Calculus Revisited
Part 1
A Self-Study Course

Study Guide

Block II
Differentiation

Center for Advanced
Engineering Study

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CALCULUS REVISITED
PART 1
A Self-Study Course

STUDY GUIDE
Block II
Differentiation

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STUDY GUIDE: Calculus of a Single Variable

TABLE OF CONTENTS

STUDY GUIDE: Calculus of a Single Variable

BLOCK II: DIFFERENTIATION

Content

STUDY GUIDE: Calculus of a Single Variable - Block II: Differentiation

PRETEST

1. Find the rate of change of $\sqrt{x^2 + 16}$ with respect to $x/(x - 1)$ at $x = 3$.

2. To compute the height h of a lamppost, the length a of the shadow of a six-foot pole is measured. The pole is 20 feet from the lamppost. If a = 15 feet, with a possible error of less than one inch, find the height of the lamppost and estimate the error in height.

3. If $x = t - t^2$, $y = t - t^3$, find the values of dy/dx and d^2y/dx^2 at $t = 1$.

4. When air expands adiabatically, the pressure p and volume v satisfy the relationship $pv^{1.4}$ = constant. At a certain instant, the pressure is 50 lb/in^2, and the volume $32in^3$ and is decreasing at the rate of $4in^3/sec$. How rapidly is the pressure changing at this instant?

5. Find the volume of the largest right circular cylinder that can be inscribed in a sphere of radius r.

6. Solve the following differential equation, subject to the prescribed initial conditions.

$$\frac{dy}{dx} = x\sqrt{x^2 - 4},\ x = 2,\ y = 3$$

7. Suppose you know that $f'(x)$ always has a value between -1 and +1. Show that

$$|f(x) - f(a)| \leqslant |x - a|.$$

STUDY GUIDE: Calculus of a Single Variable - Block II:
Differentiation

UNIT 1: Derivatives of Some Simple Functions

1. View: Lecture 2.010

2. Read: Thomas 3.1 and 3.2

3. Exercises :

2.1.1 Find the equation of the line tangent to each of the following curves at the point (1,1).

a. $y = 3x^4 - 2x^2$

b. $y = \frac{x + 1}{x^2 + 1}$

2.1.2(L)

a. Find the equation of the line which is normal to the curve $y = 2x^2$ at the point (1,2).

b. Find the equation of the line which passes through (1,2) and is normal to the curve $y = x^2/4$.

2.1.3(L) A particle projected vertically upward with an initial speed of v_o feet per second reaches an elevation $s = v_ot - 16t^2$ feet at the end of t seconds. For what value of v_o will the particle reach a maximum height of 100 feet?

2.1.4

a. A particle moves along the x-axis according to the rule $x = \frac{16t + 12}{t^2 + 1}$, $0 \leqslant t \leqslant 1$ (x in feet, t in seconds). Where is the particle and how fast is it moving when $t = 0$?

b. Where and at what time does the particle reverse its direction?

2.1.5(L) Use the definition $h'(x) = \lim_{\Delta x \to 0} \left[\frac{h(x + \Delta x) - h(x)}{\Delta x}\right]$ to show that if $h(x) = f(x)g(x)$ and f and g are both differentiable functions, then so also is h, and, in fact,

$$h'(x) = f'(x)g(x) + f(x)g'(x)$$

2.1.6 Use mathematical induction to generalize the result of 2.1.6(L). Namely, if $h(x) = f_1(x)f_2(x)\ldots.f_n(x)$ and $f_1, f_2, \ldots., f_n$ are each differentiable functions then so also is h, and moreover,

$$h'(x) = f_1'(x)f_2(x)\ldots f_n(x) + f_1(x)f_2'(x)\ldots f_n(x) + \ldots + f_1(x)\ldots f_n'(x)$$

that is, each time we differentiate a different factor of $f_1(x)f_2(x)\ldots f_n(x)$.

2.1.7(L) Suppose u and v denote differentiable functions of x. Develop a formula for $\frac{d^n(uv)}{dx^n}$, assuming that the required derivatives all exist. (The resulting formula is known as Leibniz's Rule.)

2.1.8(L) Suppose $f(x) = x^n g(x)$, where $g(x)$ is at least n-times differentiable and $g(0) \neq 0$. Show that $f(0) = f'(0) = \ldots = f^{(n-1)}(0) = 0$, but that $f^{(n)}(0) = n!g(0) \neq 0$.

STUDY GUIDE: Calculus of a Single Variable - Block II:
Differentiation

UNIT 2: Increments and Infinitesimals

1. View: Lecture 2.020
2. Read: Supplementary Notes, Chapter VI, "Infinitesimals and and Differentials", Sections A, B, and C.
3. Read: Thomas 3.4, 3.7, and 3.8
4. Exercises:

2.2.1 In the expression $\Delta y = (\frac{dy}{dx})\ \Delta x + \varepsilon\Delta x$, determine ε as a function of Δx, and show that $\lim_{\Delta x \to 0} \varepsilon = 0$ if $y = x^3$.

2.2.2 With the same notation as in 2.2.1, determine ε if $f(x) = x^4 + 2x^3 + 7$.

2.2.3(L)

a. Approximate $\sqrt{26}$ by investigating $f(x) = \sqrt{x}$ at $x = 25$ and $\Delta x = 1$.

b. In a similar way, approximate $\sqrt[3]{26}$.

2.2.4 Use differentials to find an approximate value for $(1.0006)^3 + (1.0006)^2 + \sqrt{1.0006}$.

2.2.5(L) To compute the height h of a lamppost, the length a of the shadow of a six-foot pole is measured. The pole is 20 feet from the lamppost. If a = 15 feet, with a possible error of less than one inch, find the height of the lamppost and estimate the error.

2.2.6 Approximate by differentials the error in finding the area of a circle whose radius is six feet, if the error in measuring the radius can be as great as 1/2 inch.

2.2.7 Use differentials to approximate the error in finding the volume of a sphere of radius six feet, if the maximum error in measuring the radius is 1/2 inch.

STUDY GUIDE: Calculus of a Single Variable - Block II:
Differentiation

UNIT 3: Composite Functions and the Chain Rule

1. View: Lecture 2.030

2. Read: Thomas 3.5 and 3.6

3. Exercises:

2.3.1(L) Find $f'(x)$ if $f(x) = (x^2 + 1)^3$.

2.3.2 Find $f'(x)$ if $f(x) = (3x^3 + 2x + 1)^4$; if $f(x) = [(3x^3 + 2x + 1)^4 + 2]^5$.

2.3.3(L)

a. Given that $x = f(t)$ and $y = g(t)$ and that f', f'', g', and g'' all exist, find expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{d^2y}{dt^2}$, and $\frac{d^2x}{dt^2}$.

b. The curve C is given by the pair of equations $\begin{matrix} x = t^2 \\ y = t^4 \end{matrix}$ Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

2.3.4 The curve C is defined parametrically by $\begin{matrix} x = t^2 + 1 \\ y = t^3 - t \end{matrix}$ Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point of C which corresponds to $t = 1$. Find the equation of the line tangent to C at the point corresponding to $t = 1$.

2.3.5 A curve is defined by the equations $x = t^5 + t^2 + 1$, $y = t^6 + 2t^3 + 4$. What point on the curve corresponds to the value $t = 1$? Find the equation of the line tangent to this curve at this point.

2.3.6(L) Find the instantaneous rate of change of $x^2 + 16$ with respect to $x^3 - 1$.

2.3.7(L) Find $\frac{dy}{dx}$ if it is known that $y = f(x^2)$ and that $f'(x) = 4x^3 + 1$.

2.3.8

a. Find the derivative of $x^4 + 7x^2 + 8$ with respect to $x^3 - 2x$.

b. Find $\frac{dy}{dx}$ if $y = f(x^3 + 1)$ and $\frac{df(u)}{du} = \frac{u}{u^2 + 1}$.

STUDY GUIDE: Calculus of a Single Variable - Block II:
Differentiation

UNIT 4: Differentiation of Inverse Functions

1. View: Lecture 2.040

2. Read: Supplementary notes, Chapter VI, "Infinitesimals and Differentials," Sections D and E.

3. View: Lecture 2.045

4. Read: Thomas 3.3

5. Exercises

2.4.1(L)

a. Let f be defined by $f(x) = 2x - 7$. Compare the functions g and h where:

$$g(x) = f^{-1}(x)$$

and

$$h(x) = \frac{1}{f(x)}$$

b. How do the derivatives of g and f compare?

2.4.2(L) Find $\frac{dy}{dx}$ if $x^5 + 3x^2y + y^7 = 4$.

2.4.3 Find the equation of the line which is tangent to the curve $x^7 + 5x^3y + y^6 = 7$ at the point (1,1).

2.4.4(L) Show that $\frac{d(x^n)}{dx} = nx^{n-1}$ for any <u>rational</u> number, n.

2.4.5 Find the derivative of $\sqrt{x^2 + 16}$ with respect to $\frac{x}{x-1}$ at $x = 3$.

2.4.6 Use implicit differentiation to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if $x^3 - y^3 = 1$. Check by solving for y as an explicit function of x.

2.4.7(L) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if $x = y^3 + y$.

2.4.8 A particle moves along the x-axis according to the rule $t = x^5 + x^3$ (where x is in feet and t is in seconds). Find the speed and the acceleration of the particle when $x = 1$ and $t = 2$.

2.4.9(L)

a. The general equation of the circle centered at (h,k) with radius r is $(x-h)^2 + (y-k)^2 = r^2$. How must h and k be related if the circle is tangent to the curve $y = x^2 + 1$ at (1,2)?

b. Find h, k, and r in the special case where $\frac{d^2y}{dx^2}$ at (1,2) is the same for both the circle and the curve.

STUDY GUIDE: Calculus of a Single Variable - Block II: Differentiation

UNIT 5: Continuity

1. View: Lecture 2.050

2. Read: Thomas 3.9

3. Exercises

2.5.1(L) Let f be defined by $f(x) = \frac{x^2 - 5x + 6}{x - 2}$

If f continuous at $x = 2$? Explain.

2.5.2 How must $g(3)$ be defined if $g(x) = \frac{x^2 + 2x - 15}{x - 3}$ and g is to be continuous at $x = 3$?

2.5.3(L) In this problem let dom $f = [a,b]$ and let x_1 denote any number such that $a < x_1 < b$.

a. If $f'(x_1)$ exists, show that f is continuous at $x = x_1$.

b. If f is continuous at $x = x_1$ can we conclude that $f'(x_1)$ exists? Explain.

2.5.4 Prove that every polynomial function is continuous.

2.5.5(L) Let f be defined by:

$$f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ mx + b & \text{if } x \geqslant 0, \text{ where both m and b are constants.} \end{cases}$$

a. How must m and b be chosen if f is to be continuous at $x = 0$?

b. How must m and b be chosen if f is to be differentiable at $x = 0$?

2.5.6 Define f by $f(x) = \begin{cases} x^3 \text{ if } x \geqslant 0 \\ x^4 + 1 \text{ if } x < 0 \end{cases}$

Show that f'(0) does not exist.

2.5.7

a. Show that f + g is continuous at x = c if both f and g are continuous at x = c.

b. Generalize part (a).

2.5.8(L) The function f is known to be continuous on the interval [0,1]. It is also known that f(0) is negative and that f(1) is positive. Show that the equation f(x) = 0 must have at least one real root between 0 and 1.

2.5.9 Let $f(x) = \frac{1}{2x-1}$

Then clearly f(0) is negative and f(1) is positive.

a. Does the equation f(x) = 0 have a real root between 0 and 1? Explain.

b. Why doesn't the result of Exercise 2.5.8(L) apply here?

STUDY GUIDE: Calculus of a Single Variable - Block II:
Differentiation

UNIT 6: Applications of the Derivative I

1. View: Lecture 2.060
2. Read: Thomas 4.1, 4.3, and 4.4
3. Exercises

2.6.1 (L)

a. Sketch a smooth curve $y = f(x)$ which has the properties that $f(1) = 0$, $f'(x) > 0$ for $x > 1$ and $f'(x) < 0$ for $x < 1$.

b. Sketch a smooth curve $y = f(x)$ which has the properties that $f(1) = 0$, $f''(x) < 0$ for $x < 1$ and $f''(x) > 0$ for $x > 1$.

2.6.2 The function f has the following properties: (1) its domain is the set of all real numbers and $f(x) > 0$ for all x (2) $f(0) = 3$ and $f'(0) = 0$ (3) $f'(2) = 0$ (4) $f''(1) = 0$ (5) for each x, $f(x) = f(-x)$ (6) $f'(x) > 0$ for $x > 2$ and $f'(x) < 0$ for $0 < x < 2$ (7) $f''(x) > 0$ for $x > 1$ and $f''(x) < 0$ for $0 < x < 1$.

Sketch a curve $y = f(x)$ (i.e., the graph of $f(x)$) which has each of the above properties.

2.6.3(L) Utilize any information that can be deduced from the first and second derivatives as well as other properties of the function to sketch $y = x/(x + 1)$.

2.6.4 Sketch the curve $y = x^4 - 4x^2$.

2.6.5(L)

a. Sketch the curve $y = x + \frac{1}{x}$.

b. Use the result of (a) to deduce that the sum of any positive number and its reciprocal is at least as great as 2.

2.6.6 A positive number is to be added to four times the square of its reciprocal. What is the smallest sum that can be obtained and what number yields this sum?

2.6.7(L) Sketch the curve $y = 2x^3 + 2x^2 - 2x - 1$. From your graph answer the following questions:

a. How many times and approximately where does the curve cross the x-axis (that is, how many real roots are possessed by the equation $2x^3 + 2x^2 - 2x - 1$, and what are their approximate values)?

b. How many times and approximately where would the the curve cross the x-axis if 3 were added to each y-value?

c. Similarly, how many times and approximately where would the curve cross the x-axis if 3 where subtracted from each y-value?

2.6.8(L)

a. What is the value of c if the slope of the curve $y = \frac{c}{x + 1}$ at the point (2, c/3) is equal to 1?

b. How should c be chosen if we want the line $x + y = 3$ to be tangent to the curve $y = \frac{c}{x + 1}$?

UNIT 7: Applications of the Derivative II

1. Read: Thomas 4.2

2. Exercises

2.7.1(L) When air expands adiabatically, the pressure p and the volume v are related by $pv^{1.4}$ = constant. At a certain instant, the pressure is 40 pounds per square inch and the volume is 16 cubic inches. The volume is decreasing at the rate of 2 cubic inches per second. How rapidly is the pressure changing at this instant?

2.7.2 When an ideal gas expands isothermally, the pressure p and the volume v are related by pv = constant. At a certain instant the pressure is 40 pounds per square inch while the volume is 16 cubic inches and is decreasing at the rate of 2 cubic inches per second. How rapidly is the pressure changing at this instant?

2.7.3 A raindrop is always in the form of a sphere. As it falls it accumulates moisture at a rate proportional to its surface area. Show that under these conditions the radius of the raindrop changes at a constant rate.

