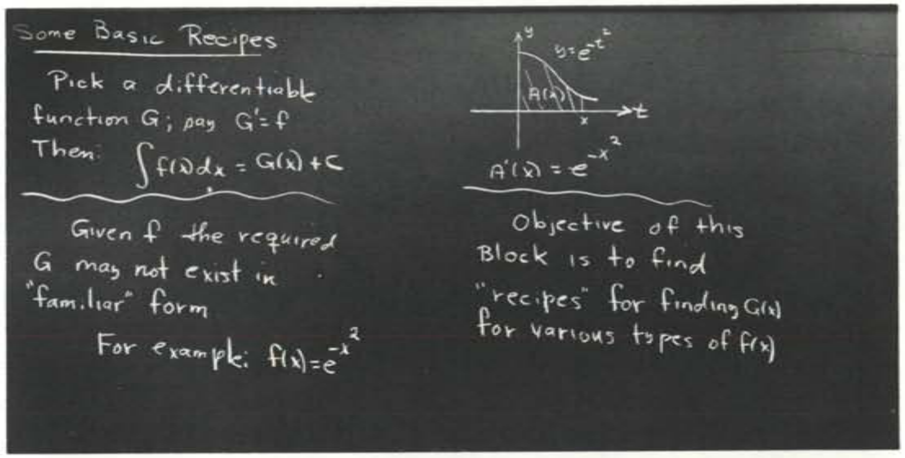

$y = \sinh^{-1} x \stackrel{?}{=} \ln(x + \sqrt{x^2+1})$
$x = \sinh y$
$= \frac{e^y - e^{-y}}{2}$
$\therefore 2x = e^y - \frac{1}{e^y}$
$\therefore e^{2y} - 2xe^y - 1 = 0$
$(e^y)^2 - 2x(e^y) - 1 = 0$
$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$
$\therefore e^y = x \pm \sqrt{x^2+1}$
$y = \ln(x + \sqrt{x^2+1})$
$y = \cosh x$
$y = C_2(x)$
$y = C_1(x)$
x
Define
$C_1(x) = \cosh x, \ x \geq 0$
$C_2(x) = \cosh x, \ x \leq 0$
Then C_1 and C_2 are each 1-1
$\therefore C_1^{-1}, C_2^{-1}$ exist

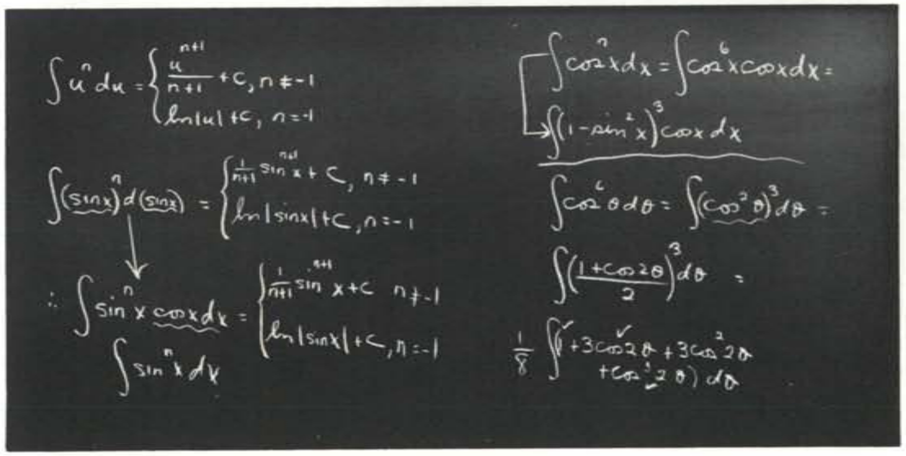

$y = C_2(x)$
$y = C_1(x)$
$y = C_1^{-1}(x)$
x
$y = x$
$y = C_2^{-1}(x)$
$\cosh^{-1} x \equiv C_1^{-1}(x)$
That is.
$y = \cosh^{-1} x$ means
$x = \cosh y$
and $y \geq 0$
(dom $\cosh^{-1} x$ is $x \geq 1$)
Find $\frac{dy}{dx}$ if $y = \cosh^{-1} x$
$x = \cosh y, \ y \geq 0$
$\frac{dx}{dy} = \sinh y, \ y \geq 0$
$\frac{dy}{dx} = \frac{1}{\sinh y}$
$\cosh^2 y - \sinh^2 y \equiv 1$
$\sinh y = \pm \sqrt{x^2-1}$
$y \geq 0 \rightarrow \sinh y \geq 0$
$\therefore \frac{d(\cosh^{-1} x)}{dx} = \frac{1}{\sqrt{x^2-1}}$
$\frac{dC_1^{-1}(x)}{dx} = \frac{1}{\sqrt{x^2-1}}$
$\frac{dC_2^{-1}(x)}{dx} = \frac{1}{-\sqrt{x^2-1}}$

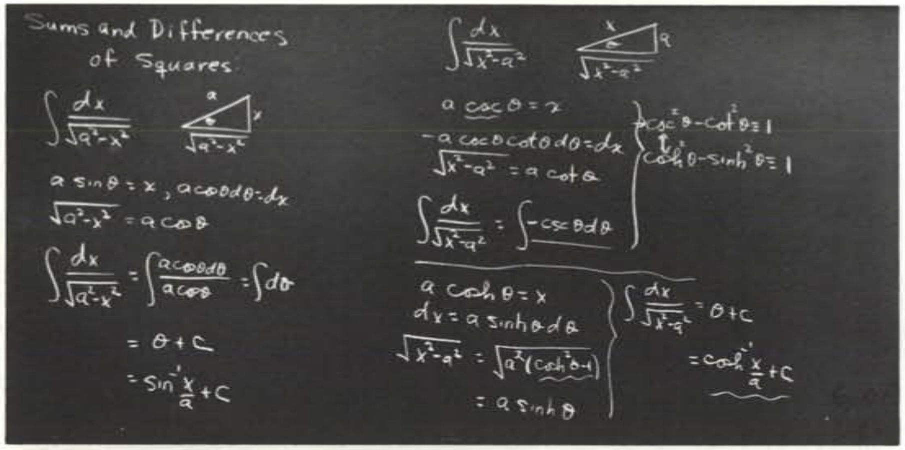
Block VI: More Integration Techniques

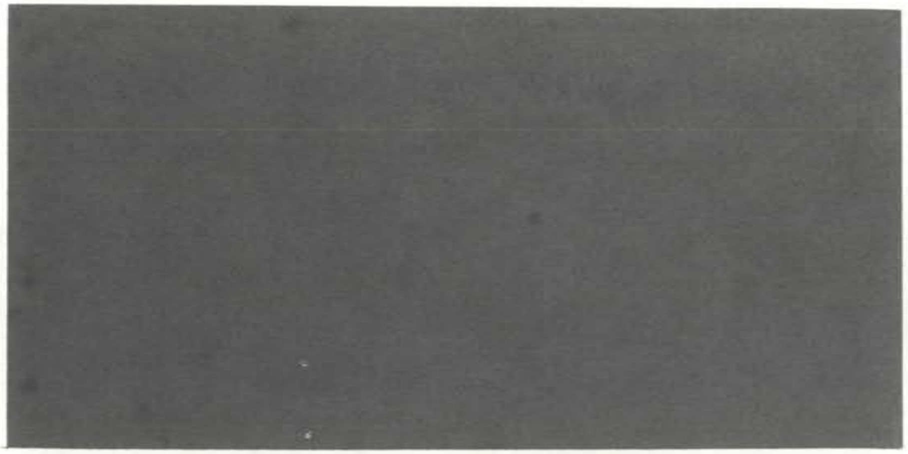
6.010 Some Basic Recipes

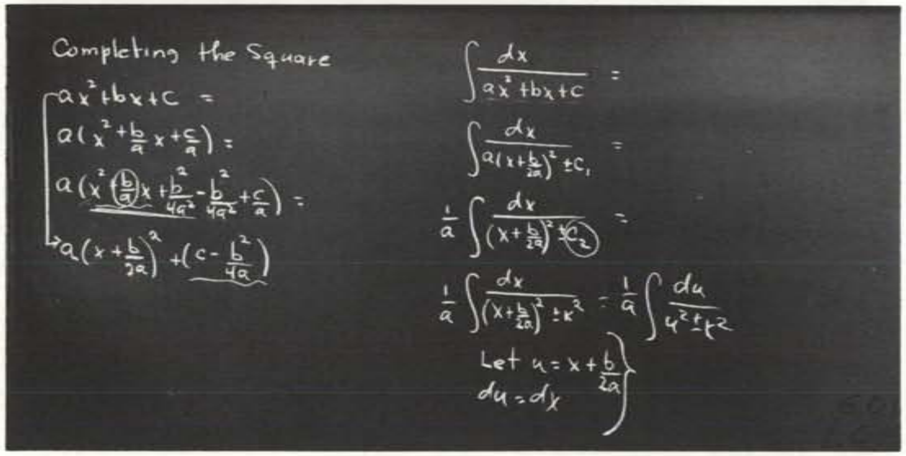
30 min.

