PROFESSOR: Hi. I'm Herb Gross, and welcome to Calculus Revisited. I guess the most difficult lecture to give with any course is probably the first one. And you're sort of tempted to look at your audience and say you're probably wondering why I called you all here. And in this sense, I have elected to entitle our first lecture simply Preface to give a double overview, an overview both of the hardware and the software that will make up this course.

To begin with, we will have a series of lectures of which this is the first. In our lectures, our main aim will be to give an overview of the material being covered, an insight as to why various computations are done, and insights as to how applications of these concepts will be made. The heart of our course will consist of a regular textbook. You see, we have our lectures. We have a textbook. The textbook is designed to supply you with deeper insights than what we can give in a lecture. In addition, recognizing the fact that the textbook may leave gaps, places where you may want some additional knowledge, we also have supplementary notes. And finally, at the backbone of our package is what we call the study guide.

The study guide consists of a breakdown of the course. It tells us what the various lectures will be, the units. There are pretests to help you decide how well prepared you are for the topic that's coming up. There is a final examination at the end of each block of material. And perhaps most importantly, especially from an engineer's point of view, in each unit that we study, the study guide will consist of exercises primarily called learning exercises, exercises which hopefully will turn you on towards wanting to be able to apply the material, and at the same time, serve as a springboard by which we can highlight why the theory and many about our lecture points are really as important as they are. So much for the hardware of our course. And now let's turn our attention to the software.

Just what is calculus? In a manner of speaking, calculus can be viewed as being high school mathematics with one additional concept called the limit concept thrown in. If you recall back to your high school days, remember that we're always dealing with things like average rate of speed. Notice I say average or constant rate of speed. The old recipe that distance equals rate times time presupposes that the rate is constant, because if the rate is varying, which rate is it that you use to multiply the time by to find the distance?

You see, in other words, roughly speaking, we can say that at least one branch of calculus known as differential calculus deals with the subject of instantaneous speed. And instantaneous speed is a rather easy thing to talk about intuitively. Imagine an object moving along this line and passing the point P. And we say to ourselves how fast was the object moving at the instant that we're at the point P? Now, you see, this is some sort of a problem. Because at the instant that you're at P, you're not in a sense moving at all because you're at P.

Of course, what we do to reduce this problem to an old one is we say, well, suppose we have a couple of observers. Let's call them O1 and O2. Let them be stationed, one on each side of P. Now, certainly what we could do physically here is we can measure the distance between O1 and O2. And we can also measure the time that it takes to go from O1 to O2. And what we can do is divide that distance by the time, and that, you see, is our old high school concept of the average speed of the particle as it moves from O1 to O2.

Now, you see, the question is, somebody says gee, that's a great answer, but it's the wrong problem. We didn't ask what was the average speed as we went from O1 to O2. We asked what was the instantaneous speed. And the idea is we say, well, lookit. The average speed and the instantaneous speed, it seems, should be pretty much the same if the observers were relatively close together.

The next observation is it seems that if we were to move the observers in even closer, there would be less of a discrepancy between O1 and O2 in the sense that-- not a discrepancy, but in the sense that the average speed would now seem like a better approximation to the instantaneous speed because there was less distance for something to go wrong in. And so we get the idea that maybe what we should do is make the observers gets closer and closer together. That would minimize the difference between the average speed and the instantaneous rate of speed, and maybe the optimal thing would happen when the two observers were together.

But the strange part is-- and this is where calculus really begins. This is what calculus is all about. As soon as the observers come together, notice that what you have is that the distance between them is 0. The time that it takes to get from one to the other is 0. And therefore, it appears that if we divide distance by time, we are going to wind up with 0/0.

Now, my claim is that 0/0 should be called-- well, I'll call it undefined, but actually, I think indeterminate would be a better word. Why do I say that? Well, here's an interesting thing. When we do arithmetic with small numbers, observe that if you add two small numbers, you expect the result to be a small number. If you multiply two small numbers, you expect the result to be a small number. Similarly, for division, for subtraction, the difference of two small numbers is a small number.