2.7.4(L) A light is at the top of a pole 50 feet high. A ball is dropped from the same height from a point 30 feet from the light. How fast is the shadow of the ball moving along the ground 1/2 second later? It is assumed that the ball falls a distance of $16t^2$ feet in t seconds.

2.7.5(L) A particle moves along the circle $x^2 + y^2 = 1$ with a velocity whose x-component is always equal to $-y$ at the point (x,y). Describe the y-component of the velocity and determine whether the particle traverses the circle in the clockwise or counter clockwise direction.

2.7.6 A particle moves along the ellipse $4x^2 + 9y^2 = 36$ in such a way that $\frac{dy}{dt} = x$. Find the x-component of the speed of the particle when it is at the point $(1,\frac{4\sqrt{2}}{3})$.

2.7.7 Two ships sail from a point O at right angles to one another. At a certain instant, ship A is 40 miles from O and moving at 20 miles per hour while ship B is 30 miles from O and moving at 25 miles per hour. How fast are the two ships separating at this point?

STUDY GUIDE: Calculus of Single Variable - Block II:
Differentiation

Unit 8: Applications of the Derivative III

1. View: Lecture 2.070
2. Read: Thomas 4.5 and 4.6
3. Exercises

2.8.1 (L) The curve $y = f(x)$ is characterized by the fact that its slope is given by $\frac{dy}{dx} = (x - 1)^4(x - 2)^3(x - 3)^6(x - 4)^8$. For what value(s) of x does the curve possess a relative low point? A relative high point?

2.8.2 A rectangle is inscribed in a circle of radius r. Show that the area of the rectangle is maximum when the rectangle is a square.

2.8.3 (L) Find the right circular cylinder of maximum volume that can be inscribed in a sphere of radius r.

2.8.4 (L)

a. The numbers $c_1, \ldots, c_n$ are recorded in an experiment. Determine the number x for which $(x - c_1)^2 + \ldots + (x - c_n)^2$ will be as small as possible.

b. The four points $(-2,-\frac{1}{2})$, $(0,1)$, $(1,2)$, and $(3,3)$ are observed to lie "fairly close" to a line of the form $y = mx + 1$. How should m be chosen if we wish to minimize the sum $(y_1 - mx_1 - 1)^2 + (y_2 - mx_2 - 1)^2 + (y_3 - mx_3 - 1)^2 + (y_4 - mx_4 - 1)^2$ where (x_1,y_1), (x_2,y_2), (x_3,y_3), and (x_4,y_4) are coordinates of the given points?

2.8.5 (L) A motorist is stranded in a desert 5 miles from a point A which is the point on a long straight road nearest to him. If he can travel 15 mph on the desert and 39 mph on the road, find the point at which he should meet the road to get to a point B on the road in the shortest time if:

[2.8.5 (L) cont'd]

a. B is 5 miles from A

b. B is 9 miles from A

c. B is 1 mile from A

2.8.6 (L) A wire of length L is cut into two pieces, one being bent to form a square and the other to form an equilateral triangle. How should the wire be cut if:

a. the sum of the two areas is to be a minimum?

b. the sum of the two areas is to be a maximum?

2.8.7 The motion of a particle which is moving in a straight line is given by $s = \lambda t - (1 + \lambda^4)t^2$ where λ is a positive constant. Show that the particle moves forward initially but eventually reverses its direction. Show also that if λ can be chosen as an arbitrary constant then in no event can the forward motion of the particle ever exceed 1/8.

STUDY GUIDE: Calculus of a Single Variable - Block II: Differentiation

UNIT 9: Rolle's Theorem and its Consequences

1. View: Lecture 2.080

2. Read: Thomas 4.7 and 4.8

3. Exercises

2.9.1 (L) Let $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ denote any two points on the parabola $y = ax^2 + bx + c$, and $P_3(x_3,y_3)$ be the point on the arc P_1P_2 at which the tangent line is parallel to the straight line which joins P_1 and P_2. Use the mean value theorem to show that $x_3 = (x_1 + x_2)/2$.

2.9.2 Using the expression $f(b) - f(a) = (b - a)f'(c)$, find c if we are given that $f(x) = \sqrt{x}$, $a = 2$, and $b = 4$.

2.9.3 We are told that h(x) has the same derivative as does f(x) where $f(x) = x^3 + 7x^2 + (2x^2 + 1)^5$. We are also told that $h(0) = 3$. Determine h(x).

2.9.4 (L) Suppose $|f'(x)| \leqslant 1$ for all real x. Show that for all real numbers a and b, $|f(b) - f(a)| \leqslant |b - a|$.

2.9.5

a. Use the mean value theorem to prove that if $f'(x) > 0$ for all x then whenever $x_1 < x_2$, $f(x_1) < f(x_2)$.

b. Show that the equation

$$x^3 + x - 11 = 0$$

has exactly one real root and that this root lies between $x = 2$ and $x = 3$.

STUDY GUIDE: Calculus of a Single Variable - Block II: Differentiation

UNIT 10: Anti-derivatives or the Indefinite Integral

1. View: Lecture 2.090

2. Read: Thomas 5.1 through 5.3

3. Exercises

2.10.1 (L)

a. Find $\int (2x + 1)^3 dx$

b. Find $\int (2x + 1)^{100} dx$

2.10.2

a. Find $\int x\sqrt{x^2 - 4}\, dx$

b. The curve $y = f(x)$ passes through the point $(2,3)$ and its slope at any point (x,y) is given by $\frac{dy}{dx} = x\sqrt{x^2 - 4}$. Determine $f(x)$.

2.10.3 The curve C passes through the point $(1,4)$ and has the property that its slope at any point $P(x,y)$ is given by $\frac{dy}{dx} = x^2\sqrt{x^3 + 1}$. What is the equation of the curve C?

2.10.4 (L) We are told that the curve C passes through the point $(2,1)$ and that its slope at any point $P(x,y)$ is given by $\frac{dy}{dx} = \frac{x^2}{y}$. Determine the equation of C.

2.10.5 A curve has the property that its slope at any point $P(x,y)$ is given by:

$$\frac{dy}{dx} = \frac{\sqrt{y^2 + 4}}{y}$$

Find the equation of the curve. What is the equation if we also know that the curve passes through the origin?

2.10.6 A particle moves along the x-axis in such a way that its speed at time t is given by $v = t^2$. At time $t = 0$ the particle is at $x = 1$. Determine x as a function of t.

2.10.7 A particle moves along the x-axis in such a way that its acceleration at any time t is given by $a = -t^2$. Express x as a function of t.

2.10.8 (L) A particle moves along the x-axis in such a way that its acceleration is given by $a = -t^2$. In addition we know that at $t = 0$, the particle is at the origin $(x = 0)$ and has a speed of 9 feet/second in the direction of the positive x-axis.

a. What is the relationship between x and t?

b. What is the maximum displacement of the particle in the direction of the positive x-axis and at what time does this displacement occur?

c. At what time does the particle return to the origin?

d. Sketch the graph of x versus t.

2.10.9 (L) A particle moves in the direction of the positive x-axis at a constant speed of 36 feet/second when it suddenly experiences a deceleration which is proportional at any instant to the square root of the velocity. We also know that the particle comes to rest in 8 seconds. How fast was the particle moving four seconds after it began to decelerate? How far has it travelled during this time? How far does it travel before it comes to a stop?

UNIT 11: The Definite Indefinite Integral

1. View: Lecture 2.100

2. Exercises

2.11.1 Compute each of the following definite integrals.

a. $\int_0^3 (t^2 - 1)dt$

b. $\int_0^2 (x + 1)^2 dx$

c. $\int_1^2 x\sqrt{x^2 + 1}\ dx$

d. $\int_{-1}^1 (4x^3 + 3x^2)dx$

2.11.2(L) A particle moves along the x-axis according to the rule $v = t - t^2$, $0 \leqslant t \leqslant 2$ (v is in f+/sec, t in seconds).

a. Where is the particle at t=2, relative to its position at t=0?

b. What was the total distance travelled by the particle during the two seconds?

2.11.3 A particle moves along the x-axis according to the rule $v = t^2-3t+2$, $0\leqslant t\leqslant 4$ (t in seconds, v in f+/sec).

a. Find the displacement of the particle.

b. Find the total distance travelled by the particle.

(2.11.4(L) Compute $\int_0^1 2(x+1)dx$ and $\int_{-1}^{1} (x+1)dx$. From this conclude that $\int_{-a}^{a} f(x)dx$ need not equal $2\int_0^a f(x)dx$.

QUIZ

1. The line L_1 is tangent to the curve $x^3 + 3xy + y^3 = 5$ at the point (1,1) and the line L_2 is tangent to the curve $y = \frac{x + 1}{x - 1}$ at the point (2,3). Find the point at which L_1 and L_2 intersect.

2. Suppose we are given that $y = t^5 + 2t + 1$ and $t = x^6 + x^5 + 3$. Compute the value of $\frac{d^2y}{dx^2}$ at $x = 1$.

3. Use the notion of differentials to estimate $5\sqrt{33}$. Is your answer greater than or less than the correct answer? Explain.

4. (a) An isosceles triangle is to be inscribed in a circle of radius r. Describe the triangle if the area of the triangle is to be as great as possible.

   (b) What volume would be obtained if the equilateral triangle inscribed in the circle of radius r were revolved through 360 degrees?

   (c) A right circular cone is to be inscribed in a sphere of radius r. Find the height and the radius of the base of the cone if the volume of the cone is to be as great as possible and what is the maximum volume?

5. Sketch the curve $y = x^4 + x^3$. If a particle moves along this curve in such a way that its horizontal speed (i.e., the x-component of its speed) is five feet per second, how fast is the particle rising (i.e., what is the y-component of its speed) at the point (1,2)?

6. Show that the equation $x^5 + 3x^2 = 2$ has exactly one real root in the range $0 < x < 1$.

7. (a) Determine the curve C if it is known that the curve passes through the point (3,4) and that its slope at any point (x,y) is the cube of its y-coordinate.

   (b) The same as (a), only now the slope is given by:

$$\frac{dy}{dx} = x\sqrt[3]{x^2 + 18}$$

8. A particle moves along the x-axis with a speed v given by $v = \frac{dx}{dt} = f(x)$.

   (a) Show that the acceleration of the particle is given by $a = f(x)f'(x)$.

   (b) If the particle starts from rest at the origin and is put in motion in the direction of the positive x-axis in such a way that its acceleration is always equal to the square of its displacement, find its speed when $x = 6$. (Assume that the units of measurement are feet and seconds.)

Calculus of a Single Variable

SOLUTIONS

SOLUTIONS: Calculus of a Single Variable - Block II: Differentiation

Pretest

1. $\frac{-12}{5}$

2. h = 14 ft. with a maximum error of 0.044 ft.

3. $\left.\frac{dy}{dx}\right|_{t=1} = \left.\frac{d^2y}{dx^2}\right|_{t=1} = 2$

4. +8.75 lbs. per in.^{2}/sec.

5. $\frac{4\sqrt{3}\pi r^3}{9}$

6. $y = \frac{1}{3}(x^2 - 4)^{3/2} + 3$

7. $|f(x) - f(a)| = |f'(c)|\ |x - a|$ and $|f'(c)| \leqslant 1 \rightarrow$

$$|f(x) - f(a)| \leqslant |x - a|$$

SOLUTIONS: Calculus of a Single Variable - Block II: Differentiation

UNIT 1: Derivatives of Some Simple Functions

2.1.1

All that has happened in this unit is that we have developed more convenient ways of computing $\frac{dy}{dx}$. We have not changed the meaning of $\frac{dy}{dx}$. Thus in this exercise all we do is compute $\left.\frac{dy}{dx}\right]_{x=1}$ to find the slope of the required line and combine this with the fact that (1,1) is on the line. We then use the "old" recipe:

$$\frac{y-1}{x-1} = \left.\frac{dy}{dx}\right]_{x=1}$$

to find the required equation.

Thus:

(a) $y = 3x^4 - 2x^2 \rightarrow \frac{dy}{dx} = 12x^3 - 4x \rightarrow \left.\frac{dy}{dx}\right]_{x=1} = 8$

$$\therefore \frac{y-1}{x-1} = 8$$

or: $\underline{y = 8x - 7}$ is the equation of the line.

(b) $y = \frac{x+1}{x^2+1} \rightarrow \frac{dy}{dx} = \frac{(x^2+1)(1)-(x+1)2x}{(x^2+1)^2} = \frac{x^2+1-2x^2-2x}{(x^2+1)^2} = \frac{1-2x-x^2}{(x^2+1)^2}$

$$\therefore \left.\frac{dy}{dx}\right]_{x=1} = \frac{1-2-1}{(1+1)^2} = \frac{-2}{4} = -\frac{1}{2}$$

$\therefore \frac{y-1}{x-1} = -\frac{1}{2}$ is the desired equation

or: $2y - 2 = -x + 1$

or: $\underline{x + 2y = 3}$

[2.1.1 cont'd]

(Actually our answer in each case should be called _an_ equation rather than _the_ equation since a line can be represented by several different equations.)

2.1.2(L)

We must, of course, first recall what is meant by a normal to a curve. A normal is roughly equivalent to a perpendicular. That is, if a curve is smooth at some point P then the normal to the curve at P is simply the line which is perpendicular to the line which is tangent to the curve at P. Thus, in terms of calculus, we proceed as in 2.1.1 to find the slope of the tangent line at P and this is $\frac{dy}{dx}$ evaluated at P. We then use the fact that the normal line has its slope equal to the negative reciprocal of the tangent line and proceed to write the equation.

In part (a) we are given that the curve is $y = 2x^2$. Hence $\frac{dy}{dx} = 4x$, and at the point (1,2) this is equal to 4. Thus the line normal to this curve at (1,2) has its slope equal to $-\frac{1}{4}$. Thus the equation of the desired line (and notice that once we have a point and the slope we compute the equation of the line just as we did in Block I, the only difference being that calculus allows us to find the slope more conveniently than by the use of the "delta 'method'") is:

$$\frac{y - 2}{x - 1} = -\frac{1}{4}$$

or:

$$x + 4y = 9.$$

[2.1.2(L) cont'd]

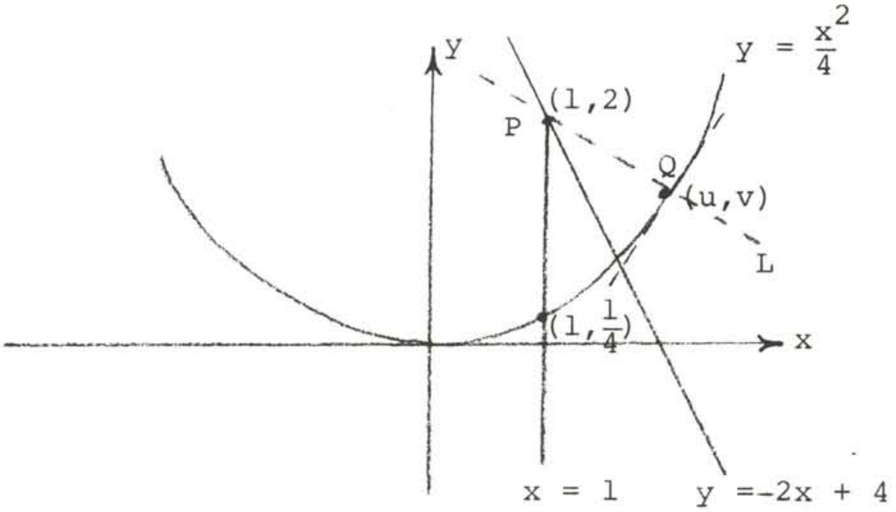
Now with the exception of having to know the meaning of a normal line, part (a) offered us no new wrinkles. Part (b) on the other hand is a horse of a different color. Among other things, let us observe that in this case (1,2) <u>isn't</u> a point on the curve.