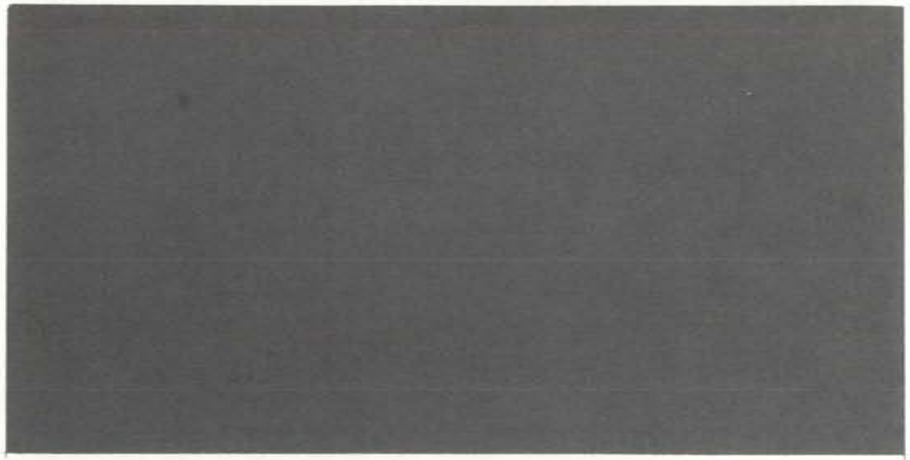




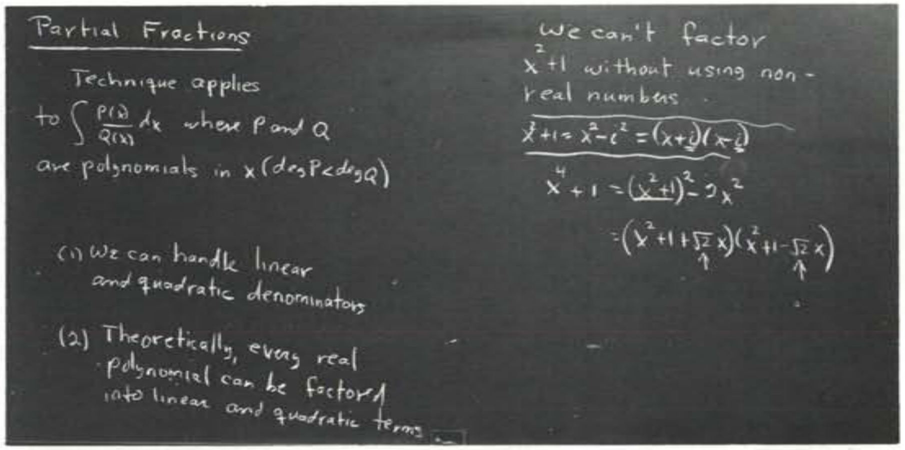

Sums and Differences of Squares
$\int \frac{dx}{\sqrt{a^2-x^2}}$
a
x
$\sqrt{a^2-x^2}$
$a \sin\theta = x$, $a\cos\theta\, d\theta = dx$
$\sqrt{a^2-x^2} = a\cos\theta$
$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta\, d\theta}{a\cos\theta} = \int d\theta$
$= \theta + C$
$= \sin^{-1}\frac{x}{a} + C$
$\int \frac{dx}{\sqrt{x^2-a^2}}$
x
a
$\sqrt{x^2-a^2}$
$a \csc\theta = x$
$-a\csc\theta\cot\theta\, d\theta = dx$
$\sqrt{x^2-a^2} = a\cot\theta$
$\int \frac{dx}{\sqrt{x^2-a^2}} = \int -\csc\theta\, d\theta$
$\csc^2\theta - \cot^2\theta \equiv 1$
$\cosh^2\theta - \sinh^2\theta \equiv 1$
$a\cosh\theta = x$
$dx = a\sinh\theta\, d\theta$
$\sqrt{x^2-a^2} = \sqrt{a^2(\cosh^2\theta - 1)}$
$= a\sinh\theta$
$\int \frac{dx}{\sqrt{x^2-a^2}} = \theta + C$
$= \cosh^{-1}\frac{x}{a} + C$

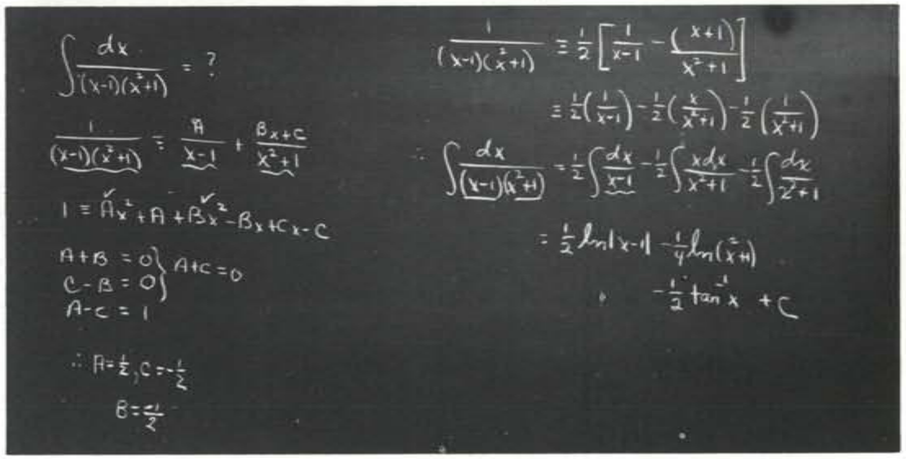


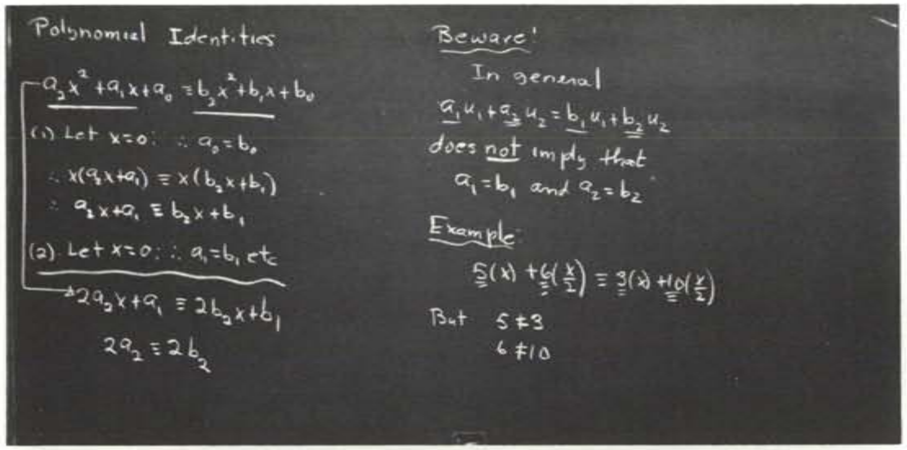

Completing the Square
$ax^2+bx+C =$
$a(x^2+\frac{b}{a}x+\frac{c}{a}) =$
$a(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}) =$
$a(x+\frac{b}{2a})^2+(c-\frac{b^2}{4a})$
$\int \frac{dx}{ax^2+bx+c} =$
$\int \frac{dx}{a(x+\frac{b}{2a})^2 \pm C_1} =$
$\frac{1}{a}\int \frac{dx}{(x+\frac{b}{2a})^2 \pm C_2} =$
$\frac{1}{a}\int \frac{dx}{(x+\frac{b}{2a})^2 \pm k^2} = \frac{1}{a}\int \frac{du}{u^2 \pm k^2}$
Let $u = x+\frac{b}{2a}$
$du = dx$



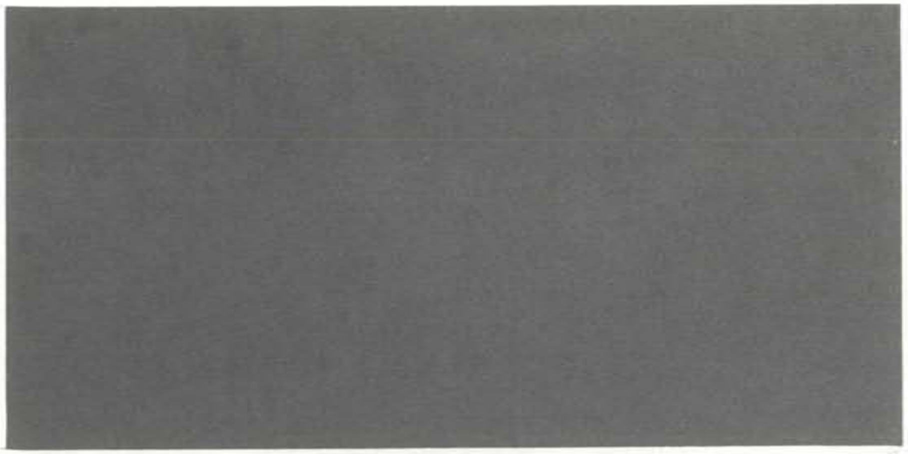
6.020 Partial Fractions 32 min.