On the other hand, the quotient of two small numbers is rather deceptive. Because it's a ratio, if one of the very small numbers happens to be very much larger compared with the other small number, the ratio might be quite large. Well, for example, visualize, say, 10 to the minus 6, 1/1,000,000, 0.000001, which is a pretty small number. Now, divide that by 10 to the minus 12th. Well, you see, 10 to the minus 12th is a small number, so small that it makes 10 to the minus sixth appear large. In fact, the quotient is 10 to the sixth, which is 1,000,000.

And here we see that when you're dealing with the ratio of small numbers, you're a little bit in trouble, because we can't tell whether the ratio will be small, or large, or somewhere in between. For example, if we reverse the role of numerator and denominator here, we would still have the quotient of two small numbers, but 10 to the minus 12th divided by 10 to the minus sixth is a relatively small number, 10 to the minus 6. Of course, this is the physical way of looking at it. Small divided by small is indeterminate. We have a more rigorous way of looking at this if you want to see it from a mathematical structure point of view. Namely, suppose we define a/b in the traditional way. Namely, a/b is that number such that when we multiply it by b we get a.

Well, what would that say as far as 0/0 was concerned? It would say what? That 0/0 is that number such that when we multiply it by 0 we get 0. Now, what number has the property that when we multiply it by 0 we get 0? And the answer is any number. This is why 0/0 is indeterminate. If we say to a person, tell me the number I must multiply by 0 to get 0, the answer is any number.

Well, the idea then is that we must avoid the expression 0/0 at all costs. What this means then is that we say OK, let the observers get closer to closer together, but never touch. Now, the point is that as long as the observers get closer and closer together and never touch, let's ask the question how many pairs of observers do we need? And the answer is that theoretically we need infinitely many pairs of observers.

Well, why is that? Because as long as there's a distance between a pair of observers, we can theoretically fit in another pair of observers. This is why in our course we do not begin with this idea, but looking backwards now, we say ah, we had better find some way of giving us the equivalent of having infinitely many pairs of observers. And to do this, the idea that we come up with is the concept called a function.

Consider the old Galileo freely falling body problem, where the distance that the body falls s equals 16t squared, where t is in seconds and s is in feet. Notice that this apparently harmless recipe gives us a way for finding s for each given t. In other words, to all intents and purposes, this recipe gives us an observer for each point of time. For each time, we can find the distance, which is physically equivalent to knowing an observer at every point.

In turn, the study of functions lends itself to a study of graphs, a picture. Namely, if we look at s equals 16t squared again, notice that we visualize a recipe here. t can be viewed as being an input, s as the output. For a given input t, we can compute the output s.

In general, if we now elect to plot the input along a horizontal line and the output at right angles to this, we now have a picture of our relationship, a picture which is called a graph. You see, we can talk about this more explicitly as far as this particular problem is concerned, just by taking a look at a picture like this. In other words, in this particular problem, the input is time t, the output is distance s. For each t, we locate a height called s by squaring t and multiplying by 16.

And now, what average speed means in terms of this kind of a diagram is the following. To find the average speed, all we have to do is on a given time interval find the distance traveled, which I call delta s, the change in distance, and divide that by the change in time. That's the average speed, which, by the way, from a geometrical point of view, becomes known as the slope of this particular straight line. In other words, average speed is to functions what slope of a straight line is to geometry.

At any rate, knowing what the average rate of speed is, we sort of say why couldn't we define the instantaneous speed to be this. We will take the change in distance divided by the change in time and see what happens. And we write this this way. Limit as delta t approaches 0. Let's see what happens as that change in time becomes arbitrarily small, but never equaling 0 because we don't want a 0/0 form here. You see, this then becomes the working definition of what we call differential calculus.

The point is that this particular definition does not depend on s equaling 16t squared. s could be any function of t whatsoever. We could have a more elaborate type of situation. The important point is what? The basic definition stays the same. What changes is the amount of arithmetic that's necessary to handle the particular relationship between s and t. This will be a major part of our course, the strange thing being that even at the very end of our course when we've gone through many, many things, our basic definition of instantaneous rate of change will have never changed from this. It will always stay like this. But what will change is how much arithmetic and algebra and geometry and trigonometry, et cetera, we will have to do in order to compute these things from a numerical point of view.