Let us also remark at this time that it is always an excellent idea to check whether a given point belongs to a given curve. It is, to say the least, quite embarrassing to find the normal to a curve at a point which is not on the curve. Moreover there are no built-in "alarm" systems in mathematics that tell us that we arrived at an answer by an incorrect method. In other words if we imitated our procedure in part (a) by assuming that (1,2) was on the curve we would obtain a numerical, although, incorrect, answer. That is, we could still see that $\frac{dy}{dx} = \frac{x}{2}$ and this in turn is $\frac{1}{2}$ when $x = 1$. We could still say that the normal line has its slope equal to -2 and we could still form the equation:

$$\frac{y - 2}{x - 1} = -2 \quad \text{or} \quad y = -2x + 4 \text{ (see Figure 1)}$$

but the error would lie in the fact that to be on the curve, the point whose x-coordinate is 1 must have its y-coordinate equal to 1/4. In other words, the point (1,2) lies ABOVE the curve $y = \frac{x^2}{4}$. Pictorially:

[2.1.2(L) cont'd]



(Figure 1)

The line we are seeking must pass through the point (1,2) and it must intersect the curve at some point, say, (u,v). Calling the line L, we see that the slope of L is given by

$$m_L = \frac{v - 2}{u - 1} \tag{1}$$

Now $\left(\frac{dy}{dx}\right)_{(u,v)} = \frac{u}{2}$ $\left\{ \text{That is, } y = \frac{x^2}{4} \rightarrow \frac{dy}{dx} = \frac{x}{2} \rightarrow \left(\frac{dy}{dx}\right)_{x=u} = \frac{u}{2} \right.$

Therefore since L is normal to the curve, its slope at (u,v) must be given by

$$m_L = \frac{-2}{u} \tag{2}$$

[2.1.2(L) cont'd]

Combining (1) and (2) we obtain, since a line can have only one slope:

$$\frac{v - 2}{u - 1} = \frac{-2}{u} \tag{3}$$

Simplifying (3) yields

$$uv = 2 \tag{4}$$

Finally, since (u,v) is on the curve $x^2 = 4y$, it follows that $u^2 = 4v$; or $v = \frac{u^2}{4}$. Putting this result into (4) we obtain

$$u(\frac{u^2}{4}) = 2 \quad \text{or} \quad u^3 = 8 \quad \text{or} \quad \underline{u = 2}$$

$$\therefore v = \frac{2^2}{4} = 1.$$

Hence our normal line meets the curve at (2,1). Its slope is given by $\frac{\Delta y}{\Delta x} = \frac{2-1}{1-2} = -1$, and its equation becomes $\frac{y-2}{x-1} = -1$ (or $\frac{y-1}{x-2} = -1$ since both (1,2) and (2,1) are on the line). Thus the required line has as its equation, $\underline{x + y = 3}$. (An alternative way to see this is that since $m_L = \frac{-2}{u}$ and $u = 2$, $m_L = -1$. Therefore $\frac{y-2}{x-1} = -1$ or $x + y = 3$.)

By the way, if we refer again to Figure 1, notice that given the point P we located the point Q analytically by use of calculus techniques. An interesting related question is how could we have located Q by purely geometric means. Notice that

[2.1.2(L) cont'd]

from P there are (infinitely) many lines that can be drawn to the curve. It is quite natural to define the distance from P to a curve as the length of the shortest straight line that can be drawn from the point to the curve.

Thus, from an experimental point of view, we could draw a line from P to the curve and approximate the position that gives us the shortest line. Without actually doing this experiment it might still not be too difficult to visualize the plausibility of the result that if C is "smooth" (i.e., differentiable) then the shortest line which joins P to C is normal to the curve at the point at which it meets the curve. (This is an extension of the result that the shortest distance from a point to a line is the length of the perpendicular from the point to the line.)

As an application of this observation, we can solve our present exercise by using it. Namely, let (u,v) denote a typical point on $y = \frac{x^2}{4}$. Then $\sqrt{(u - 1)^2 + (v - 2)^2}$ denotes the distance from (1,2) to the curve $y = \frac{x^2}{4}$ at (u,v). To minimize this distance, it is sufficient to minimize the square of the distance (that is, for positive numbers a and b, $a < b$ if and only if $a^2 < b^2$).

If we also recall that $v = u^2/4$, which is the meaning of (u,v) being on the curve $y = \frac{x^2}{4}$, we find that the square of the distance is given by:

$$f(u) = (u - 1)^2 + (\frac{u^2}{4} - 2)^{2*}$$

---

*Clearly we could have worked with $g(u) = \sqrt{(u-1)^2+(\frac{u^2}{4} - 2)^2}$, but "lifting" the square root symbol makes the differentiation simpler from a computational point of view. For this reason it is quite a common practice when we want to minimize (or maximize) a distance, that we work with the square of the distance. The key factor, of course, is that for positive numbers f(x), f(x) is minimum (maximum) if and only if $f^2(x)$ is minimum (maximum).

[2.1.2(L) cont'd]

Recalling that f is smallest when f' is 0 (why?), we differentiate the above relation to obtain:

$$\begin{aligned} f'(u) &= 2(u - 1) + 2(\frac{u^2}{4} - 2)(\frac{2u}{4}) \\ &= 2u - 2 + \frac{u^3}{4} - 2u \\ &= (u^3 - 8)/4 \end{aligned}$$

Therefore $f'(u) = 0$ if and only if $u = 2$ and we have obtained the same result as we obtained in our earlier solution.

2.1.3(L)

In many ways this problem affords us a good review but also presents one or two new computational-type wrinkles. We recall that the particle reaches its maximum height when its velocity is 0. We also know that at any time t its velocity is defined by ds/dt, so that in this case,

$$\frac{ds}{dt} = v_o - 32t \qquad (1)$$

Thus the velocity is 0 when $t = v_o/32$. This, in turn, means that the maximum height is obtained when $t = v_o/32$; and this means that for a given value of v_o, we can find the maximum height of the particle by replacing t by $v_o/32$ in the expression $s = v_ot - 16t^2$. Denoting the maximum height by s_{max}, we have:

[2.1.3(L) cont'd]

$$s_{max} = v_o(v_o/32) - 16(v_o/32)^2$$

$$= v_o^2/64 \qquad (2)$$

Notice that (2) allows us to compute much more than what was asked for in this problem. Namely (2) allows us to compute the maximum height very quickly for any given initial velocity.

Notice, of course, that we can "invert" (2) and write:

$$v_o = 8\sqrt{s_{max}} \qquad (3)$$

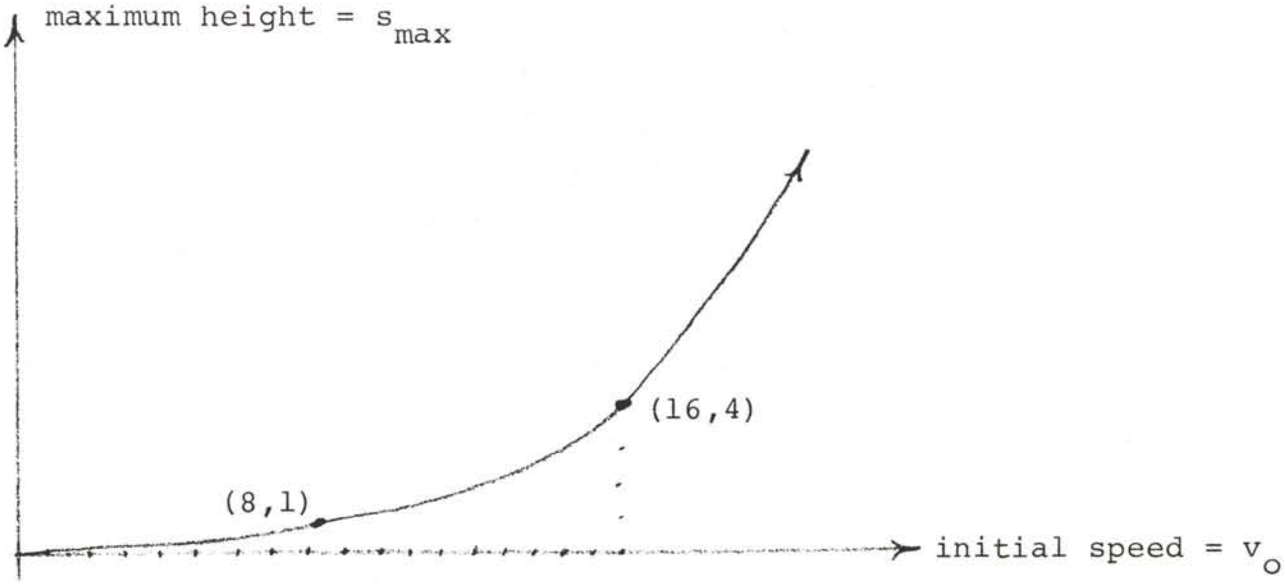
and (3) shows us how to choose v_o once s_{max} is specified. In particular, we obtain the answer to our present problem quite trivially now simply by replacing s_{max} by 100 in (3).

Thus

$$v_o = 8(\sqrt{100}) = 8(10) = 80 \text{ feet Q.E.D.}$$

Also notice how (2) [and for this matter (3)] tells us a great deal in terms of experimental value. For example we see from (2) that the maximum height is proportional to the SQUARE of the initial velocity. Hence we are forewarned that especially for fairly large initial speeds, a small change in speed can produce a large change in maximum height. In terms of a graph:

[2.1.3(L) cont'd]



2.1.4

At any time t the speed of the particle is given by $\frac{dx}{dt}$. Since $x = \frac{16t + 12}{t^2 + 1}$,

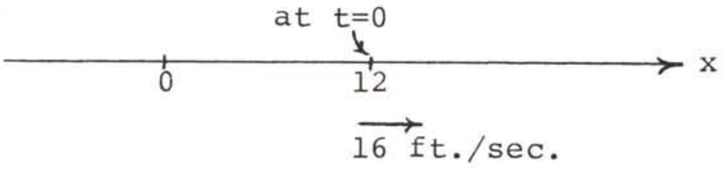
$$\frac{dx}{dt} = \frac{(t^2 + 1)16 - (16t + 12)2t}{(t^2 + 1)^2} = \frac{16 - 24t - 16t^2}{(t^2 + 1)^2}$$

Thus

$$\left.\begin{aligned} x &= \frac{16t + 12}{t^2 + 1} \qquad &(1a) \\ v &= \frac{dx}{dt} = \frac{16 - 24t - 16t^2}{(t^2 + 1)^2} \qquad &(1b) \end{aligned}\right\}$$

[2.1.4 cont'd]

Therefore when $t = 0$, $\begin{cases} x = \dfrac{0+12}{0+1} = 12 \text{ feet} \\ \\ v = \dfrac{16-0-0}{(1+0)^2} = 16 \text{ feet/sec.} \end{cases}$



(b) The particle changes direction when its speed is zero (why?). From equation (1b)

$$\begin{aligned} v = 0 \quad &\leftrightarrow \quad 16-24t-16t^2 = 0 \\ &\leftrightarrow \quad 8(2-3t-2t^2) = 0 \\ &\leftrightarrow \quad 8(1-2t)\ (2+t) = 0 \\ &\leftrightarrow \quad t = \frac{1}{2} \text{ or } t = -2 \end{aligned}$$

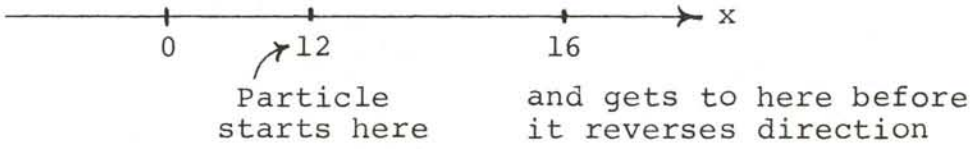
But $t = -2$ is not in the given domain $0 \leqslant t \leqslant 1$.

$\therefore$ Particle reverses direction when $t = \frac{1}{2}$ and from equation (1a) this occurs when

$$x = \frac{16(\frac{1}{2}) + 12}{(\frac{1}{2})^2 + 1} = 16 \text{ feet}$$

[2.1.4 cont'd]

Thus:



2.1.5(L)

Here we wish to derive a result given in the text but in terms of our original definition of a derivative.

Letting $h(x) = f(x)g(x)$, we have

$$h'(x) = \lim_{\Delta x \to 0} \left[\frac{h(x+\Delta x)-h(x)}{\Delta x}\right] = \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x)g(x+\Delta x)-f(x)g(x)}{\Delta x}\right] \qquad (1)$$

Now since f' and g' exist, we know that

$$\lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right] = f'(x)$$

while

$$\lim_{\Delta x \to 0} \left[\frac{g(x+\Delta x)-g(x)}{\Delta x}\right] = g'(x)$$

Somehow or other we would like to make use of these results in (1). To do this we employ our trick of adding 0 in a clever way. This time we write it as $-f(x)g(x+\Delta x) + f(x)g(x+\Delta x)$ to obtain:

[2.1.5(L) cont'd]

$$\frac{f(x+\Delta x)g(x+\Delta x)-f(x)g(x)}{\Delta x}$$

$$= \frac{f(x+\Delta x)g(x+\Delta x)-f(x)g(x+\Delta x)+f(x)g(x+\Delta x)-f(x)g(x)}{\Delta x}$$

$$= \left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]g(x+\Delta x) + f(x)\left[\frac{g(x+\Delta x)-g(x)}{\Delta x}\right]$$

Our last expression gives us a strong hint that we are home free; since as $\Delta x \to 0$,

$$\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right] \to f'(x), \quad \left[\frac{g(x+\Delta x)-g(x)}{\Delta x}\right] \to g'(x)$$

and $g(x+\Delta x) \to g(x)$.

Cleaning things up a bit, we have:

$$h'(x) = \lim_{\Delta x\to 0}\left[\frac{h(x+\Delta x)-h(x)}{\Delta x}\right] = \lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)g(x+\Delta x)-f(x)g(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x\to 0}\left[\left\{\frac{f(x+\Delta x)-f(x)}{\Delta x}\right\} g(x+\Delta x) + f(x)\left\{\frac{g(x+\Delta x)-g(x)}{\Delta x}\right\}\right]$$

We now apply our limit theorems from Block I (and in this sense, differential calculus is a corollary of the limit theorems since we prove all of the formulas for differentiation by writing derivatives as limits and applying the limit theorems) to obtain:

[2.1.5(L) cont'd]

$$h'(x) = \lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]\lim_{\Delta x\to 0} g(x+\Delta x) + \lim_{\Delta x\to 0} f(x)\lim_{\Delta x\to 0}\left[\frac{g(x+\Delta x)-g(x)}{\Delta x}\right]$$

$$= f'(x)g(x)^{*} + f(x)g'(x)$$

Finally observe that the feeling that we might expect $[f(x)g(x)]' = f'(x)g'(x)$ is incorrect. Logically speaking, the hope that the derivative of a product is the product of the derivatives is inconsistent with out definition of a derivative.