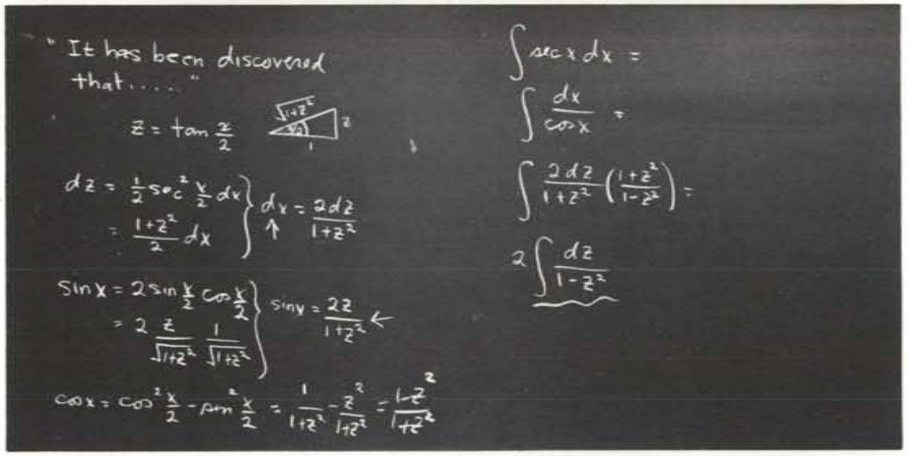




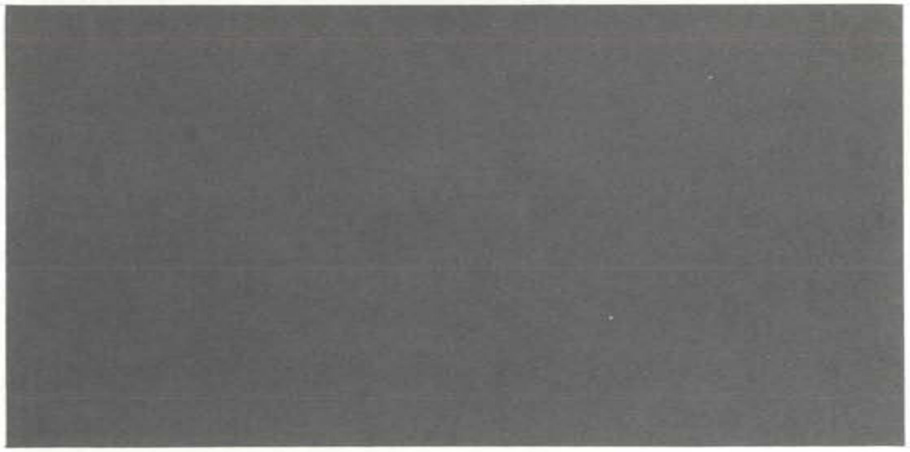


Polynomial Identities
$a_2x^2+a_1x+a_0 \equiv b_2x^2+b_1x+b_0$
(1) Let $x=0$: $\therefore a_0=b_0$
$\therefore x(a_2x+a_1) \equiv x(b_2x+b_1)$
$\therefore a_2x+a_1 \equiv b_2x+b_1$
(2) Let $x=0$: $\therefore a_1=b_1$ etc
$2a_2x+a_1 \equiv 2b_2x+b_1$
$2a_2 \equiv 2b_2$
Beware!
In general
$a_1u_1+a_2u_2=b_1u_1+b_2u_2$
does not imply that
$a_1=b_1$ and $a_2=b_2$
Example
$5(x)+6\left(\frac{x}{2}\right) \equiv 3(x)+10\left(\frac{x}{2}\right)$
But $5 \neq 3$
$6 \neq 10$

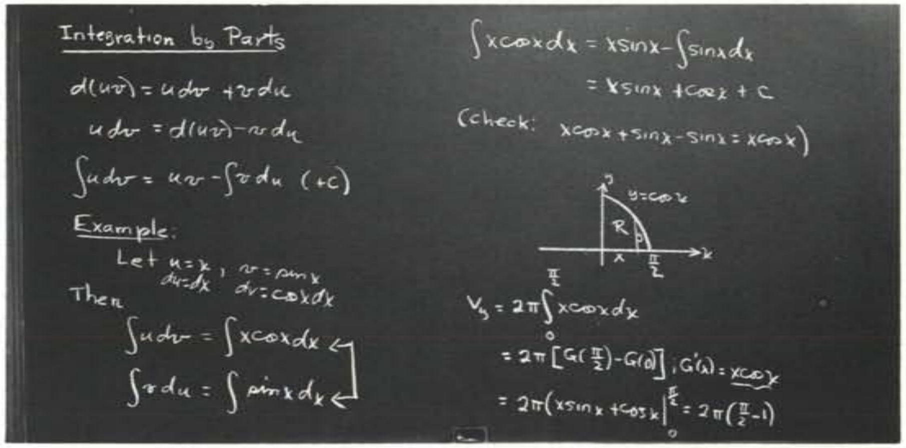


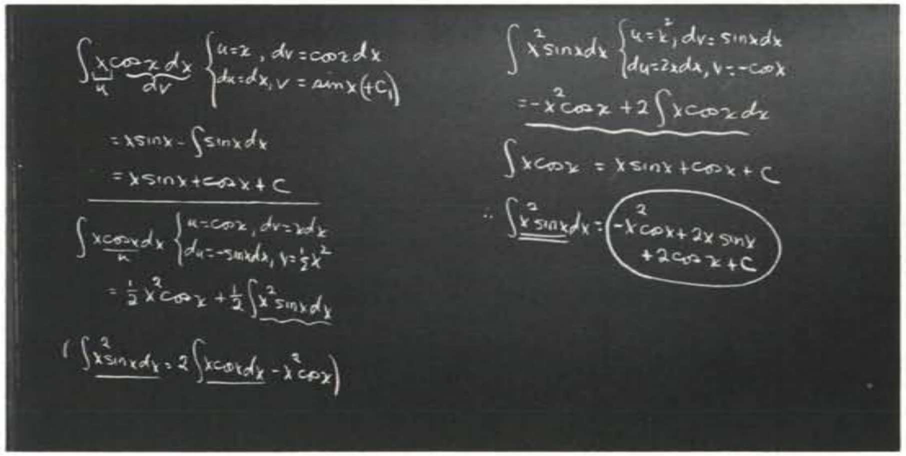


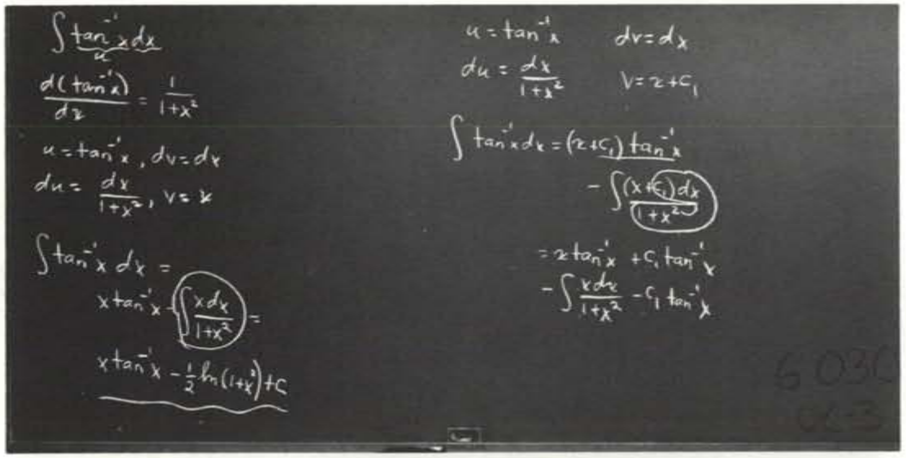
"It has been discovered that....."
$z=\tan\frac{x}{2}$
$\sqrt{1+z^2}$
z
$x/2$
1
$dz=\frac{1}{2}\sec^2\frac{x}{2}\,dx = \frac{1+z^2}{2}\,dx$
$dx=\frac{2\,dz}{1+z^2}$
$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\frac{z}{\sqrt{1+z^2}}\frac{1}{\sqrt{1+z^2}}$
$\sin x=\frac{2z}{1+z^2}$
$\cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}=\frac{1}{1+z^2}-\frac{z^2}{1+z^2}=\frac{1-z^2}{1+z^2}$
$\int \sec x\,dx =$
$\int \frac{dx}{\cos x} =$
$\int \frac{2\,dz}{1+z^2}\left(\frac{1+z^2}{1-z^2}\right) =$
$2\int \frac{dz}{1-z^2}$

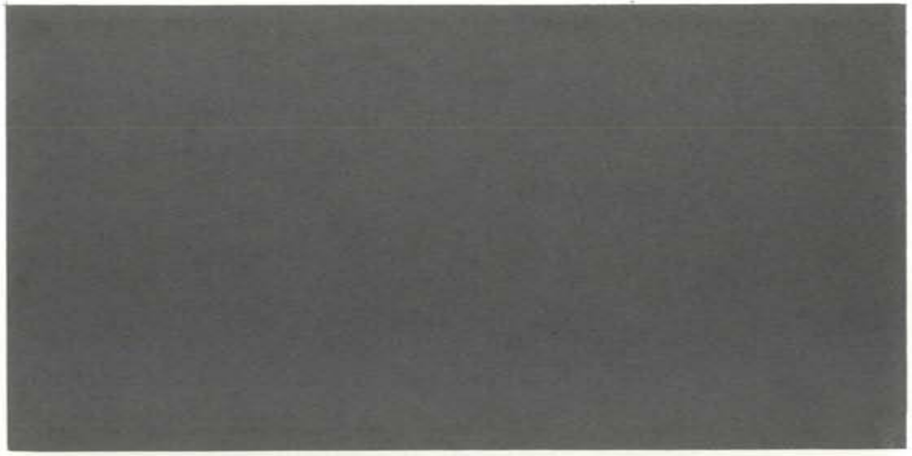


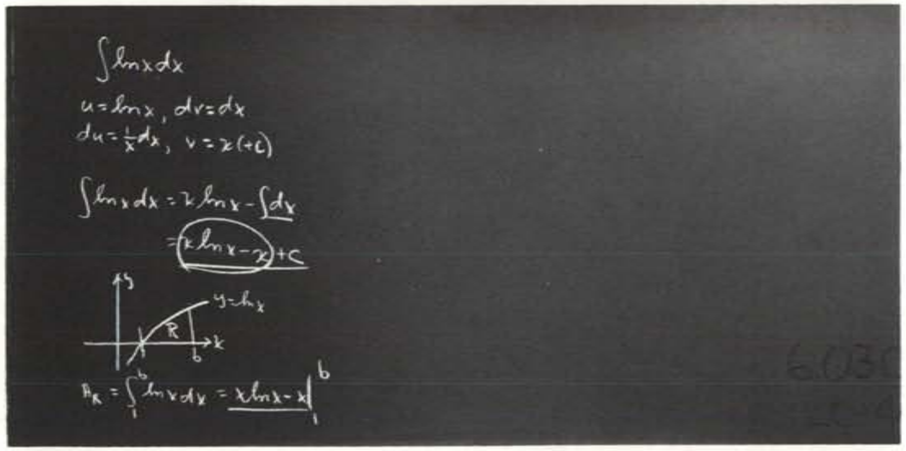
6.030 **Integration by Parts** **26 min.**