Well, so much for the first phase of calculus called differential calculus. A second phase of calculus, one which was developed by the Ancient Greeks by 600 BC, the subject that ultimately becomes known as integral calculus, concerns problem of finding area under a curve. Here, I've elected to draw the parabola y equals x squared on the interval from 0, 0 to 1, 0. And the question basically is what is the area bounded by this sort of triangular region? Let's call that region R, and what we would like to find is the area of the region R.

And the Ancient Greeks had a rather interesting title for this type of approach for finding the area. It is both figurative and literal, I guess. It's called the method of exhaustion. What they did was to -- They would divide the interval, say, into n equal parts. And picking the lowest point in each interval, they would inscribe a rectangle. Knowing that the area of the rectangle was the base times the height, they would add up the area of each of these rectangles, and know that whatever that area was, that would have to be too small to be the right answer because that region was contained in R. And that would be labeled A sub n-- lower bar, say-- to indicate that this was a sum of rectangles which was too small to be the right answer.

Similarly, they would then find the highest point in each rectangle, get an overapproximation by adding up the sum of those areas, which they would call A sub n upper bar, and now know that the area of the regions they were looking for was squeezed in between these two. Then what they would do is make more and more divisions, and hopefully, and I think you can see this sort of intuitively happening here, each of the lower approximations gets bigger and fills out the space from inside. Each of the upper approximations gets smaller and chops off the space from outside here. And hopefully, if both of these bounds sort of converge to the same value L, we get the idea that the area of the region R must be L.

This is not anything new. In other words, this is a technique that is some 2,500 years old, used by the Ancient Greeks. Of course, what happens with engineering students in general is that one frequently says, but I'm not interested in studying area. I am not a geometer. I am a physicist. I am an engineer. What good is the area under a curve? And the interesting point here becomes that if we label the coordinate axis rather than x and y, give them physical labels, it turns out that area under a curve has a physical interpretation.

Consider the same problem. Only now, instead of talking about y equals x squared, let's talk about v, the velocity, equaling the square of the time. And say that the time goes to 0 to 1. In other words, if we plot v versus t, we get a picture like this. And the question that comes up is what do we mean by the area under the curve here? And again, without belaboring this point, not because it's not important, but because this is just an overview and we'll come back to all of these topics later in our course, the point I just want to bring out here is, notice that the area under the curve here is the distance that this particle would travel moving at this speed if the time goes from 0 to 1.

And notice what we're saying here. Again, suppose we divide this interval into n equal parts and inscribe rectangles. Notice that each of these rectangles represents a distance. Namely, if a particle moved at the speed over this length of time, the area under the curve would be the distance that it traveled during that time interval. In other words, what we're saying is that if the particle moved at this speed from this time to this time, then moved at this speed from this time to this time, the sum of these two areas would give the distance that the particle traveled, which obviously is less than the distance that the particle truly traveled, because notice that the particle was moving at a speed which at every instance from here to here was greater than this and at every instant from here to here was greater than this.

In other words, in the same way as before, that area of the region R was whittled in between A sub n upper bar and A sub n lower bar, notice that the distance traveled by the particle can now be limited or bounded in the same way. And in the same way that we found area as a limit, we can now find distance as a limit.

And these two things, namely, what? Instantaneous speed and area under a curve are the two essential branches of calculus, differential calculus being concerned with instantaneous rate of speed, integral calculus with area under a curve. And the beauty of calculus, surprisingly enough, in a way is only secondary as far as these two topics are concerned. The true beauty lies in the fact that these apparently two different branches of calculus, one of which was invented by the Ancient Greeks as early as 600 BC, the other of which-- differential calculus-- was not known to man until the time of Isaac Newton in 1690 AD are related by a rather remarkable thing. That remarkable thing, which we will emphasize at great length during our course, is that areas and rates of change are related by area under a curve.