In fact even if we didn't know the correct answer we could still show that $f'(x)g'(x) \neq [f(x)g(x)]'$ by means of an example. For instance, let $f(x) = x^2$ and $g(x) = x^3$.

---

*It is not trivial to conclude that $\lim_{\Delta x\to 0} g(x+\Delta x) = g(x)$. In fact if g is not differentiable the result doesn't even have to be true. For example define g by

$$g(x) = \begin{cases} 2 & x \geqslant 1 \\ 0 & x < 1 \end{cases}$$

Then $g(1) = 2$, but $\lim_{\Delta x\to 0^-} g(1+\Delta x) = 0$ (since, then, $1+\Delta x < 1$)

$\therefore \lim_{\Delta x\to 0^-} g(1+\Delta x) \neq g(1)$. However, if g is differentiable then

$$\lim_{\Delta x\to 0}\left[\frac{g(x+\Delta x)-g(x)}{\Delta x}\right] = g'(x)$$

Then the identity $g(x+\Delta x) = \left[\frac{g(x+\Delta x)-g(x)}{\Delta x}\right]\Delta x + g(x)$ yields

$$\lim_{\Delta x\to 0} g(x+\Delta x) = \lim_{\Delta x\to 0}\left[\frac{g(x+\Delta x)-g(x)}{\Delta x}\right]\lim_{\Delta x\to 0}\Delta x + \lim_{\Delta x\to 0} g(x)$$

$$= [g'(x)]\ (0) + g(x)$$

$$= g(x)$$

In other words it is important that we know $g'(x)$ exists in order for our proof to work.

[2.1.5(L) cont'd]

Then $f'(x) = 2x$, $g'(x) = 3x^2$

$$\therefore f'(x)g'(x) = 6x^3 \tag{2}$$

On the other hand,

$$f(x)g(x) = x^2x^3 = x^5$$

$$\therefore [f(x)g(x)]' = 5x^4 \tag{3}$$

A comparison of (2) and (3) shows that $f'(x)g'(x)$ and $[f(x)g(x)]'$ are <u>not</u> synonyms.

Thus what we have really done in this exercise is to show how $[f(x)g(x)]'$ <u>must</u> be defined if we are to be consistent with our definition. Unless we agree to change the definition we cannot avoid the conclusion obtained in this exercise. It is like saying that once we agree on the number of inches in a foot and on the number of feet in a yard, we are no longer free to decide arbitrarily on how many inches there are in a yard.

2.1.6

From Exercise 2.1.5(L) we know the result is true for $n = 2$. So we assume the result to be true for $n = k$, and we need only prove that this assumption implies the result is true for $n = k + 1$.

[2.1.6 cont'd]

Thus we need only show that

$$[f_1(x)\dots f_{k+1}(x)]' = f_1'(x)\dots f_{k+1}(x)+\dots+f_1(x)\dots f_{k+1}'(x)$$

<u>once it is assumed that</u>:

$$[f_1(x)\dots f_k(x)]' = f_1'(x)\dots f_k(x)+\dots+f_1(x)\dots f_k'(x)$$

Well:

$$[f_1(x)\dots f_{k+1}(x)]'$$

$$= [(f_1(x)\dots f_k(x))f_{k+1}(x)]'$$

$$= [f_1(x)\dots f_k(x)]'f_{k+1}(x) + [f_1(x)\dots f_k(x)]f_{k+1}'(x) \quad \text{(by Exercise 2.1.6)}$$

$$= [f_1'(x)\dots f_k(x)+\dots+f_1(x)\dots f_k'(x)]f_{k+1}(x) + [f_1(x)\dots f_k(x)]f_{k+1}'(x) \quad \text{(by the inductive assumption)}$$

$$= f_1'(x)\dots f_k(x)f_{k+1}(x)+\dots+f_1(x)\dots f_k'(x)f_{k+1}(x) + f_1(x)\dots f_k(x)f_{k+1}'(x)$$

which is exactly what we had to show!

2.1.7(L)

Here we get one of our first chances to derive an inductive result in which the desired proposition is not nearly as obvious as in our previous examples. At the same time it gives us more drill with taking derivatives.

To begin with, we already know that:

$$\frac{d'(uv)}{dx} = \frac{du}{dx}v + u\frac{dv}{dx} \tag{1}$$

From (1) it follows that:

$$\begin{aligned}
\frac{d^2(uv)}{dx^2} &= \frac{d}{dx}\left[\frac{d}{dx}(uv)\right] = \frac{d}{dx}\left[\frac{du}{dx}v + u\frac{dv}{dx}\right] \\
&= \left(\frac{d^2u}{dx^2}v + \frac{du}{dx}\frac{dv}{dx}\right) + \left(\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}\right)^* \\
&\qquad\qquad\qquad \searrow \quad \swarrow \\
&= \frac{d^2u}{dx^2}v + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2} \qquad (2)
\end{aligned}$$

(We draw the arrows to emphasize that terms from the different sets of parentheses can be combined.)

---

*Remember that du/dx and dv/dx are also functions of x, hence the product rule also applies to du/dx v and u dv/dx. Also realize that the derivative of a differentiable function need not be differentiable. That is, the fact that dv/dx exists does not in itself guarantee that d^2v/dx^2 exists. For example, if $y = x^{3/2}$ then $dy/dx = 3/2\ x^{1/2}$ which exists at x = 0, but $d^2v/dx^2 = 3/4\ x^{-1/2} = 3/4\sqrt{x}$ which doesn't exist at x = 0. This is why the problem must include the statement that the required derivatives exist.

[2.1.7(L) cont'd]

At any rate, using (2) we may obtain:

$$\frac{d^3(uv)}{dx^3} = \frac{d}{dx}\left[\frac{d^2(uv)}{dx^2}\right] = \frac{d}{dx}\left[\frac{d^2u}{dx^2}v + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}\right]$$

$$= \left(\frac{d^3u}{dx^3}v + \frac{d^2u}{dx^2}\frac{dv}{dx}\right) + \left(2\frac{d^2u}{dx^2}\frac{dv}{dx} + 2\frac{du}{dx}\frac{d^2v}{dx^2}\right) + \left(\frac{du}{dx}\frac{d^2v}{dx^2} + u\frac{d^3v}{dx^3}\right)$$

$$\searrow \quad \swarrow \qquad \swarrow \qquad \swarrow$$

$$= \frac{d^3u}{dx^3}v + 3\frac{d^2u}{dx^2}\frac{dv}{dx} + 3\frac{du}{dx}\frac{d^2v}{dx^2} + u\frac{d^3v}{dx^3} \tag{3}$$

If we compare (1), (2), and (3) we notice the so-called binomial coefficients:

1
1 1
1 2 1
1 3 3 1

If we let C(n,k) (often written as $\binom{n}{k}$) denote the number of ways of choosing k objects from a collection of n objects without regard to order, it can be shown that:

$$C(n,k) = \frac{n!}{k!(n-k)!}^{*} \text{ (see the appendix on "combinations")}$$

In fact with this new notation, (3) becomes

[2.1.7(L) cont'd]

$$\frac{d^3(uv)}{dx^3} = C(3,0)\frac{d^3u}{dx^3}v + C(3,1)\frac{d^2u}{dx^2}\frac{dv}{dx} + C(3,2)\frac{du}{dx}\frac{d^2v}{dx^2} + C(3,3)u\frac{d^3v}{dx^3}$$

$$= \sum_{k=0}^{3} C(3,k)\frac{d^{3-k}u}{dx^{3-k}}\frac{d^kv}{dx^k}$$

(where $\frac{d^\circ u}{dx^\circ}$ is defined to be u and $\frac{d^\circ v}{dx^\circ}$ is defined to be v.)

Based on these results, our conjecture is that

$$\frac{d^n(uv)}{dx^n} = \sum_{k=0}^{n} C(n,k)\frac{d^{n-k}u}{dx^{n-k}}\frac{d^kv}{dx^k} \qquad (4)$$

We know that this result is true when n = 2 or when n = 3. To complete our inductive proof we need only show that (4) is true when n is replaced by n + 1. More symbolically, we must show that:

if $\frac{d^n(uv)}{dx^n} = \sum_{k=0}^{n} C(n,k)\frac{d^{n-k}u}{dx^{n-k}}\frac{d^kv}{dx^k}$ then

$$\frac{d^{n+1}(uv)}{dx^{n+1}} = \sum_{k=0}^{n+1} C(n+1,k)\frac{d^{(n+1)-k}u}{dx^{(n+1)-k}}\frac{d^kv}{dx^k}$$

[2.1.7(L) cont'd]

Let us concentrate on a typical pair of bracketed terms in obtaining $\frac{d^{n+1}(uv)}{dx^{n+1}}$ from $\frac{d^n(uv)}{dx^n}$.

We have:

$$\frac{d^n(uv)}{dx^n} = \frac{d^nu}{dx^n}v + \ldots + C(n,k-1)\frac{d^{n-k+1}u}{dx^{n-k+1}}\frac{d^{k-1}v}{dx^{k-1}} + C(n,k)\frac{d^{n-k}u}{dx^{n-k}}\frac{d^kv}{dx^k} + \ldots + u\frac{d^nv}{dx^n}$$

Now one term when we differentiate $C(n,k-1)\frac{d^{n-k+1}u}{dx^{n-k+1}}\frac{d^{k-1}v}{dx^{k-1}}$ is:

$$\boxed{C(n,k-1)\frac{d^{n-k+1}u}{dx^{n-k+1}}\frac{d^kv}{dx^k}}$$

while one term when we differentiate $C(n,k)\frac{d^{n-k}u}{dx^{n-k}}\frac{d^kv}{dx^k}$ is:

$$\boxed{C(n,k)\frac{d^{n-k+1}u}{dx^{n-k+1}}\frac{d^kv}{dx^k}}$$

Adding these two terms yields:

$$\left[C(n,k-1) + C(n,k)\right]\frac{d^{n-k+1}u}{dx^{n-k+1}}\frac{d^kv}{dx^k} =$$

[2.1.7(L) cont'd]

$$= \boxed{C(n+1,k)\ \frac{d^{(n+1)-k}u}{dx^{(n+1)-k}}\ \frac{d^k v}{dx^k}}\ ^*$$

which is precisely what our typical term in the recipe for $\frac{d^{n+1}(uv)}{dx^{n+1}}$ must look like for our inductive proof to hold q.e.d.

(For those who would like to see the details done completely in terms of sigma-notation, we need invoke the additional property that

$$\sum_{k=1}^{n} (a_k+b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

For:
$$\begin{aligned}\sum_{k=1}^{n} (a_k+b_k) &= (a_1+b_1) + (a_2+b_2) +\ldots+ (a_n+b_n)\\ &= (a_1+a_2 +\ldots+ a_n) + (b_1+b_2 +\ldots+ b_n)\\ &= \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k\end{aligned}$$

---

*The result that C(n+1,k) = C(n,k-1) + C(n,k) can be obtained directly from the factorial definition of C(n,k). More intuitively, if we wish to choose k objects from n+1 we may focus our attention on one of the objects. If this object is chosen we must still choose k-1 from the remaining n and this can be done in C(n,k-1) ways. If this object is not chosen then we must still pick k objects from the remaining n and this can be done in C(n,k) ways. Since exactly one of these two conditions must previal there are in all C(n,k-1) + C(n,k) ways of choosing k objects from (n+1) objects.

[2.1.7(L) cont'd]

Then starting with $\dfrac{d^n(uv)}{dx^n} = \displaystyle\sum_{k=0}^{n} C(n,k)\,\frac{d^{n-k}u}{dx^{n-k}}\,\frac{d^k v}{dx^k}$ we obtain

$$\frac{d^{n+1}(uv)}{dx^{n+1}} = \frac{d}{dx}\left(\sum_{k=0}^{n} C(n,k)\,\frac{d^{n-k}u}{dx^{n-k}}\,\frac{d^k v}{dx^k}\right)$$

Recalling that the derivative of a sum is the sum of the derivatives, we then obtain:

$$\frac{d^{n+1}(uv)}{dx^{n+1}} = \sum_{k=0}^{n} \frac{d}{dx}\left[C(n,k)\frac{d^{n-k}u}{dx^{n-k}}\,\frac{d^k v}{dx^k}\right]$$

$$= \sum_{k=0}^{n} \left\{C(n,k)\frac{d^{n-k+1}u}{dx^{n-k+1}}\,\frac{d^k v}{dx^k} + C(n,k)\frac{d^{n-k}u}{dx^{n-k}}\,\frac{d^{k+1}v}{dx^{k+1}}\right\}$$

$$= \sum_{k=0}^{n} C(n,k)\frac{d^{n-k+1}u}{dx^{n-k+1}}\,\frac{d^k v}{dx^k} + \sum_{k=0}^{n} C(n,k)\frac{d^{n-k}u}{dx^{n-k}}\,\frac{d^{k+1}v}{dx^{k+1}}$$

[2.1.7(L) cont'd]

But $$\sum_{k=0}^{n} C(n,k)\frac{d^{n-k}u}{dx^{n-k}}\,\frac{d^{k+1}v}{dx^{k+1}} = \sum_{k=1}^{n+1} C(n,k-1)\frac{d^{n-k+1}u}{dx^{n-k+1}}\,\frac{d^{k}v}{dx^{k}}$$

where we merely reduced k by 1 inside the sigma-sign and adjusted for this in the index of summation. That is:

$$\sum_{k=1}^{n} a_k = \sum_{k=2}^{n+1} a_{k-1} = a_1 + \ldots + a_n$$

Also since

$$\sum_{k=1}^{n} a_k = a_1 + \sum_{k=2}^{n} a_k \text{ etc.,}$$

we have that

$$\sum_{k=0}^{n} C(n,k)\frac{d^{n-k+1}u}{dx^{n-k+1}}\,\frac{d^{k}v}{dx^{k}} = C(n,0)\frac{d^{n+1}u}{dx^{n+1}}v + \sum_{k=1}^{n} C(n,k)\frac{d^{n-k+1}u}{dx^{n-k+1}}\,\frac{d^{k}v}{dx^{k}}$$

[2.1.7(L) cont'd]

while

$$\sum_{k=1}^{n+1} C(n,k-1)\frac{d^{n-k+1}u}{dx^{n-k+1}}\frac{d^kv}{dx^k} = \sum_{k=1}^{n} C(n,k-1)\frac{d^{n-k+1}u}{dx^{n-k+1}}\frac{d^kv}{dx^k} +$$

$$C(n,n)\frac{d^\circ u}{dx^\circ}\frac{d^{n+1}v}{dx^{n+1}} \quad \left(\text{observe } C(n,n) = 1 \text{ and } \frac{d^\circ u}{dx^\circ} \text{ means } u\right)$$

Putting all these results together yields the desired result. For example:

$$\frac{d^5(uv)}{dx^5} = \frac{d^5u}{dx^5}v + 5\frac{d^4u}{dx^4}\frac{dv}{dx} + 10\frac{d^3u}{dx^3}\frac{d^2v}{dx^2} + 10\frac{d^2u}{dx^2}\frac{d^3v}{dx^3} + 5\frac{du}{dx}\frac{d^4v}{dx^4} + u\frac{d^5v}{dx^5}$$

While this result may be interesting in its own right, we will give an application of it in the next exercise to obtain another interesting result.