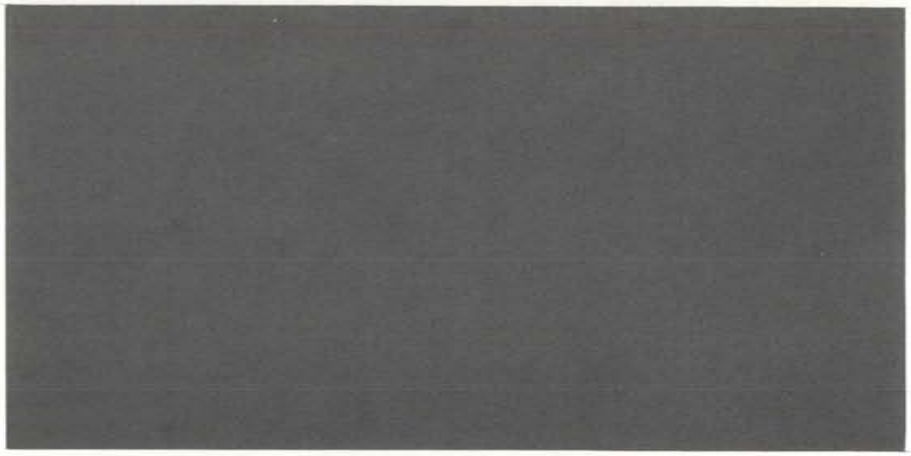




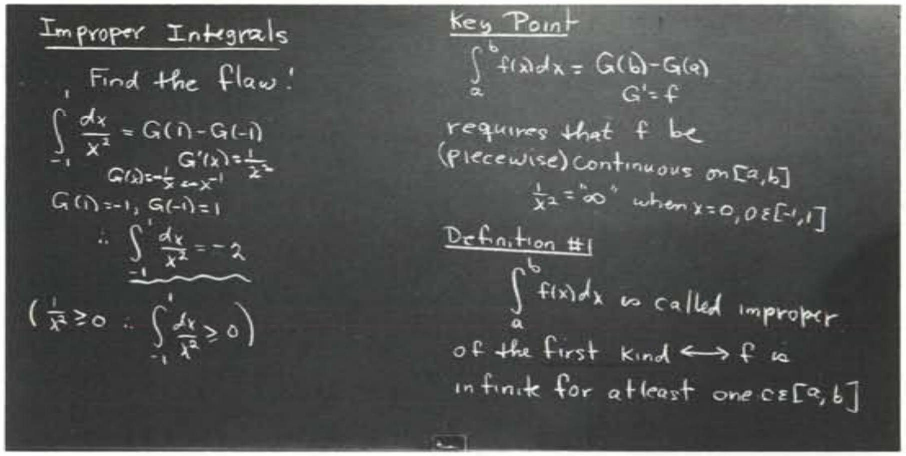

$\int \underbrace{\tan^{-1} x}_{u}\, \underline{dx}$
$\frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}$
$u = \tan^{-1} x,\ dv = dx$
$du = \frac{dx}{1+x^2},\ v = x$
$\int \tan^{-1} x\, dx =$
$x \tan^{-1} x - \int \frac{x\,dx}{1+x^2} =$
$x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$
$u = \tan^{-1} x \qquad dv = dx$
$du = \frac{dx}{1+x^2} \qquad V = x + C_1$
$\int \tan^{-1} x\, dx = (x + C_1) \tan^{-1} x$
$- \int \frac{(x + C_1)\, dx}{1+x^2}$
$= x \tan^{-1} x + C_1 \tan^{-1} x$
$- \int \frac{x\,dx}{1+x^2} - C_1 \tan^{-1} x$

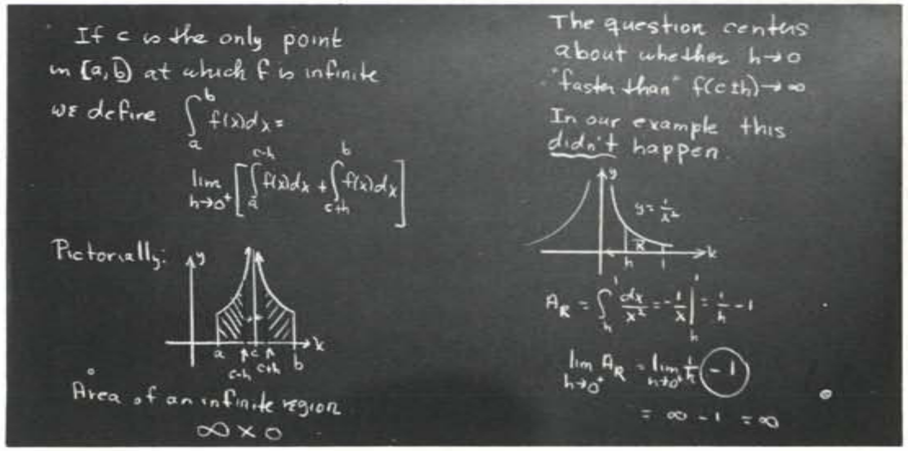


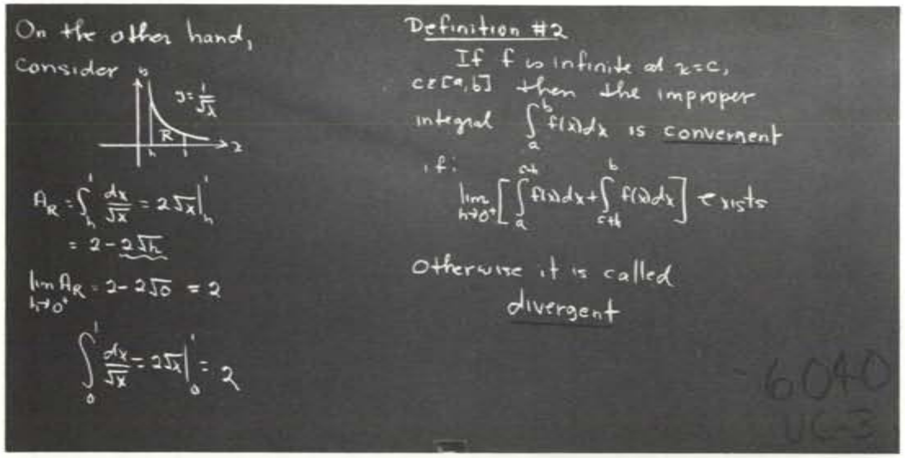

$\int \ln x\, dx$
$u = \ln x,\ dv = dx$
$du = \frac{1}{x} dx,\ v = x (+C)$
$\int \ln x\, dx = x \ln x - \int dx$
$= x \ln x - x + C$
y
$y = \ln x$
R
x
1
b
$A_R = \int_1^b \ln x\, dx = x \ln x - x \Big|_1^b$

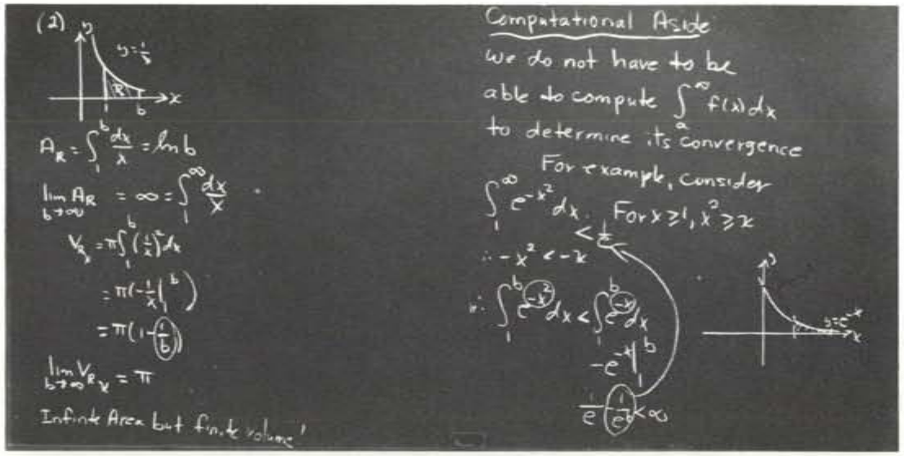


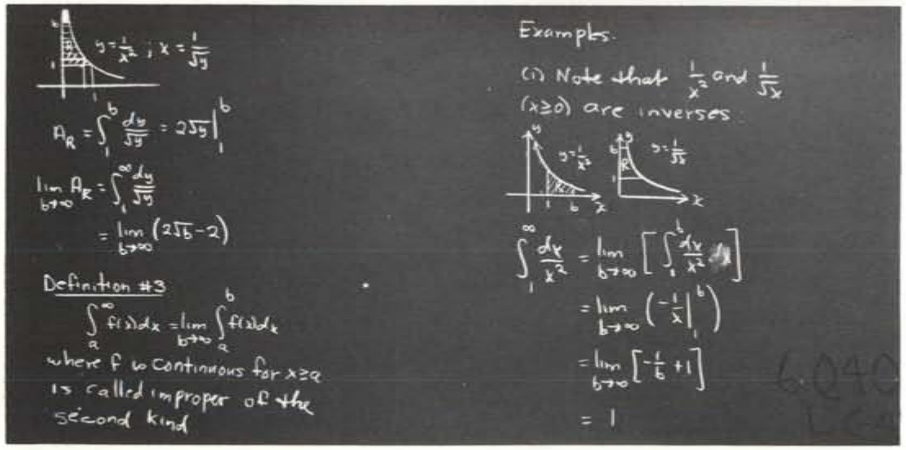
6.040 **Improper Integrals** **29 min.**