Now, I don't know how to draw this so that you see this thing as vividly as possible, but the idea is this. Think of area being swept out as we take a line and move it, tracing out the curve this way towards the right. Notice that if we have a certain amount of area, if we now move a little bit further to the right, notice that the new area somehow depends on what the height of this curve is going to be. That somehow or other, it seems that the area under the curve must be related to how fast the height of this line is changing.

Or to look at it inversely, how fast the area is changing should somehow be related to the height of this line. And just what that relationship is will be explored also in great detail in the course. And we will show the beautiful marriage between this differential and integral calculus through this relationship here, which becomes known as the fundamental theorem of integral calculus.

At any rate then, what this should show us is that calculus hinges-- whether it's differential calculus or integral calculus, that calculus hinges on something called the limit concept. Again, by way of a very quick review, one of the limit concepts-- and I think it's easy to see geometrically rather than analytically. Imagine that we have a curve, and we want to find the tangent of the curve at the point P. What we can do is take a point Q and draw the straight line that joins P to Q. We could then find the slope of the line PQ.

The trouble is that PQ does not look very much like the tangent line. So we say OK, let Q move down so it comes closer to P. We can then find the slopes of PQ1. We could find the slope of PQ2. But in each case, we still do not have the slope of the line tangent to the curve at P. But we get the idea that as Q gets closer and closer to P, the slope, or the secant line that joins P to Q, becomes a better and better approximation to the line that would be tangent to the curve at P.

In fact, it's rather interesting that in the 16th century, the definition that was given of a tangent line was that a tangent line is a line which passes through two consecutive points on a curve.

Now, obviously, a curve does not have two consecutive points. What they really meant was what? That as Q gets closer and closer to P, the secant line becomes a better and better approximation for the tangent line, and that in a way, if the two points were allowed to coincide, that should give us the perfect answer.

The trouble is, just like you can't divide 0 by 0, if P and Q coincide, how many points do you have? Just one point. And it takes two points to determine a straight line. No matter how close Q is to P, we have two distinct points. As soon as Q touches P, we lose this. And this is what was meant by ancient man or medieval man by his notion of two consecutive points. And I should put this in double quotes because I think you can see what he's begging to try to say with the word "consecutive," even though from a purely rigorous point of view, this has no geometric meaning.

Now, the other form of limit has to do with adding up areas of rectangles under curves. Namely, we divided the curve up into n parts. We inscribed n rectangles, and then we let n increase without bound. In other words, this is sort of a discrete type of limit. Namely, we must add up a whole number of areas, but the sum is endless in the sense that the number of rectangles becomes greater than any number we want to preassign. And the basic question that we must contend with here is how big is an infinite sum? You see, when we say infinite sum, that just tells you how many terms you're combining. It doesn't tell you how big each term, how big the sum will be.

For example, look at the following sum. I will start with 1. Then I'll add 1/2 on twice. Then I'll add 1/3 on three times. And without belaboring this point, let me then say I'll had on 1/4 four times, 1/5 five times, 1/6 six times, et cetera. Notice as I do this that each time the terms gets smaller, yet the sum increases without any bound. Namely, notice that this adds up to 1. This adds up to 1. The next four terms will add up to 1. And as I go out further and further, notice that this sum can become as great is I want, just by me adding on enough 1's.

On the other hand, let's look at this one. 1 plus 1/2 plus 1/4 plus 1/8 plus 1/16 plus 1/32. In other words, I start with 1 and each time add on half the previous number. See, 1 plus 1/2 plus 1/4 plus 1/8. You may remember this as being the geometric series whose ratio is 1/2.

The interesting thing is that now this sum gets as close to 2 as you want without ever getting there. And rather than prove this right now, let's just look at the geometric interpretation here. Take a line which is 2 inches long. Suppose you first go halfway. You're now here. Now go half the remaining distance. That's what? 1 plus 1/2. That puts you over here. Now go half the remaining distance. That means add on 1/4. Now go half the remaining distance. That means add on on 1/8. Now go half the remaining distance. Add up this on 1/16, you see. And ultimately, what happens? Well, no matter where you stop, you've become closer and closer to 2 without ever getting there. And as you go further and further, you can get as close to 2 as you want.