2.1.8(L)

This exercise affords us an excellent application of Leibniz's Rule (Exercise 2.1.7(L)). By Leibniz's Rule we saw that:

$$\frac{d^m(uv)}{dx^m} = \sum_{k=0}^{m} C(m,k)\frac{d^{m-k}u}{dx^{m-k}}\frac{d^kv}{dx^k} \qquad (1)$$

[2.1.8(L) cont'd]

To apply (1) to our present exercise we need only let $u = x^n$ and $v = g(x)$. Equation (1) then becomes:

$$\frac{d^m[x^n g(x)]}{dx^m} = \sum_{k=0}^{m} C(m,k)\frac{d^{m-k}(x^n)}{dx^{m-k}}\, g^{(k)}(x) \qquad (2)$$

Let us next investigate the appearance of $\frac{d^r(x^n)}{dx^r}$. We have:

$$\frac{d(x^n)}{dx} = nx^{n-1},\ \frac{d^2(x^n)}{dx^2} = n(n-1)x^{n-2},\ \ldots\ \frac{d^3(x^n)}{dx^3} = n(n-1)(n-2)x^{n-3}$$

and in general

$$\frac{d^r(x^n)}{dx^r} = n(n-1)(n-2)\cdots(n-[r-1])x^{n-r} \qquad (r \leqslant n) \quad (3)$$

In particular, then, if $r < n$, $n-r$ is positive; hence, x^{n-r} is 0 when $x = 0$. (The point is that if $n-r$ were negative x^{n-r} would have the form $\frac{1}{0}$ when $x = 0$.) This, in turn, from (3), means that $\frac{d^r(x^n)}{dx^r}$ is 0 if $r < n$.

Referring now to (2), we have that if $m < n$ so also are $m-0, m-1, m-2, \ldots, m-m$ (that is $m-k$ where $k = 0,1,\ldots,m$). From our previous observations about (3), then, since $m-k < n$ $\frac{d^{m-k}(x^n)}{dx^{m-k}} = 0$ when $x = 0$. In still other words, if $m < n$ every

[2.1.8(L) cont'd]

term in the right hand side of (2) has 0 as a factor when $x = 0$. Thus the right hand side of (2) is equal to zero when $x = 0$. That is:

$$\boxed{\text{If } m < n \text{ then } \left.\frac{d^m[x^n g(x)]}{dx^m}\right]_{x=0} = 0} \qquad (4)$$

If we now introduce the notation of this exercise - namely, $f(x) = x^n g(x)$, equation (4) says:

$$\underline{f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0}$$

Finally if we let $m = n$, equation (2) becomes

$$\left.\frac{d^n[x^n g(x)]}{dx^n}\right]_{x=0} = \left.\sum_{k=0}^{n} C(n,k)\frac{d^{n-k}(x^n)}{dx^{n-k}}\, g^{(k)}(x)\right]_{x=0}$$

$$= \left.C(n,0)\frac{d^n(x^n)}{dx^n}\, g^{(0)}(x)\right]_{x=0} + 0$$

(since every remaining term contains $\left.\frac{d^r(x^n)}{dx^r}\right]_{x=0}$ with $r < n$)

Now, $g^{(0)}(x)$ means $g(x)$; hence $\left.g^{(0)}(x)\right]_{x=0}$ is $g(0)$.

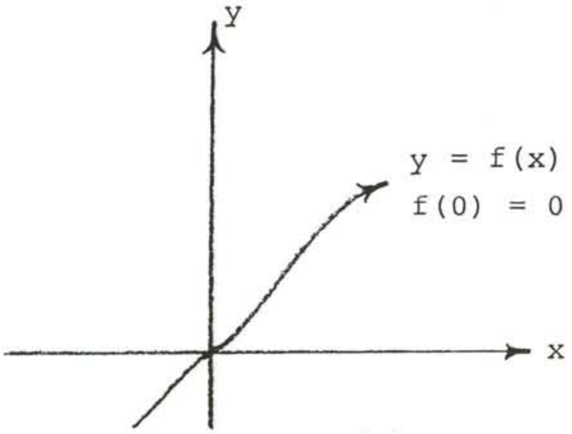
[2.1.8(L) cont'd]

Moreover $\frac{d^n(x^n)}{dx^n} = n!$ (This can be obtained directly from (3) with n=r.) Also C(n,0) = 1. Hence

$$f^{(n)}(0) = \left.\frac{d^n[x^n g(x)]}{dx^n}\right]_{x=0} = n!g(0) \qquad (5)$$

and since we are told that $g(0) \neq 0$, equation (5) tells us that $f^{(n)}(0) \neq 0$.

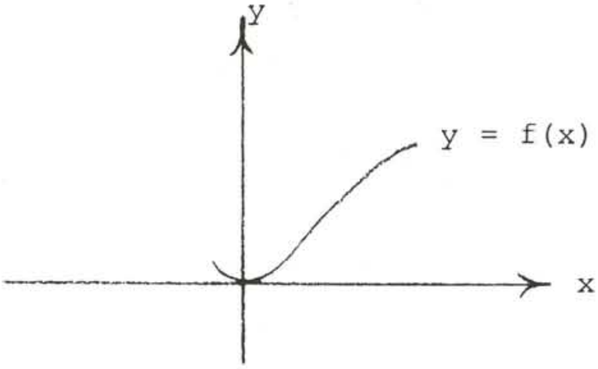
This result has some interesting interpretations. Let us study the graph of $y = f(x)$ where we know only that f is any differentiable function of x and $f(0) = 0$. Then we know that the graph passes through the origin. That is:



(a)

On the other hand suppose we now know that f(0) and f'(0) are both 0. Then the curve must not only pass through (0,0) but its slope there must also be 0. This means that the graph looks like:

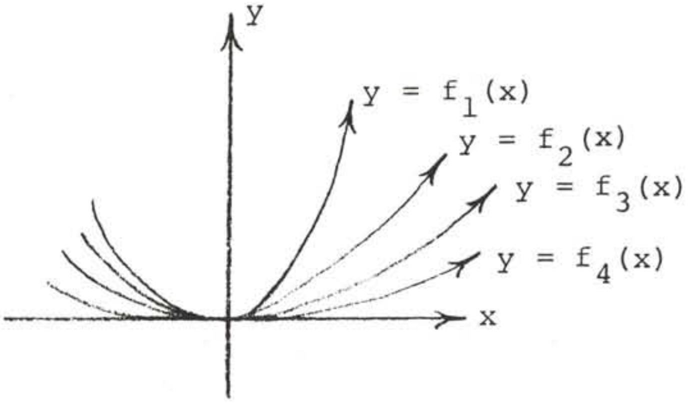
[2.1.8(L) cont'd]



(b)

If we compare (a) and (b) we should sense that in a suitable neighborhood of x = 0 the x-axis serves as a better approximation to the curve in (b) than it does in (a).

Without belaboring the point (we will belabor it much later in our course) the concept of "more" tangent seems to exist. For example in (c) each curve is tangent to the x-axis at (0,0), yet each curve seems to "fit" the x-axis better than the previous ones in a neighborhood of x = 0.

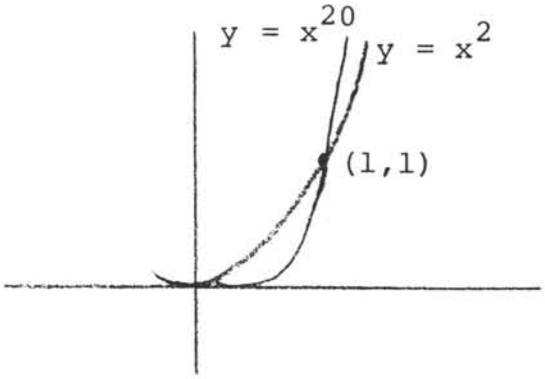


(c)

[2.1.8(L) cont'd]

That is, each curve seems to have a different degree of contact with the x-axis at $x = 0$.

As a more specific illustration, $y = x^2$ and $y = x^{20}$ are both tangent to the x-axis at $x = 0$. Yet at, say, $x = \frac{1}{2}$, $y = \frac{1}{4}$ on the curve $y = x^2$ but $(\frac{1}{2})^{20}$ on the curve $y = x^{20}$. Since $2^{20} > 1{,}000{,}000$, $y < 0.000001$ when $x = \frac{1}{2}$ in the latter case. In other words, assuming a reasonable drawn line we could not distinguish $(\frac{1}{2}, 2^{-20})$ from the x-axis but we could distinguish $(\frac{1}{2}, \frac{1}{4})$ from the x-axis. That is:



$y = x^2$ and $y = x^{20}$ both pass through (1,1) but $y = x^{20}$ "hugs" the x-axis for quite some time

In summary the number of derivatives of $f(x)$ which vanish at $x = 0$ determines the degree of contact that the curve has with its tangent line at that point.

If we refer to the function rather than to the graph our above remarks are related to multiple roots of an equation. For example, both $x = 0$ and $x^{20} = 0$ have $x = 0$ as the lone distinct root. Yet one, somehow or other, is told to count 0 <u>twenty times</u> as a root in $x^{20} = 0$.

[2.1.8(L) cont'd]

Again without belaboring the details, what we are saying is that if r is a root of $f(x) = 0$, the greater the multiplicity of r the better $f(r)$ serves as an approximation for $f(x)$ in a neighborhood of $x = r$. By way of a specific example, let $f_1(x) = (x - 1)x$, $f_2(x) = (x - 1)^2x$, and $f_3(x) = (x - 1)^3x$. Then in each case we have

$$f_1(1) = f_2(1) = f_3(1) = 0$$

(It is also true that in our example $f_1(0) = f_2(0) = f_3(0) = 0$ but this is not in line with the point we are trying to make.) Now:

$$f_1(1.01) = (1.01 - 1)(1.01) = (0.01)(1.01) = 0.0101$$

$$f_2(1.01) = (1.01 - 1)^2(1.01) = 0.0001(1.01) = 0.000101$$

$$f_3(1.01) = (1.01 - 1)^3(1.01) = (0.01)^3(1.01) = 0.0000000101$$

and we see how much better $f(1)$ approximates $f(1.01)$ as the multiplicity of the root $r = 1$ increases.

While these points are interesting in their own right, they play a very important role later in our course when we shall talk about approximating certain curves by polynomials and the technique we shall use is directly related to our discussion of this exercise.

UNIT 2: Increments and Infinitesimals

2.2.1

Given that $f(x) = x^3$, we have that

$$f(x+\Delta x)-f(x) = (x+\Delta x)^3-x^3 = x^3+3x^2\Delta x + 3x\overline{\Delta x}^2 + \overline{\Delta x}^3-x^3$$

$$= 3x^2\Delta x + (3x\Delta x + \overline{\Delta x}^2)\Delta x$$

Recalling that $\Delta y = f(x+\Delta x)-f(x)$, we have:

$$\boxed{\Delta y = 3x^2\Delta x + (3x\Delta x + \overline{\Delta x}^2)\Delta x} \qquad (1)$$

On the other hand $y = x^3$ implies that $\frac{dy}{dx} = 3x^2$. Hence

$$\boxed{\Delta y_{tan} = \frac{dy}{dx}\,\Delta x = 3x^2\Delta x} \qquad (2)$$

Comparing (1) and (2), and observing that $\Delta y-\Delta y_{tan} = \varepsilon\Delta x$, we have:

$$\varepsilon = (3x\Delta x + \overline{\Delta x}^2) = (3x+\Delta x)\Delta x$$

$$\therefore \lim_{\Delta x\to 0} \varepsilon = \lim_{\Delta x\to 0}(3x+\Delta x) \cdot \lim_{\Delta x\to 0}\Delta x$$

$$= (3x)\ (0)$$

$$= 0$$

2.2.2

$$\Delta y = f(x+\Delta x)-f(x)$$

$$= \left[(x+\Delta x)^4 + 2(x+\Delta x)^3 + 7\right] - (x^4 + 2x^3 + 7)$$

$$= 4x^3\Delta x + 6x^2\overline{\Delta x}^2 + 4x\overline{\Delta x}^3 + \overline{\Delta x}^4 + 6x^2\Delta x + 6x\overline{\Delta x}^2 + 2\overline{\Delta x}^3$$

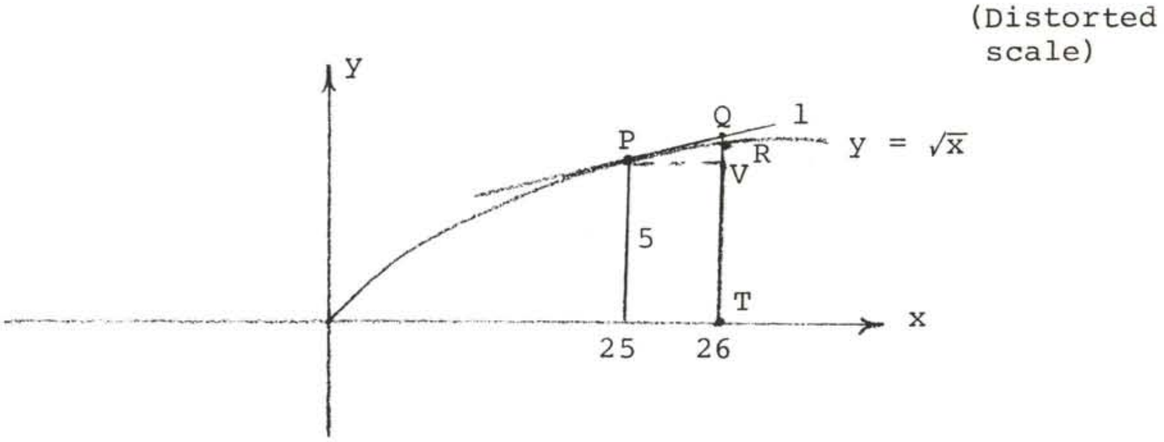
$$= (4x^3 + 6x^2)\Delta x + 6x^2\overline{\Delta x}^2 + 4x\overline{\Delta x}^3 + \overline{\Delta x}^4 + 6x\overline{\Delta x}^2 + 2\overline{\Delta x}^3$$

$$= \left(\frac{dy}{dx}\right)\Delta x + (6x^2\Delta x + 4x\overline{\Delta x}^2 + \overline{\Delta x}^3 + 6x\Delta x + 2\overline{\Delta x}^2)\Delta x$$

$$\therefore \underline{\varepsilon = 6x^2\Delta x + 4x\overline{\Delta x}^2 + \overline{\Delta x}^3 + 6x\Delta x + 2\overline{\Delta x}^2}$$

2.2.3(L)

(a) We may view $\sqrt{26}$ as meaning the value of y if $y = \sqrt{x}$ and x = 26. When x = 25, y = 5. So pictorially:



The equation of the tangent line, l, is given by

$$\frac{y-5}{x-25} = \left.\frac{d\sqrt{x}}{dx}\right|_{x=25}$$

[2.2.3(L) cont'd]

$$\therefore \frac{y-5}{x-25} = \frac{1}{2}x^{-\frac{1}{2}}\Bigg]_{x=25} = \frac{1}{10}$$

$$\therefore y - 5 = \frac{x}{10} - \frac{25}{10}; \quad \text{or} \quad y = \frac{x}{10} + \frac{25}{10}$$

$$\therefore \text{when } x = 26,\ y = \frac{26}{10} + \frac{25}{10} = \frac{51}{10} = 5.1$$

But this names the point (26,5.1) which is on the tangent line not the curve. That is $\overline{QT} = 5.1$ and we want $\overline{RT}$

$$\therefore \sqrt{26} = \overline{RT} \approx \overline{QT} = 5.1$$

(Check 5.1 x 5.1 = 26.01.)