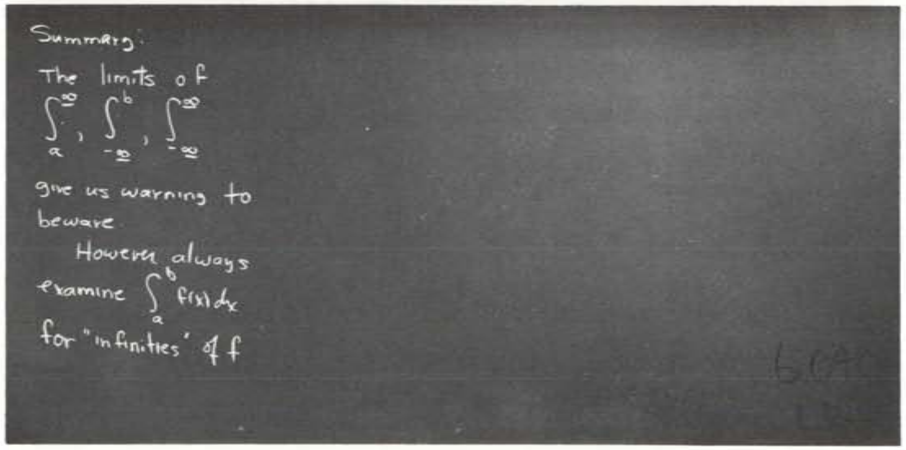





On the other hand,
consider
$y = \frac{1}{\sqrt{x}}$
$A_R = \int_h^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_h^1$
$= 2 - 2\sqrt{h}$
$\lim_{h\to 0^+} A_R = 2 - 2\sqrt{0} = 2$
$\int_0^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_0^1 = 2$
Definition #2
If f is infinite at x=c, c∈[a,b] then the improper integral $\int_a^b f(x)dx$ is convergent
if:
$\lim_{h\to 0^+}\left[\int_a^{c-h} f(x)dx + \int_{c+h}^b f(x)dx\right]$ exists
Otherwise it is called
divergent
6040
UC-3

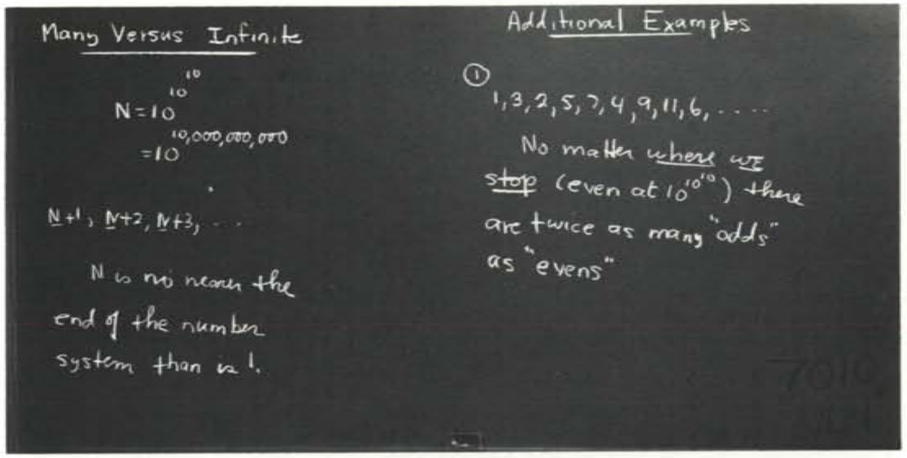

(2)
$y = \frac{1}{x}$
$A_R = \int_1^b \frac{dx}{x} = \ln b$
$\lim_{b\to\infty} A_R = \infty = \int_1^\infty \frac{dx}{x}$
$V_{R_x} = \pi\int_1^b \left(\frac{1}{x}\right)^2 dx$
$= \pi\left(-\frac{1}{x}\Big|_1^b\right)$
$= \pi\left(1 - \frac{1}{b}\right)$
$\lim_{b\to\infty} V_{R_x} = \pi$
Infinite Area but finite volume!
Computational Aside
We do not have to be able to compute $\int_a^\infty f(x)dx$ to determine its convergence
For example, consider
$\int_1^\infty e^{-x^2}dx$. For $x \geq 1$, $x^2 \geq x$
$\therefore -x^2 < -x$
$\therefore \int_1^b e^{-x^2}dx < \int_1^b e^{-x}dx$
$-e^{-x}\Big|_1^b$
$\frac{1}{e} - \frac{1}{e^b} < \infty$
$< \frac{1}{e}$
$y = e^{-x}$

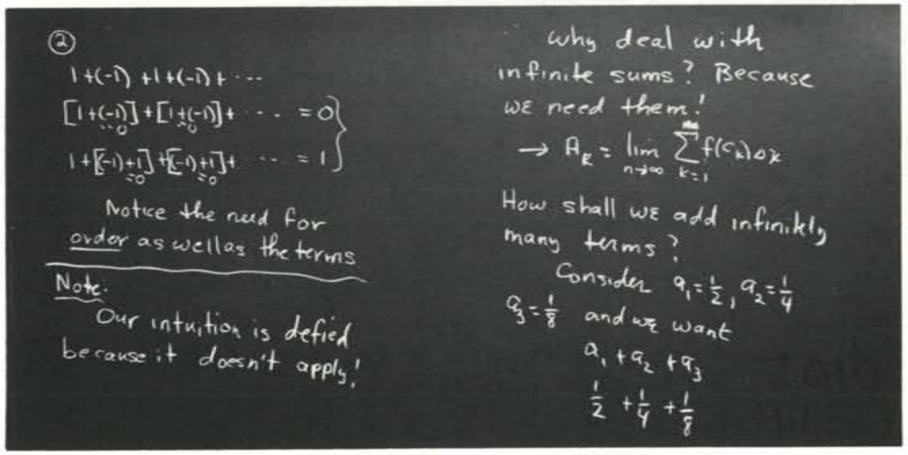

$y = \frac{1}{x^2}$; $x = \frac{1}{\sqrt{y}}$
$A_R = \int_1^b \frac{dy}{\sqrt{y}} = 2\sqrt{y}\Big|_1^b$
$\lim_{b\to\infty} A_R = \int_1^\infty \frac{dy}{\sqrt{y}}$
$= \lim_{b\to\infty}(2\sqrt{b} - 2)$
Definition #3
$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$
where f is continuous for x≥a is called improper of the second kind
Examples.
(1) Note that $\frac{1}{x^2}$ and $\frac{1}{\sqrt{x}}$ (x≥0) are inverses
$y = \frac{1}{x^2}$
$y = \frac{1}{\sqrt{x}}$
$\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty}\left[\int_1^b \frac{dx}{x^2}\right]$
$= \lim_{b\to\infty}\left(-\frac{1}{x}\Big|_1^b\right)$
$= \lim_{b\to\infty}\left[-\frac{1}{b} + 1\right]$
$= 1$
6040


Summary:
The limits of
$\int_a^\infty, \int_{-\infty}^b, \int_{-\infty}^\infty$
give us warning to beware.
However always examine $\int_a^b f(x)dx$
for "infinities" of f

Block VII: Infinite Series

7.010 **Many Versus Infinite** **26 min.**





$s_1 = \frac{1}{2}$

$s_2 = \frac{1}{2} + \frac{1}{4} = s_1 + a_2 = \frac{3}{4}$

$s_3 = s_2 + a_3 = \frac{3}{4} + \frac{1}{8} = \frac{7}{8}$

$= a_1 + a_2 + a_3$

(Do not confuse the a's and <u>s's</u>. The sequence of numbers being added is a_1, a_2, a_3 $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right)$ while the partial sums is the sequence s_1, s_2, s_3 $\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{8}\right)$. The sum is <u>the number</u> $s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$)

Generalized, if $a_n = \frac{1}{2^n}$

then $s_n = a_1 + \cdots + a_n$

$= \frac{1}{2} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$

$= 1 - \frac{1}{2^n}$

For example:

$$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{1024} = 1 - \frac{1}{1024}$$

So, how about

$$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots$$

ENDLESSLY?

$s_n = 1 - \frac{1}{2^n}$

$\lim_{n\to\infty} s_n = 1$

Define $\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{4} + \cdots\right) = 1$

Notice same limit theorems as before apply.

Example:

$$\lim_{n\to\infty} \left(\frac{2n+3}{5n+7}\right) = \lim_{n\to\infty} \left[\frac{2 + \frac{3}{n}}{5 + \frac{7}{n}}\right]$$

$$= \frac{2}{5}$$

$\sum_{n=1}^{\infty} \left(\frac{2n+3}{5n+7}\right) = \infty$ since "after a while" each term behaves like $\frac{2}{5}$. That is,

$$\frac{5}{12} + \frac{7}{17} + \frac{9}{22} + \frac{11}{27} + \cdots$$

That is:

$\sum_{n=1}^{\infty} a_n$ converges $\rightarrow \lim_{n\to\infty} a_n = 0$

(For if $\lim_{n\to\infty} a_n = L \neq 0$, then

$\sum_{n=k}^{\infty} a_n$ "behaves like" $\sum_{n=k}^{\infty} L$

On the other hand $\lim_{n\to\infty} a_n = 0$ is not enough to guarantee the convergence of $\sum_{n=1}^{\infty} a_n$

<u>Example</u>

$$\underbrace{\frac{1}{1}}_{} + \underbrace{\frac{1}{2} + \frac{1}{2}}_{1} + \underbrace{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}}_{1} + \underbrace{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}}_{1} + \cdots$$

Pictorially:

After "a while" all s_n's are within ϵ of 1

Basic Definition

The infinite sequence $b_1, b_2, b_3, \ldots = \{b_n\}$ is said to converge to the limit L (written $\lim_{n\to\infty} b_n = L$) $\Longleftrightarrow$ for $\epsilon > 0$, we can find $N(\epsilon)$ such that

$$n > N \rightarrow |a_n - L| < \epsilon$$

Again Pictorially

$\lim_{n\to\infty} a_n = L$

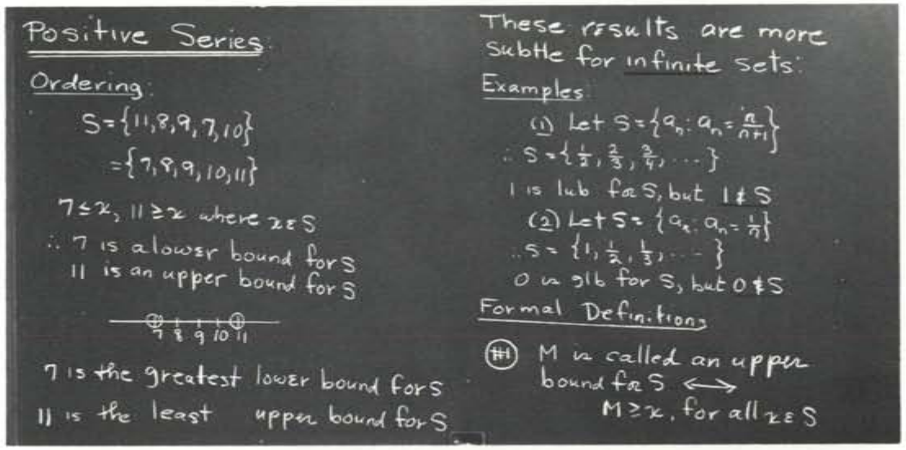
limit : infinite sequence

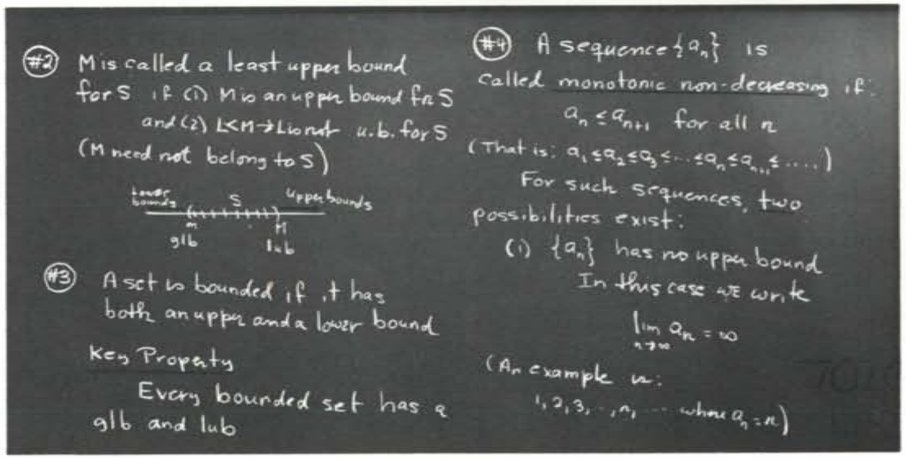
as

last term: finite sequence

Limit replaces infinitely many points by a finite number of points ~ a "dot"

7.020 **Positive Series** **34 min.**





(2) $\{a_n\}$ has an upper bound

Then $\lim_{n\to\infty} a_n = L$ where

$L = \text{lub}\{a_n\}$

"Proof"

at least one a_n, say a_N, is in $(L-\epsilon, L)$

$L-\epsilon \quad L \quad L+\epsilon$

no a_n is in $(L, L+\epsilon)$

$n > N \rightarrow a_n \geq a_N$

$\therefore n \geq N \longrightarrow L-\epsilon < a_N \leq a_n \leq L$

$\rightarrow |a_n - L| < \epsilon$

$\rightarrow \lim_{n\to\infty} a_n = L$

Positive Series

If $a_n \geq 0$ for each n, then $\sum_{n=1}^{\infty} a_n$ is called a positive series.

In this case the sequence of partial sums is monotonic non-decreasing.

Therefore:

If $\sum_{n=1}^{\infty} a_n$ is a positive series, it either diverges to ∞ or it converges to the limit L where L is the lub for the sequence of partial sums

Comparison Test

Suppose $\sum_{n=1}^{\infty} c_n$ is a convergent positive series and

$0 \leq u_n \leq c_n$ for each n

Then $\sum_{n=1}^{\infty} u_n$ also converges

Proof:

Let $\left.\begin{matrix} T_n = c_1 + \cdots + c_n \\ S_n = u_1 + \cdots + u_n \end{matrix}\right\}$ $S_n \leq T_n$ for each n

$\lim_{n\to\infty} T_n = T = \text{lub}\{T_n\}$

$\therefore S_n \leq T$ for each n

$\therefore S_n$ is bounded (and monotone-non-decreasing) $\therefore \lim_{n\to\infty} S_n$ exists

Notes:

(1) The condition $0 \leq u_n \leq c_n$ for all n can be weakened to $0 \leq u_n \leq c_n$ for all $n \geq N$ (i.e. for n "sufficiently large") since convergence depends on the "end" of the sequence

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

$\rightarrow 10^6 + \frac{1}{3} + \frac{1}{5} + \frac{1}{9}\Big\{ + \frac{1}{17} + \cdots$

(2) If $u_n \geq d_n$ where $\sum_{n=1}^{\infty} d_n$ is a positive divergent series then $\sum_{n=1}^{\infty} u_n$ also diverges (since its convergence would imply $\sum_{n=1}^{\infty} d_n$ converges)

Ratio Test

Given $\sum_{n=1}^{\infty} a_n$, form the sequence $\left\{\frac{a_{n+1}}{a_n}\right\} = \{u_n\}$

(For example, given $\sum_{n=1}^{\infty} \frac{10^n}{n!}$, u_n is given by $\frac{10^{n+1}}{(n+1)!} \div \frac{10^n}{n!} = \frac{10}{n+1}$)

Now, assuming $\lim_{n\to\infty} u_n$ exists, call it ρ. Then:

$\sum_{n=1}^{\infty} a_n$ converges if $\rho < 1$

diverges if $\rho > 1$

If $\rho = 1$, test fails

Note: Even if $u_n < 1$ for every n, ρ may still equal 1.

For example, $\lim_{n\to\infty} \frac{n}{n+1} = 1$, but $\frac{n}{n+1} < 1$

Geometric "Proof" for $\rho < 1$

$\rho-\epsilon \quad \rho+\epsilon$

$\rho \quad r \quad 1$

There exists N such that

$n \geq N \rightarrow \frac{a_{n+1}}{a_n} < r \; (<1)$

$\therefore \frac{a_{N+1}}{a_N} < r \quad \therefore a_{N+1} < r a_N$

$\frac{a_{N+2}}{a_{N+1}} < r \quad \therefore a_{N+2} < r a_{N+1} < r^2 a_N$

$\sum_{N+1}^{\infty} a_n < a_N \sum_{N+1}^{\infty} r^n$ Convergent geometric series

Integral Test

Series and Improper Integrals

Suppose there is a decreasing continuous function $f(x)$ such that $f(n) = u_n$ is the n^{th} term of the positive series $u_1 + u_2 + \cdots + \cdots$

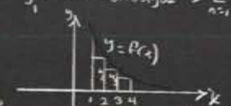
Then: $\sum_{n=1}^{\infty} u_n$ converges

if and only if $\int_1^{\infty} f(x)\,dx$ converges

"Proof"

$u_1 + \cdots + u_n > \int_1^{n+1} f(x)\,dx$

$\therefore \int_1^{\infty} f(x)\,dx$ diverges $\rightarrow \sum_{n=1}^{\infty} u_n$ diverges

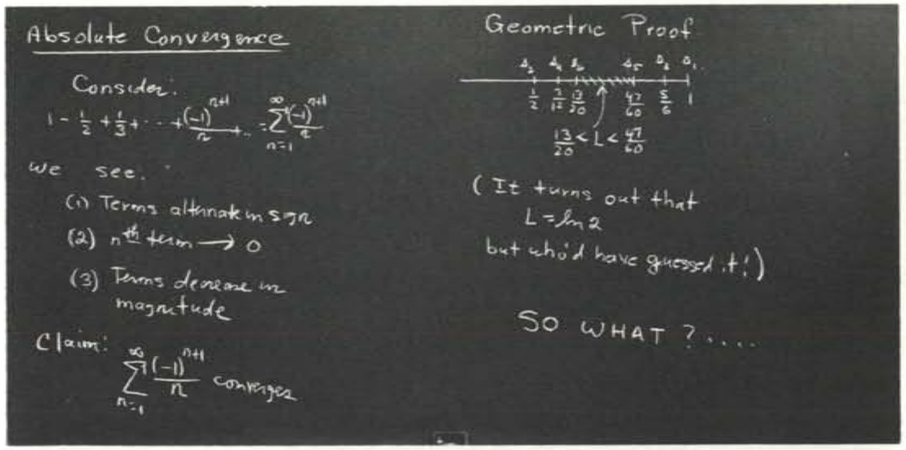


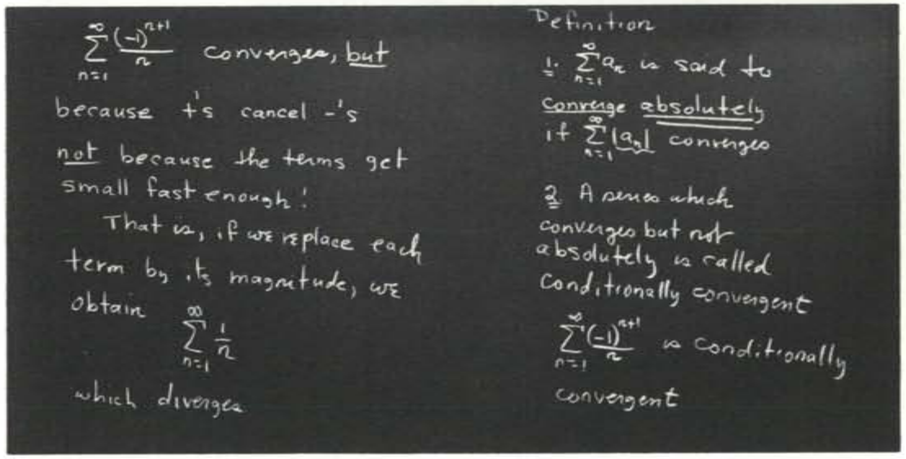
$u_2 + \cdots + u_n < \int_1^{n} f(x)\,dx \nearrow \int_1^{\infty} f(x)\,dx$

$\sum_{n=1}^{\infty} u_n \leftarrow u_1 + \cdots + u_n < u_1 + \int_1^{n} f(x)\,dx$

$\therefore \int_1^{\infty} f(x)\,dx$ converges $\rightarrow \sum_{n=1}^{\infty} u_n$ converges

7.030 Absolute Convergence **21 min.**





The Subtlety of Conditional Convergence

The sum of a cond. conv. series <u>depends</u> on the <u>order</u> of the terms

<u>Example</u>

Divide the terms of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ into 2 "teams" (sets)