In other words, here are infinitely many terms whose infinite sum is 2. Here are infinitely many terms whose infinite sum is infinity, we should say, because it increases without bound. And this was the problem that hung up the Ancient Greek. How could you do infinitely many things in a finite amount of time? In fact, at the same time that the Greek was developing integral calculus, the famous greek philosopher Zeno was working on things called Zeno's paradoxes. And Zeno's paradoxes are three in number, of which I only want to quote one here. But it's a paradox which shows how Zeno could not visualize quite what was happening.

You see, it's called the Tortoise and the Hare problem. Suppose that you give the Tortoise a 1 yard head start on the Hare. And suppose for the sake of argument, just to mimic the problem that we were doing before, suppose it's a slow Hare and a fast Tortoise so that the Hare only runs twice as fast as the Tortoise. You see, Zeno's paradox says that the Hare can never catch the Tortoise. Why? Because to catch the Tortoise, the Hare must first go the 1 yard head start that the Tortoise had.

Well, by the time the Hare gets here, the Tortoise has gone 1/2 yard because the Tortoise travels half as fast. Now, the Hare must make up the 1/2 yard. But while the Hare makes up the 1/2 yard, the Tortoise goes 1/4 of a yard. When the Hare makes up the 1/4 of a yard, the Tortoise goes 1/8 of a yard. And so, Zeno argues, the Hare gets closer and closer to the Tortoise but can't catch him. And this, of course, is a rather strange thing because Zeno knew that the Tortoise would catch the Hare. That's it's called a paradox. A paradox means something which appears to be true yet is obviously false.

Now, notice that we can resolve Zeno's paradox into the example we were just talking about. For the sake of argument, notice what's happening here with the time. For the sake of argument, let's suppose that the Tortoise travels at 1 yard per second. Then what you're saying is-- I mean, the Hare travels at 1 yard per second. What you're saying is it takes the Hare 1 second to go this distance. Then it takes him 1/2 a second to go this distance, then 1/4 of a second to go this distance. And what you're saying is that as he's gaining on the Tortoise, these are the time intervals which are transpiring. And this sum turns out to be 2.

Now, of course, those of us who had eighth grade algebra know an easier way of solving this problem. We say lookit, let's solve this problem algebraically. Namely, we say give the Tortoise a 1 yard head start. Now call x the distance of a point at which the Hare catches the Tortoise. Now, the Hare is traveling 1 yard per second. The Tortoise is traveling 1/2 yard per second, OK?

So if we take the distance traveled and divided by the rate, that should be the time. And since they both are at this point at the same time, we get what? x/1 equals x minus 1 divided by 1/2. And assuming as a prerequisite that we have had algebra, it follows almost trivially that x equals 2. In other words, what this says is, in reality, that the Hare will not overtake the Tortoise until he catches him, which is obvious. But what's not so obvious is what? That these infinitely many terms can add up to a finite sum.

Well, at any rate, this complete the overview of what our course will be like. And to help you focus your attention on what our course really says, what we shall do computationally is this. In review, we shall start with functions, and functions involve the modern concept of sets because they're relationships between sets of objects. We'll talk about limits, derivatives, rate of change, integrals, area under curves. This will be our fundamental building block.

Once this is done, these things will never change. But the remainder of our course will be to talk about applications, which is the name of the game as far as engineering is concerned. More elaborate functions, namely, how do we handle tougher relationships. Related to the tougher relationships will come more sophisticated techniques. And finally, we will conclude our course with the topic that we were just talking about: infinite series, how do we get a hold of what happens when you add up infinitely many things, each of which gets small.

At any rate, that concludes our lecture for today. We will have a digression in the sense that the next few lessons will consist of sets, things that you can read about at your leisure in our supplementary notes. Learn to understand these because the concept of a set is the building block, the fundamental language of modern mathematics. And then we will return, once we have sets underway, to talk about functions. And then we will build gradually from there. Hopefully, when our course ends, we will have in slow motion gone through today's lesson. This completes our presentation for today. And until next time, goodbye.

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