We also see that (and the check verifies this) 5.1 should be more than the correct answer in this case since the tangent line lies above the curve $y = \sqrt{x}$.

Notice that we have used a slightly different approach than that in our notes.

The differential approach whereby we compute

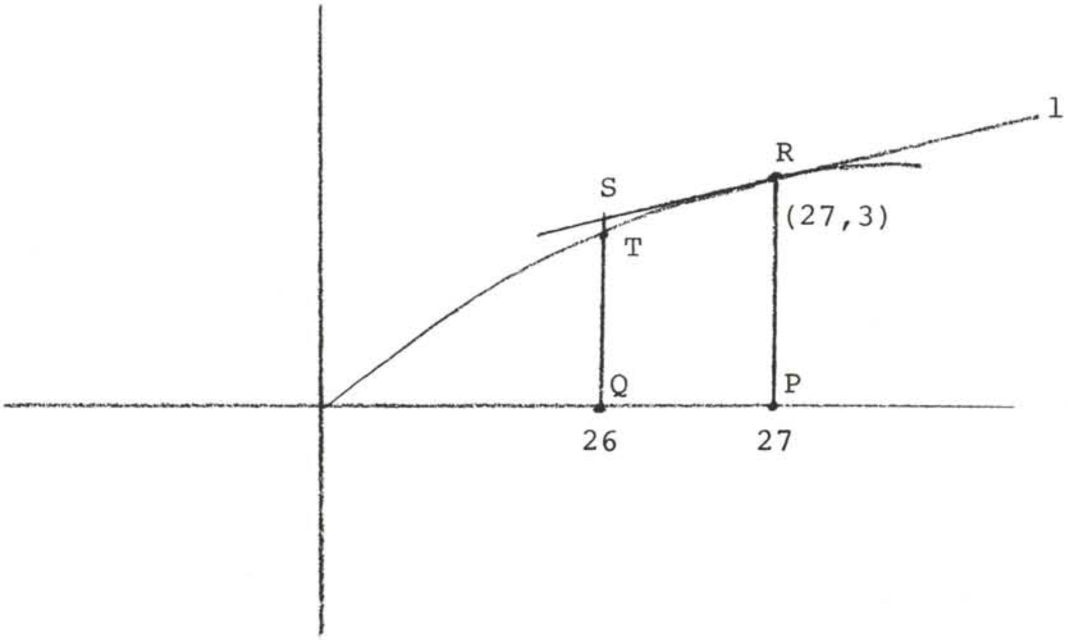
$$\left(\frac{dy}{dx}\right)_{x=25} \Delta x = \left(\frac{1}{10}\right)(1) = 0.1$$

yields the length $\overline{QV}$. Thus,

$$\overline{RT} \approx \overline{QT} = \overline{QV} + \overline{VT} = 0.1 + 5 = 5.1$$

[2.2.3(L) cont'd]

(b) To approximate $\sqrt[3]{26}$ we utilize the fact that $\sqrt[3]{27} = 3$. Thus we talk about the line tangent to the curve $y = \sqrt[3]{x}$ at $x = 27$. That is:



$$\sqrt[3]{26} = \overline{QT} \approx \overline{QS}$$

$$m_l = \left.\frac{d(x^{\frac{1}{3}})}{dx}\right|_{x=27} = \left.\frac{1}{3}x^{-\frac{2}{3}}\right|_{x=27} = \frac{1}{27}$$

$\therefore$ Equation of (l) is:

$$\frac{y-3}{x-27} = \frac{1}{27} \qquad \therefore 27y - 81 = x - 27$$

or

$$y = \frac{x+54}{27}$$

[2.2.3(L) cont'd]

when $x = 26$, $y = \frac{80}{27} = 2\frac{26}{27} = Q\bar{S}$. (As a check $(\frac{80}{27})^3 \approx 26.01$.)

In terms of differentials, $\left.\frac{dy}{dx}\right]_{x=27} \Delta x = \frac{1}{27}(-1) = \frac{-1}{27}$

$$\therefore \quad QS \approx \sqrt[3]{27} = 3 - \frac{1}{7}$$

2.2.4

We look at $y = x^3 + x^2 + \sqrt{x}$ with $x = 1$ and $\Delta x = 0.0006$.

We obtain:

$$\left.\frac{dy}{dx}\right]_{x=1} = \left. 3x^2 + 2x + \frac{1}{2\sqrt{x}}\right]_{x=1} = 5\frac{1}{2} = \frac{11}{2}$$

$$\Delta y_{tan} = \frac{11}{2}(0.0006) = 11(0.0003) = 0.0033$$

Now when $x = 1$, $y = 3$

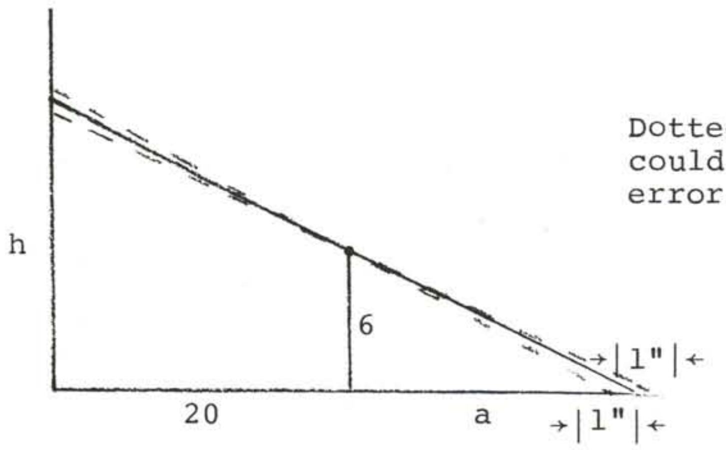
$$\therefore y = 3 + \Delta y \approx 3 + \Delta y_{tan} = 3.0033$$

$$\therefore (1.0006)^3 + (1.0006)^2 + \sqrt{1.0006} \approx 3.0033$$

2.2.5(L)

This gives us a more practical application of the use of differentials. First of all, let us observe that until the maximum error in the height of the shadow was mentioned, we had a simple pre-calculus problem. In fact, all we had was an exercise in Similar Triangles. Namely:

[2.2.5(L) cont'd]



Dotted lines indicate how we could find the exact maximum error.

$$\frac{h}{6} = \frac{20 + a}{a}$$

$$h = 6\left(\frac{20 + a}{a}\right) = 6\left(1 + \frac{20}{a}\right) \qquad (1)$$

Thus $a = 15 \rightarrow h = 6\left(1 + \frac{20}{15}\right) = 6\left(\frac{7}{3}\right) = 14$ feet

In other words, if we assumed that the shadow was EXACTLY 15 feet, then the post would have been exactly 14 feet.

Now, to utilize the known error in the length of the shadow (and certainly a maximum of one inch is reasonable in such a case since all we are doing is measuring the length of a shadow along the ground) we compute dh/da from (1) to obtain:

$$\frac{dh}{da} = 6\left[0 + \frac{-20}{a^2}\right]$$

$$\therefore dh = \frac{-120}{a^2}\, da$$

[2.2.5(L) cont'd]

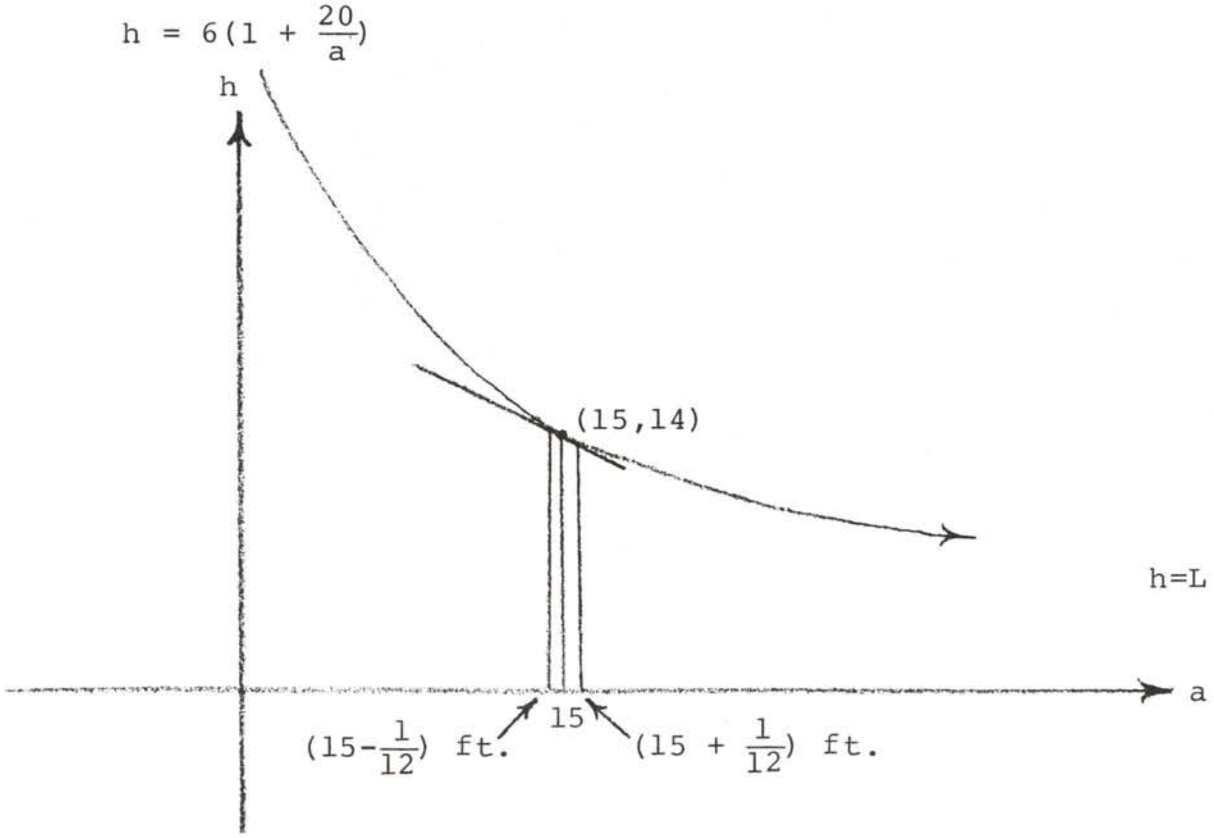
$$|dh| = |\Delta h_{tan}| = \frac{120}{a^2}|da| = \frac{120}{a^2}\Delta a$$

So with a = 15, da = $\left|\frac{1}{12}\right|$ (since all other measurements are in <u>feet</u> not inches.)

$$|dh| = \frac{120}{225}\,\frac{1}{12} = \frac{10}{225} = 0.044 \text{ feet}$$

$$\therefore h = 14 \pm 0.044$$

In terms of a graph (1) yields:



2.2.6

$$A = \pi r^2$$

$$dA = 2\pi r\ dr$$

We are interested in the case $r = 6$ feet, $dr = \frac{1}{2}$inch $= \frac{1}{24}$feet

$$\therefore\ dA = 2\pi(6)\ \frac{1}{24} = \frac{\pi}{2}$$

Now recalling that $dr = \Delta r$, $dA = \Delta A_{\tan}$, we have that the approximate error in A (ΔA) is given by $\frac{\pi}{2}$ (square feet).

As a check, the exact error is $\pi(6 \pm \frac{1}{24})^2 - \pi(6)^2$

or

$$\boxed{\pm \frac{\pi}{2}} \quad + \quad \underline{\underline{\frac{\pi}{576}}}$$ error in the approximation

2.2.7

$$V = \frac{4}{3}\pi r^3$$

$$\begin{aligned} dV &= 4\pi r^2 dr \\ &= 4\pi(6)^2\ (\tfrac{1}{24}) \\ &= 6\pi \approx 19 \text{ cubic feet} \end{aligned}$$

That is, an error of $\frac{1}{2}$ inch in the radius led to an error of <u>19 cubic feet</u>.

[2.2.7 cont'd]

Again as a check

$$\frac{4}{3}\pi\left(6 \pm \frac{1}{24}\right)^3 - \frac{4}{3}\pi(6)^3 =$$

$$\frac{4}{3}\pi\left[\mp 3(6)^2\,\frac{1}{24} + 3(6)\,\left(\frac{1}{24}\right)^2 \mp 3\left(\frac{1}{24}\right)^3\right] =$$

$$\mp 6\pi + \frac{\pi}{24} \mp \pi\,\frac{4}{(24)^3}$$

The fact that small errors can lead to large ones is the basis of a rather well-known mathematical riddle. The story is told that Hercules wanted to tie a piece of string around the equator of the earth (25,000 miles). In cutting the string, he made it just one yard too long so he decided to slip it over the earth and center it so that the excess of a yard would be uniformly distributed over the entire equator. The riddle asks how far above the ground the string stands. The answer lies in the fact that since $C = 2\pi r$,

$$\Delta C = 2\pi\Delta r.$$

Hence (in inches)

$$\Delta r = \frac{36}{2\pi}$$

or approximately 6 inches. In other words an error in the circumference of one yard in 25,000 miles causes the string to lie six inches above the earth all the way around the equator.

This riddle can be omitted from our present discussion with no great loss of enrichment but the result seemed worthwhile to pass on.

UNIT 3: Composite Functions and the Chain Rule

2.3.1(L)

The most common error in this type of problem is to mechanically invoke the "rule" of "bringing down the exponent and replacing it by one less."

That is, we might be tempted to write

$$\frac{d(x^2 + 1)^3}{dx} = 3(x^2 + 1)^2$$

What we must remember is that the rule

$$\frac{d(x^n)}{dx} = nx^{n-1}$$

requires that the variable which is being raised to the n^{th} power be precisely the same as the variable with respect to which we are differentiating. More symbolically,

$$\frac{d([\ ]^n)}{d([\ ])} = n[\ ]^{n-1} \qquad (1)$$

With regard to (1), what is true is that

$$\frac{d(x^2 + 1)^3}{d(x^2 + 1)} = 3(x^2 + 1)^2 \qquad (2)$$

but this was not what was asked for in this problem.