$\rightarrow P = \{1, \frac{1}{3}, \frac{1}{5}, \ldots \frac{1}{2n-1}, \ldots\}$

$\rightarrow N = \{-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{6}, \ldots -\frac{1}{2n}\}$

Both $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ and $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverge to ∞

Let us, for example, make the sum $\frac{3}{2}$

We start with P and write $1 + \frac{1}{3} + \frac{1}{5} + \cdots$ until the sum first exceeds (or equals) $\frac{3}{2}$

(This must happen since $1 + \frac{1}{3} + \frac{1}{5} + \cdots$ diverges to ∞)

In particular:

$1 + \frac{1}{3} + \frac{1}{5} = \frac{23}{15} > \frac{3}{2}$

We next "annex" members of N until the sum falls below $\frac{3}{2}$

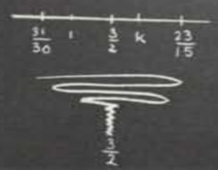
$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} = \frac{31}{30} < \frac{3}{2}$

Continue with P,

$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} = \frac{247}{210} < \frac{3}{2}$

$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \cdots + \frac{1}{13} = \frac{131{,}093}{90{,}090} < \frac{3}{2}$

$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \cdots + \frac{1}{15} = \frac{2{,}056{,}485}{1{,}351{,}350} = k > \frac{3}{2}$



($\frac{3}{2}$ was not important – though the arithmetic gets "messier" if we choose a larger number)

<u>Summary – So Far</u>:

If $\sum_{n=1}^{\infty} a_n$ is cond. conv, <u>its limit exists</u>, but the limit changes as the order of the terms is changed. That is, rearranging the terms changes the series.

This is not true in "finite" arithmetic!

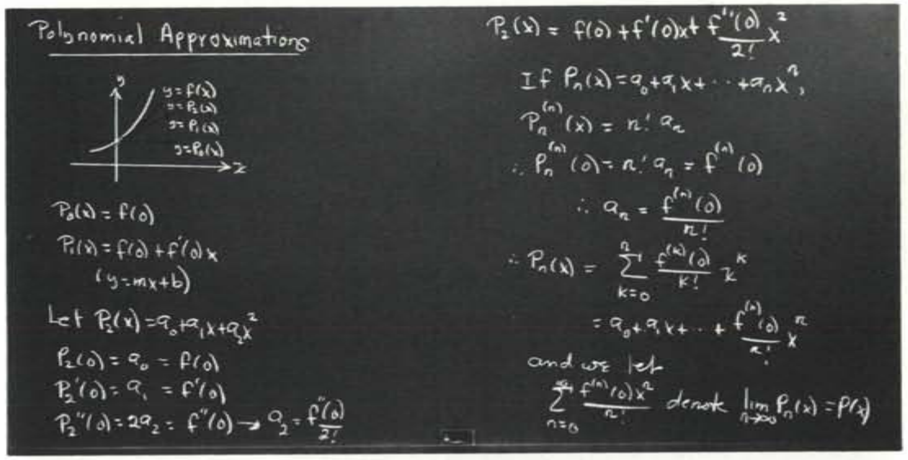
The moral:

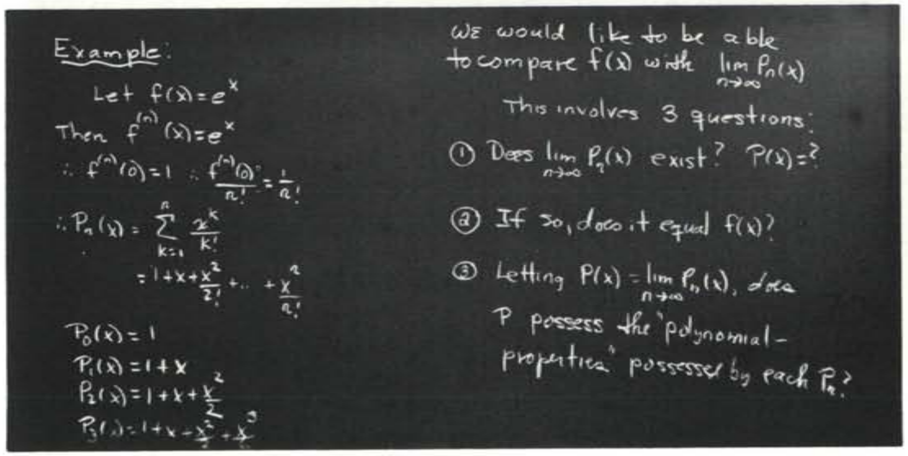
DON'T "MONKEY" WITH CONDITIONAL CONVERGENCE

The beauty of absolute convergence is that the sum is the same for every rearrangement of the terms!

Details are left to the supplementary notes!

7.040 **Polynomial Approximations** 32 min.





Counter-Examples

① Let $f(x)=\frac{1}{1-x}$

$f'(x)=(1-x)^{-2}$

$f''(x)=2(1-x)^{-3}$

$f'''(x)=3!(1-x)^{-4}$

$f^{(n)}(x)=n!(1-x)^{-n-1}$

$\frac{f^{(n)}(0)}{n!}=1$

$\therefore f(x)=\frac{1}{1-x}\to P_n(x)=1+x+\cdots+x^n$

Now:

$f(2)=-1$

$P(2)=\lim_{n\to\infty}P_n(2)=\infty$

② Let $f(x)=\begin{cases}x^2 & \text{if } x\le 2\\ 4 & \text{if } x>2\end{cases}$

$f(0)=0,\ f'(0)=0,\ f''(0)=2$

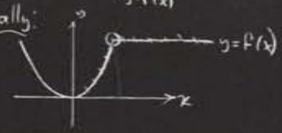
$f^{(n)}(0)=0,\ n>2$

$\therefore P_0(x)=0,\ P_1(x)=0,\ P_2(x)=x^2$

$P_n(x)=x^2,\ n>2$

$\therefore P(x)=\lim_{n\to\infty}P_n(x)=x^2\not\equiv f(x)$

Pictorially:



② $\sum\frac{|x|^n}{n!}$ yields

$u_n=\frac{|x|^n}{n!},\ u_{n+1}=\frac{|x|^{n+1}}{(n+1)!}$

$\frac{u_{n+1}}{u_n}=\frac{|x|}{n+1}$

$\rho=\lim_{n\to\infty}\frac{|x|}{n+1}=0<1$ for all x

$\therefore \sum\frac{x^n}{n!}$ converges for $|x|<\infty$

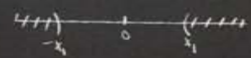
In general, given $\sum a_nx^n$ there exists M such that

$\sum a_nx^n\begin{cases}\text{convs. abs} & \text{if } |x|<M\\ \text{diverges} & \text{if } |x|>M\end{cases}$

Pictorially:

$\sum a_nx^n$

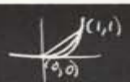
(If $\sum a_nx_1^n$ converges then $\sum a_nx^n$ converges abs if $|x|<|x_1|$)



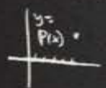
(If $\sum a_nx_1^n$ diverges then $\sum a_nx^n$ diverges if $|x|>|x_1|$

③ Let $P_n(x)=x^n$

dom $P_n=[0,1]$



Then $P(x)=\lim_{n\to\infty}P_n(x)$



$=\begin{cases}0, & \text{if } 0\le x<1\\ 1, & \text{if } x=1\end{cases}$

$\therefore$ Each P_n is continuous when $x=1$ but P isn't!

The Ratio Test and Absolute Convergence:

$\sum_{n=0}^{\infty}a_nx^n$ converges if $\sum_{n=0}^{\infty}|a_nx^n|$ does.

But we may test $\sum|a_nx^n|$ for convergence by the ratio test etc.

Examples

(i) $\sum_{n=0}^{\infty}|x|^n$ converges $\Longleftrightarrow |x|<1$

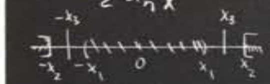
(Let $u_n=|x|^n,\ u_{n+1}=|x|^{n+1}$

$\therefore \frac{u_{n+1}}{u_n}=|x|$

$\therefore \rho=\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}|x|<1$)

$\therefore \sum_{n=0}^{\infty}x^n$ converges (absolutely) if $|x|<1$

$\sum a_nx^n$



Suppose $\sum a_nx_1^n$ conv

$\sum a_nx_2^n$ div

"Taylor's Theorem with Remainder"

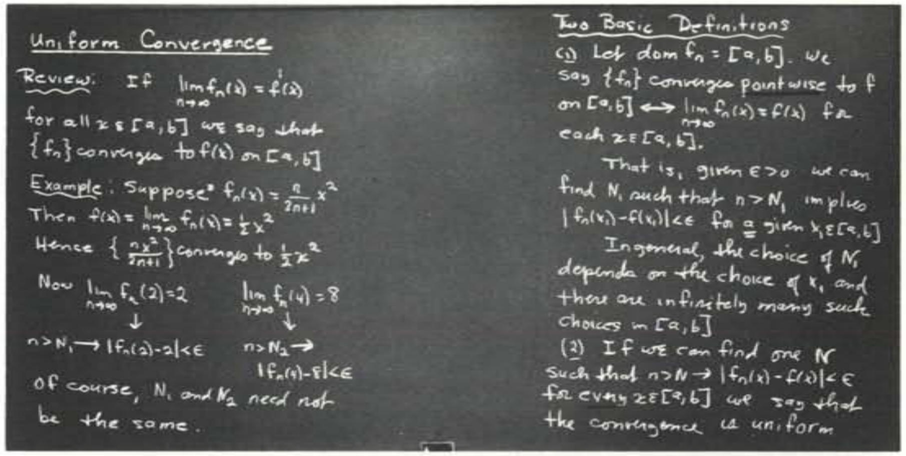
Using integration by parts repeatedly (as shown in our text) it follows that if f and its first $n+1$ derivatives exist at $x=0$, then