[2.3.1(L) cont'd]

(If the use of brackets seems confusing in (1), we may equivalently think in terms of a substitution or change of variable. For example, we may let u, say, equal $x^2 + 1$, whereupon (2) says

$$\frac{du^3}{du} = 3u^2$$

where now u takes the place of the brackets.)

At any rate, we bring the chain rule into play by observing that

$$\frac{d(x^2 + 1)^3}{dx} = \frac{d(x^2 + 1)^3}{d(x^2 + 1)} \frac{d(x^2 + 1)}{dx} \qquad (3)$$

From (2) we have that

$$\frac{d(x^2 + 1)^3}{d(x^2 + 1)} = 3(x^2 + 1)^2$$

while from our previous knowledge we have that $\frac{d(x^2 + 1)}{dx} = 2x$.

Putting this into (3), we obtain

$$\boxed{\frac{d(x^2 + 1)^3}{dx} = 3(x^2 + 1)^2 2x = 6x(x^2 + 1)^2}$$

Quite in general, then, since $\frac{du^n}{du} = nu^{n-1}$, if u is a differentiable function of x,

[2.3.1(L) cont'd]

$$\frac{d(u^n)}{dx} = \frac{d(u^n)}{du}\ \frac{du}{dx} \qquad \text{(by the chain rule)}$$

$$= nu^{n-1}\frac{du}{dx} \qquad (4)$$

In other words, to "adjust" for the fact that what is being raised to the n^{th} power (u) is not the same as the variable with respect to which we are differentiating, we must multiply by $\frac{du}{dx}$. This is just a restatement of the chain rule.

If we insist on doing this type of problem mechanically, (4) tells us that we can differentiate "one layer at a time" so to speak. (Some texts refer to this as "peeling the onion!")

That is, given $(x^2 + 1)^3$ we differentiate as if the parentheses were the only variable to obtain $3(x^2 + 1)^2$ and then we go "inside" the parentheses to obtain 2x as the derivative, whereupon $3(x^2 + 1)^2 2x$ is our answer.

If we had wished to differentiate $[(x^2 + 1)^3 + 1]^4$ we would first obtain $4[(x^2 + 1)^3 + 1]^3$. Then we would go inside the brackets to obtain $3(x^2 + 1)^2$, and then inside the parentheses to obtain 2x.

Thus,

$$\frac{d}{dx}[(x^2 + 1)^3 + 1]^4 = 4[(x^2 + 1)^3 + 1]^3\ 3(x^2 + 1)^2 2x$$

$$= 24x(x^2 + 1)^2\ [(x^2 + 1)^3 + 1]^3$$

(In terms of the chain rule this is a minor extension that can be proved inductively if we wish. Namely, if

[2.3.1(L) cont'd]

$$y = y(u_1),\ u_1 = u_1(u_2),\ \ldots,\ u_n = u_n(x),$$

then $\frac{dy}{dx} = \frac{dy}{du_1}\frac{du_1}{du_2}\frac{du_2}{du_2} \cdots \frac{du_n}{dx}$.)

In this problem, we can let $u = x^2 + 1$; then $y = [(x^2 + 1)^3 + 1]4$ becomes $y = [u^3 + 1]^4$. We can then let $v = u^3 + 1$ whereupon $y = [(x^2 + 1)^3 + 1]^4$ becomes $y = v^4$. Finally we may let $w = v^4$. Putting this all together we get that $y = [(x^2 + 1)^3 + 1]^4$ breaks down as

$$\left.\begin{array}{l} y = w \\ w = v^4 \\ v = u^3 + 1 \\ u = x^2 + 1 \end{array}\right\} \rightarrow \begin{array}{l} \frac{dw}{dv} = 4v^3 \\ \frac{dv}{du} = 3u^2 \\ \frac{du}{dx} = 2x \end{array}$$

$$\frac{dy}{dx} = \frac{dw}{dx} = \frac{dw}{dv}\frac{dv}{du}\frac{du}{dx}$$

$$= (4v^3)\ (3u^2)\ (2x) = 4(u^3 + 1)^3 \left(3(x^2 + 1)^2\right) (2x)$$

$$= 4[(x^2 + 1)^3 + 1]^3\ 3(x^2 + 1)^2 2x =$$

$$24x(x^2 + 1)^2\ [(x^2 + 1)^3 + 1]^3$$

and this checks with our previous result.

As a final note, $(x^2 + 1)^3$ lends itself to a simple expansion. That is,

[2.3.1(L) cont'd]

$$(x^2 + 1)^3 = (x^2)^3 + 3(x^2)^2 + 3(x^2) + 1$$

$$= x^6 + 3x^4 + 3x^2 + 1$$

$$\therefore \frac{d}{dx}(x^2 + 1)^3 = \frac{d}{dx}(x^6 + 3x^4 + 3x^2 + 1)$$

$$= 6x^5 + 12x^3 + 6x = 6x(x^4 + 2x^2 + 1) =$$

$$6x(x^2 + 1)^2 \qquad (5)$$

Equation (5) shows us that $\frac{d}{dx}(x^2 + 1)^3 \neq 3(x^2 + 1)^2$. In fact it is easy to see that the necessary "correction" factor is 2x, which is precisely the derivative of $(x^2 + 1)$ with respect to x.

2.3.2

$$\frac{d}{dx}(3x^3 + 2x + 1)^4 = \frac{d(3x^3 + 2x + 1)^4}{d(3x^3 + 2x + 1)} \cdot \frac{d(3x^3 + 2x + 1)}{dx}$$

$$= \underline{4(3x^3 + 2x + 1)^3 [9x^2 + 2]}$$

Alternative way:

$$y = (3x^3 + 2x + 1)^4 \text{ means: } \begin{cases} y = u^4 \\ u = 3x^3 + 2x + 1 \end{cases}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 4u^3 (9x^2 + 2)$$

$$= 4(3x^2 + 2x + 1)^3 (9x^2 + 2)$$

[2.3.2 cont'd]

Next, if $y = [(3x^3 + 2x + 1)^4 + 2]^5$ we may write:

$$\left.\begin{aligned} u &= 3x^3 + 2x + 1 \\ v &= u^4 + 2 \\ y &= v^5 \end{aligned}\right\} \quad \therefore \frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{du}\frac{du}{dx}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= (5v^4)(4u^3)(9x^2 + 2) \\ &= 5(u^4 + 2)^4\ 4(3x^3 + 2x + 1)^3\ (9x^2 + 2) \\ &= \underline{20[(3x^3 + 2x + 1)^4 + 2]^4\ (3x^3 + 2x + 1)^3\ (9x^2 + 2)} \end{aligned}$$

In terms of "peeling the onion":

$$\underline{5[(3x^3 + 2x + 1)^4 + 2]^4\ [4(3x^3 + 2x + 1)^3]\ [9x^2 + 2]}$$

2.3.3(L)

One form of the chain rule tells us that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Namely $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$ implies the above result.

The main problem seems to be that many suspect that

$$\frac{d^2y}{dx^2} = \frac{d^2y/dt^2}{d^2x/dt^2}$$

Certainly, one can compute $\frac{d^2y}{dt^2}$ and $\frac{d^2x}{dt^2}$, and one can form the

[2.3.3(L) cont'd]

quotient $\frac{d^2y}{dt^2} \div \frac{d^2x}{dt^2}$ but, in general, this quotient is not equal to $\frac{d^2y}{dx^2}$.

In the form of an aside, notice the notation of how the "2's" are placed in $\frac{d^2y}{dx^2}$. If we could treat $\frac{d^2y}{dt^2}$ and $\frac{d^2x}{dt^2}$ as fractions, we would obtain:

$$\frac{d^2y}{dt^2} \div \frac{d^2x}{dt^2} = \frac{d^2y}{d^2x} \quad \underline{\text{not}} \quad \frac{d^2y}{dx^2}$$

Of course, the above discourse is more of an excuse than a reason. The more important observation is that if $\frac{dy}{dt}$ and $\frac{dx}{dt}$ are functions of t then so also is $\frac{dy}{dx}$ $(= \frac{dy}{dt} \div \frac{dx}{dt})$. Hence we may apply the chain rule to $\frac{dy}{dx}$. That is:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[\frac{dy/dt}{dx/dt}\right] = \frac{d}{dt}\left[\frac{dy/dt}{dx/dt}\right]\frac{dt}{dx} \qquad (1)$$

$\left(\text{That is, let } \frac{dy/dt}{dx/dt} = g(t). \text{ Then } \frac{dg(t)}{dx} = \frac{dg(t)}{dt}\frac{dt}{dx}.\right)$

Applying the quotient rule, we obtain:

$$\frac{d}{dt}\left[\frac{dy/dt}{dx/dt}\right] = \frac{\left(\frac{dx}{dt}\right)\left(\frac{d^2y}{dt^2}\right) - \left(\frac{dy}{dt}\right)\left(\frac{d^2x}{dt^2}\right)}{\left(\frac{dx}{dt}\right)^2} \qquad (2)$$

[2.3.3(L) cont'd]

Coupling (2) with the fact that $\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$* (1) becomes:

$$\frac{d^2y}{dx^2} = \left[\frac{(\frac{dx}{dt})\ (\frac{d^2y}{dt^2}) - (\frac{dy}{dt})\ (\frac{d^2x}{dt^2})}{(\frac{dx}{dt})^2}\right]\left(\frac{1}{\frac{dx}{dt}}\right) = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{(\frac{dx}{dt})^3}$$

To see how this works in a particular problem, we investigate part (b).

$$\left.\begin{aligned} x = t^2 \rightarrow \frac{dx}{dt} = 2t \\ y = t^4 \rightarrow \frac{dy}{dt} = 4t^3 \end{aligned}\right\} \therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t^3}{2t} = 2t^2 \qquad (3)$$

From (3) we next obtain:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}(2t^2) = \frac{d}{dt}(2t^2)\frac{dt}{dx} = 4t/\frac{dx}{dt} = 4t/2t = \underline{2} \qquad (4)$$

---

*This, too, is a corollary of the chain rule. Namely when we say that $y = f(x)$ and $x = g(t)$ imply that $dy/dt = dy/dx\ dx/dt$ nothing excludes the possibility that $y = t$. That is, the first and third variables can be equal. In this event

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

becomes, when we replace t by y:

$$\frac{dy}{dy} = \frac{dy}{dx}\frac{dx}{dy} \quad \text{or} \quad 1 = (\frac{dy}{dx})\ (\frac{dx}{dy}) \quad \therefore \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

and we see still another resemblance of derivatives with fractions.

[2.3.3(L) cont'd]

(Beware that you don't say $\frac{d}{dx}(2t^2) = 4t$, but $\frac{d}{dt}(2t^2) = 4t$.)

As a check observe that we can eliminate t and obtain y directly in terms of x. Namely $y = t^4 = (t^2)^2 = x^2$. But

$$y = x^2 \rightarrow \begin{cases} \frac{dy}{dx} = 2x \text{ (and since } x = t^2\text{, this agrees with (3))} \\ \\ \frac{d^2y}{dx^2} = 2 \text{ (and this, clearly, agrees with (4))} \end{cases}$$

In general it is either difficult or impossible to eliminate t to find y in terms of x. (For example, see Exercise 2.3.5.) We chose an example in which t could be eliminated so that we could see what was really happening. However while we are on the topic, notice that something is "lost" when we eliminate t. For example, when we eliminate t, both

$$\begin{cases} y = t^2 \\ x = t \end{cases} \quad \text{and} \quad \begin{cases} y = t^4 \\ x = t^2 \end{cases} \quad \text{become } y = x^2$$

but if we think of t as time, all we are saying is that the path traced out by the particle P(x,y) is the same ($y = x^2$) but the particles traverse the curve differently - for example if $\left.\begin{array}{l} y = t^2 \\ x = t \end{array}\right\}$ then at t = 2 we are at the point (2,4) while if $\left.\begin{array}{l} y = t^4 \\ x = t^2 \end{array}\right\}$ then at t = 2 we are at (4,16).

2.3.4

$$\left.\begin{aligned} x &= t^2 + 1 \\ y &= t^3 - t \end{aligned}\right\} \rightarrow \left.\begin{aligned} \frac{dx}{dt} &= 2t \\ \frac{dy}{dt} &= 3t^2 - 1 \end{aligned}\right\} \rightarrow \frac{dy}{dx} = \frac{3t^2 - 1}{2t} \qquad (1)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{3t^2 - 1}{2t}\right) = \frac{d}{dt}\left(\frac{3t^2 - 1}{2t}\right)\frac{dt}{dx} = \left(\frac{d}{dt}\left[\frac{3t^2 - 1}{2t}\right]\right)\frac{1}{2t}$$

$$= \left[\frac{2t(6t) - (3t^2 - 1)(2)}{4t^2}\right]\frac{1}{2t}$$

$$= \frac{6t^2 + 2}{8t^3} = \frac{3t^2 + 1}{4t^3} \qquad (2)$$

With $t = 1$, (1) and (2) become:

$$\left.\begin{aligned} \left.\frac{dy}{dx}\right]_{x=1} &= \frac{3 - 1}{2} = 1 \\ \left.\frac{d^2y}{dx^2}\right]_{x=1} &= \frac{3 + 1}{4} = 1 \end{aligned}\right\}\underline{\text{ans.}}$$

(Note: We must find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in general and then let $t = 1$. In other words $f''(1)$ does not mean $[f'(1)]'$, since this would always be 0 because $f'(1)$ is a constant. Rather $f''(1)$ means $\left.f''(x)\right]_{x=1}$.)

Now the point on the curve corresponding to $t = 1$ is $x = 1^2 + 1$, $y = 1^3 - 1$ or $(2,0)$. The slope of the curve at this point is $\left.\frac{dy}{dx}\right]_{x=1} = 1$.

[2.3.4 cont'd]

Hence the equation of the tangent line is

$$\frac{y-0}{x-2} = 1 \qquad \text{or} \qquad \underline{\underline{y = x - 2}}$$

2.3.5

$$\left.\begin{aligned} x = t^5 + t^2 + 1 &\rightarrow \frac{dx}{dt} = 5t^4 + 2t \\ y = t^6 + 2t^3 + 4 &\rightarrow \frac{dy}{dt} = 6t^5 + 6t^2 \end{aligned}\right\} \text{Also when } t = 1,\ x=3,\ y=7$$

$$\therefore \frac{dy}{dx} = \frac{6t^5 + 6t^2}{5t^4 + 2t}$$

$$\left.\frac{dy}{dx}\right]_{t=1} = \frac{12}{7}$$

Thus the point on the curve is (3,7) and the slope at this point on the curve is $\frac{12}{7}$.

$$\therefore \frac{y-7}{x-3} = \frac{12}{7}$$

$$\text{or:} \quad 7y - 49 = 12x - 36$$

$$\text{or:} \quad \underline{7y = 12x + 13}$$

[2.3.5 cont'd]

Notice again that while it may be difficult to compute $\frac{dy}{dx}$ in certain cases, it still is always a slope. Secondly, observe the difficulty in trying to eliminate t from this pair of parametric equations.