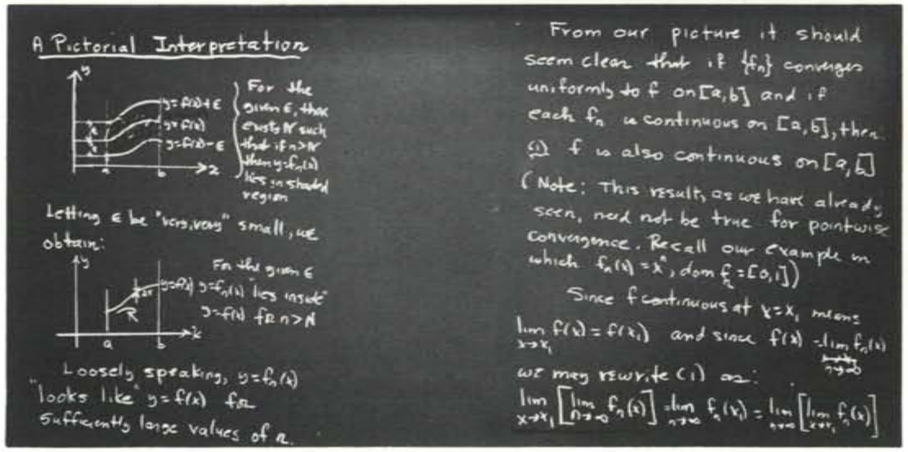
$$f(x)=\underbrace{f(0)+\cdots+\frac{f^{(n)}(0)x^n}{n!}}_{P_n(x)}+R_n(x)$$

where $R_n(x)=\int_0^x\frac{(x-t)^n}{n!}f^{(n+1)}(t)\,dt$

$$\boxed{f(x)=P(x)\Longleftrightarrow\lim_{n\to\infty}R_n(x)=0}$$

7.050 Uniform Convergence **28 min.**





(2) $\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$

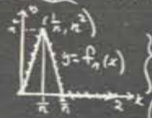
$= \int_a^b \lim_{n\to\infty} f_n(x)\,dx$

These two results are proven as theorems in the supplementary notes. A theorem about differentiation is also proved.

(Differentiation is in a way more subtle than integration since it requires "smoothness" as well as "unbrokenness")

__Note__: (2) need not be true if the convergence is not uniform

__Example__: Define f_n on $[0,2]$ by:



$$\left.\begin{array}{l} f_n(\frac{1}{n}) = n^2 \\ f_n(x_0) = 0 \text{ if } x_0 \geq \frac{2}{n} \\ f_n(0) = 0 \text{ for each } n \end{array}\right.$$

(a) $\lim_{n\to\infty} f_n(0) = 0$

(b) If $0 < x_0 \leq 2$, we may find N such that $n > N \to \frac{2}{n} < x_0$

$\therefore n > N \to f_n(x_0) = 0$

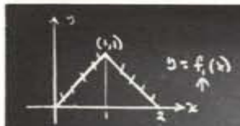
$\therefore \lim_{n\to\infty} f_n(x_0) = 0$

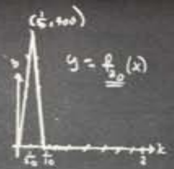
Hence if $\left\{\sum_{k=0}^{n} a_k x^k\right\}$ converges uniformly to $\sum_{n=0}^{\infty} a_n x^n$ then, for example, $\sum_{n=0}^{\infty} a_n x^n$ must be continuous since each $\sum_{k=0}^{n} a_k x^k$ is.

Also if $[a,b]$ is included in the interval of uniform convergence then $\int_a^b \left(\sum_{n=0}^{\infty} a_n x^n\right) dx = \sum_{n=0}^{\infty}\left(\int_a^b \sum_{k=0}^{n} a_k x^k\,dx\right)$

Next time we shall show that within the interval of absolute convergence $\left\{\sum_{k=0}^{n} a_k x^k\right\}$ converges uniformly to $\sum_{n=0}^{\infty} a_n x^n$

In other words within the radius of convergence $\sum_{n=0}^{\infty} a_n x^n$ "enjoys" the "usual" polynomial properties associated with $\sum_{k=0}^{n} a_k x^k$





$\int_0^2 f_n(x)\,dx = A_R = \frac{1}{2}\left(\frac{2}{n}\right)n^2 = n$

$\lim_{n\to\infty}\int_0^2 f_n(x)\,dx = \lim_{n\to\infty} n = \infty$

$\left\{\begin{array}{l}\lim_{n\to\infty} f_n(x_0) = 0 \\ \quad \text{all } x_0 \in [0,2]\end{array}\right.$

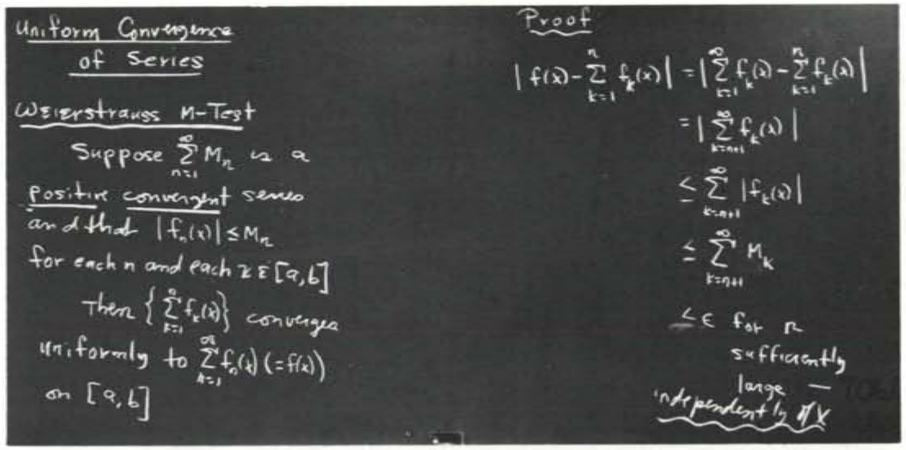
$\therefore \int_0^2 \lim_{n\to\infty} f_n(x)\,dx = \int_0^2 0\,dx = 0$

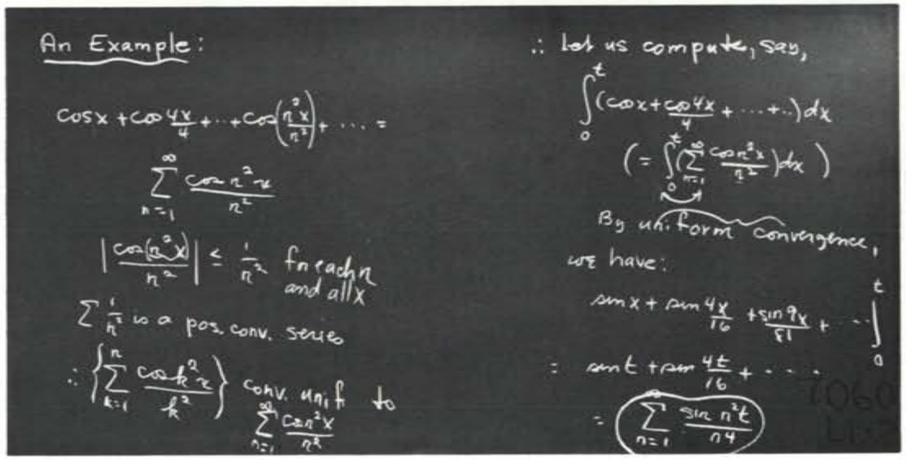
Application of Uniform convergence to series

Recall that $\sum_{n=0}^{\infty} a_n x^n$ is an abbreviation for $\lim_{n\to\infty}\left(\sum_{k=0}^{n} a_k x^k\right)$

That is, $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$ represents the limit of the sequence: a_0, $a_0 + a_1 x$, $a_0 + a_1 x + a_2 x^2$, $\underbrace{a_0 + a_1 x + a_2 x^2 + a_3 x^3, \ldots}_{\text{polynomials}}$

7.060 Uniform Convergence of Series **27 min.**





Application to Power Series

Let $|x_0| < |x_1| < R$ where

$\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < R$

$$\lim_{n\to\infty} a_n x_1^n = 0 \quad \left(\text{Since } \sum a_n x_1^n \text{ converges}\right)$$

$\therefore$ Given $M > 0$, there exists N such that $n > N \rightarrow |a_n x_1^n| < M$

Key idea is:

$$|a_n x_0^n| = \left|a_n \left(\frac{x_0}{x_1}\right)^n x_1^n\right| = |a_n x_1^n| \left|\frac{x_0}{x_1}\right|^n \leq M \left|\frac{x_0}{x_1}\right|^n$$

Now $\left|\frac{x_0}{x_1}\right|$ is a positive constant < 1

$\therefore \sum_{n=0}^{\infty} M \left|\frac{x_0}{x_1}\right|^n$ is a positive convergent (geometric) series

$\therefore$ By Weierstrass M-test

$\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent if $|x| < R$ where R is the radius of convergence for $\sum_{n=0}^{\infty} a_n x^n$

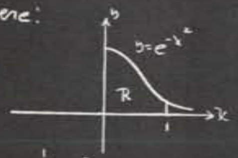
In other words

$$A_R = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} + \cdots$$

$$\doteq 0.748$$

Example:

Find the area of R where:



$A_R = \int_0^1 e^{-x^2}\,dx$, but we do not know (explicitly) $g(x)$ such that $g'(x) = e^{-x^2}$

we do know:

$$e^u = 1 + u + \frac{u^2}{2!} + \cdots + \frac{u^n}{n!} + \cdots \quad \text{for all } u$$

$$\therefore e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} + \cdots + \frac{(-1)^n x^{2n}}{n!}$$

$$\therefore \int_0^1 e^{-x^2}\,dx = \int_0^1 \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\right) dx$$

$$= \sum_{n=0}^{\infty} \left(\int_0^1 \frac{(-1)^n x^{2n}}{n!}\,dx\right) \quad \text{(by unif. conv)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\,n!}\Bigg|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,n!}$$