2.3.6(L)

All that happens here is that the chain rule takes on a different look. We are being asked to find

$$\frac{d(x^2 + 16)}{d(x^3 - 1)}$$

Mechanically, if we use differential notation we obtain:

$$\frac{2x\,dx}{3x^2dx} = \frac{2}{3x}$$

If we wish to see the chain rule more concretely we may let $y = x^2 + 16$ and $u = x^3 - 1$. Then, in this language, the problem is asking us to find $\frac{dy}{du}$. By the chain rule:

$$\frac{dy}{du} = \frac{dy/dx}{du/dx} = \frac{2x}{3x^2}$$

In any event the answer is $\frac{2}{3x}$.

2.3.7(L)

When we write $y = f(x^2)$* we mean:

$$y = f(u) \text{ and } u = x^2 \tag{1}$$

If we only assume that f is a differentiable function, the chain rule applied to (1) allows us to conclude that:

$$\frac{dy}{dx} = \frac{df(u)}{du}\ \frac{du}{dx}$$

$$= f'(u)2x \tag{2}$$

Without further information, (2) is as far as we can go in our attempt to determine $\frac{dy}{dx}$ explicitly. In this problem, however, we are given the additional information that $f'(x) = 4x^3 + 1$. Recalling that the name of the variable is irrelevant (that is, $f'(x) = 4x^3 + 1$ should be thought of as $f'([\ ]) = 4[\ ]^3 + 1$), we have that $f'(u) = 4u^3 + 1$. With this, equation (2) becomes:

$$\frac{dy}{dx} = (4u^3 + 1)2x \tag{3}$$

and finally since $u = x^2$, (3) can be rewritten explicitly as a function of x. That is:

$$\frac{dy}{dx} = (4[x^2]^3 + 1)2x = 2x(4x^6 + 1)$$

---

*$y = f(x^2)$ is a special type of function of x. The notation $f(x^2)$ emphasizes that the variable, x, always appears in the form x^2. For example $f(x^2)$ could denote $(x^2)^3 + x^2 + \sqrt{x^2 + 1}$. Another way of denoting this is to write $y = f(u)$ and $u = x^2$. With regard to our present illustration if $f(u) = u^3 + u + \sqrt{u + 1}$ then $f(x^2)$ is $(x^2)^3 + x^2 + \sqrt{x^2 + 1}$.

2.3.8

(a) We want:

$$\frac{d(x^4 + 7x^2 + 8)}{d(x^3 - 2x)} = \frac{(4x^3 + 14x)dx}{(3x^2 - 2)dx}$$

$$= \frac{4x^3 + 14x}{3x^2 - 2} \quad \text{ans.}$$

In terms of the chain rule, we let

$$y = x^4 + 7x^2 + 8$$

$$u = x^3 - 2x$$

Then $\frac{dy}{du} = \frac{dy/dx}{du/dx} = \frac{4x^3 + 14x}{3x^2 - 2}$

(b) $y = f(u)$

$$u = x^3 + 1$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= f'(u)3x^2$$

$$= \left(\frac{u}{u^2 + 1}\right) 3x^2$$

$$= \left[\frac{x^3 + 1}{(x^3 + 1)^2 + 1}\right] 3x^2$$

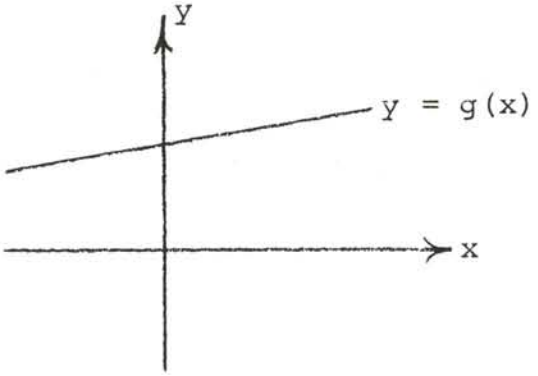
$$= \frac{3x^2(x^3 + 1)}{x^6 + 2x^3 + 2} \quad \underline{\text{ans.}}$$

UNIT 4: Inverse Functions

2.4.1(L)

We have already solved part (a) as Exercise 1.4.9(L). Of course, we've come a long way since then. In particular, with our knowledge of derivatives we can sketch the graphs y = g(x) and y = h(x).

In particular, since $f^{-1}(x) = \frac{x + 7}{2}$, the curve y = g(x) is simply the straight line whose slope is $\frac{1}{2}$ and whose y-intercept is $\frac{7}{2}$. That is:



On the other hand, since $h(x) = \frac{1}{2x - 7} = (2x - 7)^{-1}$, we see that:

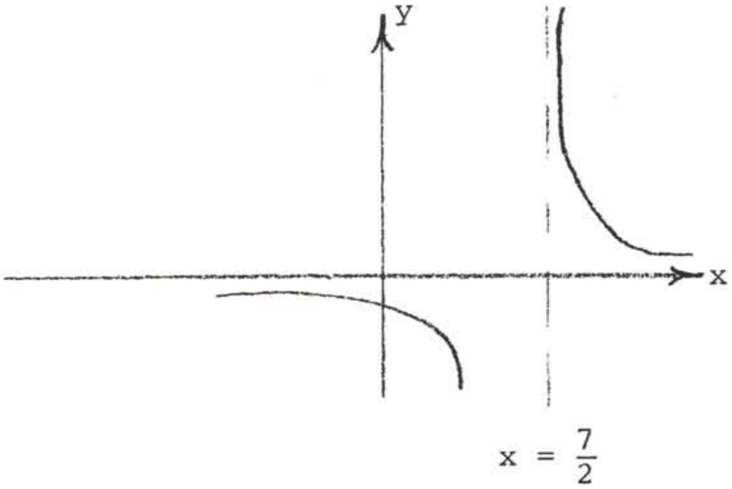
$$h'(x) = -2(2x - 7)^{-2}$$

and

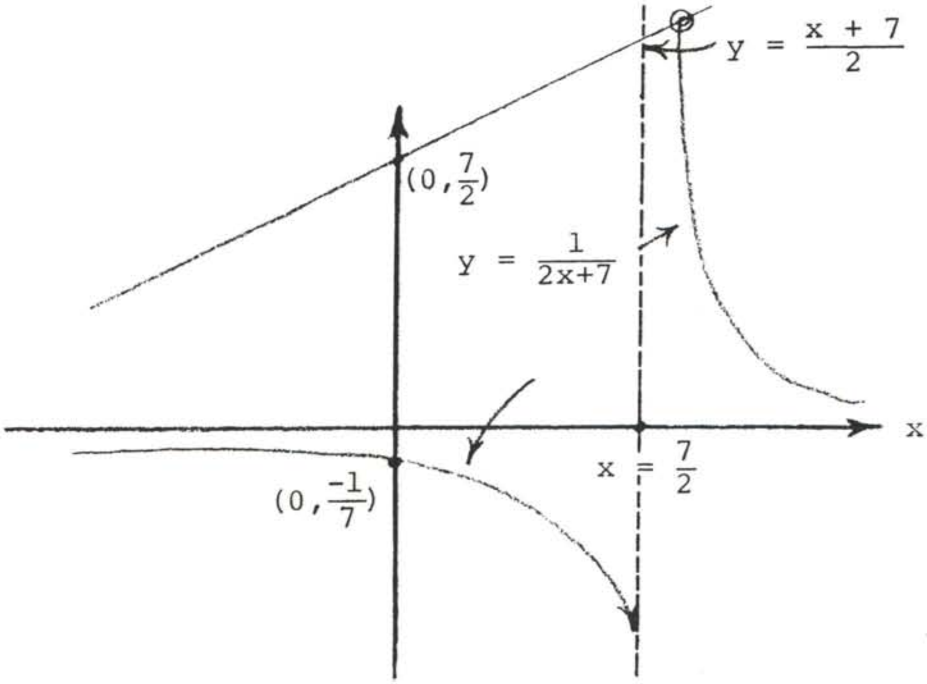
$$h''(x) = 8(2x - 7)^{-3}$$

Thus the curve y = h(x) is always falling, it "holds water" if $x > \frac{7}{2}$, spills water if $x < \frac{7}{2}$, and $x = \frac{7}{2}$ cannot be the x-coordinate of a point on the curve. Thus:

[2.4.1(L) cont'd]



Superimposed in a single diagram, we see quite vividly that the curves y = g(x) and y = h(x) are quite different:



[2.4.1(L) cont'd]

(The curves meet at two points and these are determined analytically by solving $\frac{x+7}{2} = \frac{1}{2x-7}$, or $2x^2 + 7x - 51 = 0$.

$$\therefore x = \frac{-7 \pm \sqrt{49 + 408}}{4} = \frac{-7 \pm \sqrt{457}}{4} \approx \frac{-7 \pm 21}{4} \text{ etc.)}$$

Again this portion of the exercise is designed to emphasize the fact that $f^{-1}(x)$ and $\frac{1}{f(x)}$ are very different from a conceptual point of view.

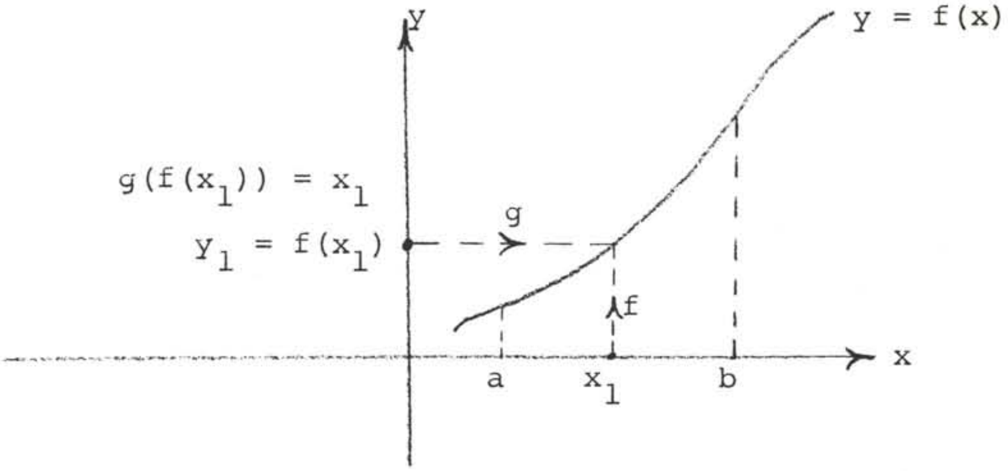
On the other hand, part (b) is designed to reinforce the idea that as differentials $\frac{dy}{dx}$ and $\frac{dx}{dy}$ are indeed reciprocals and that since as far as derivatives are concerned, we have shown that $\frac{dy}{dx}$ and $\frac{dx}{dy}$ are reciprocals, we CAN conclude that f and f^{-1} are characterized by the fact that their derivatives are reciprocals. In this particular exercise we chose a rather simple computational problem. Namely if $f(x) = 2x - 7$, it is clear at once that $f'(x) = 2$, while since $g(x) = f^{-1}(x) = \frac{x+7}{2}$, it is equally clear that $g'(x) = \frac{1}{2}$. This shows, at least in this example, that f' and g' are reciprocals.

Moreover, it is not too difficult to generalize the result of this exercise and this is exactly what we did in our discussion of the chain rule. The only "special case" aspect of this exercise was that both f' and g' were constants. In this example, it was correct to say that $g'(x) = \frac{1}{f'(x)}$. In the general case we must be more careful in specifying the variables. That is, what is true in general is that:

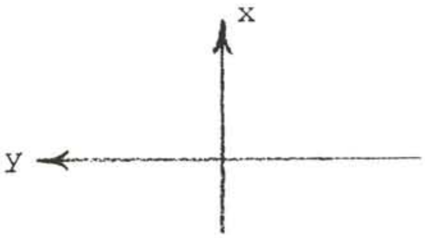
$$g'(y_1) = \frac{1}{f'(x_1)} \quad \text{where } y_1 = f(x_1)$$

[2.4.1(L) cont'd]

Pictorially, since g undoes f, it is clear that g "operates" on f(x). That is:



(Notice also in the above diagram that we do not invert merely by interchanging the x and the y axes. If we did this we would obtain



and this is an incorrect orientation of the axes.)

2.4.2(L)

This exercise serves as an introduction to the subject known as implicit differentiation. That is, from the given equation we see that x and y cannot be chosen independently. So we assume,

[2.4.2(L) cont'd]

whether we can explicitly do it or not, that y is defined as a single-valued function of x. With regard to this exercise we assume that y is precisely that function of x which makes $x^5 + 3x^2y + y^7 = 4$ an IDENTITY. That is, we assume that y is that function of x for which 4 and $x^5 + 3x^2y + y^7$ are synonyms.

We can conclude that:

$$\frac{d(x^5 + 3x^2y + y^7)}{dx} = \frac{d(4)}{dx} \text{ *}$$

Thus:

$$\frac{d(x^5)}{dx} + \frac{d(3x^2y)}{dx} + \frac{d(y^7)}{dx} = 0$$

This, in turn, says that:

$$5x^4 + \left[3x^2\left(\frac{dy}{dx}\right) + 6xy\right] + 7y^6\left(\frac{dy}{dx}\right) = 0$$ (where we find $\frac{d(3x^2y)}{dx}$ by the product rule)

(Again, beware it is $\frac{d(y^7)}{dy}$, not $\frac{d(y^7)}{dx}$, which equals $7y^6$;

$$\frac{d(y^7)}{dx} = \frac{d(y^7)}{dy}\ \frac{dy}{dx}$$ by the chain rule)

---

*This result is not as trivial as it might seem. It is crucial that we have an identity. For example, given 2x = 5 if we differentiate both sides with respect to x, we obtain 2 = 0 which is absurd. The point is that 2x and 5 are <u>not</u> synonyms. That is, 2x = 5 is not an identity but a <u>conditional</u> equality.

On the other hand, if $f(x) = x^2 - 1$ and $g(x) = (x+1)(x-1)$ then f and g are identical and for this reason, since they are just different names for the same thing, they have, among other things, the same derivative. As a check, $f'(x) = 2x$ while by the product rule, $g'(x) = (x+1)(1) + (x-1)(1) = 2x$.

[2.4.2(L) cont'd]

Thus:

$$(5x^4 + 6xy) + (3x^2 + 7y^6)(\frac{dy}{dx}) = 0, \text{ whereupon:}$$

$$\frac{dy}{dx} = \frac{-(5x^4 + 6xy)}{(3x^2 + 7y^6)}$$

2.4.3

As usual, we need only know the slope $(\frac{dy}{dx})$ of the tangent line at (1,1). In this case, we find $\frac{dy}{dx}$ by differentiating implicitly. Thus:

$$\frac{d(x^7 + 5x^3y + y^6)}{dx} = \frac{d(7)}{dx}$$

This leads to:

$$7x^6 + 15x^2y + 5x^3(\frac{dy}{dx}) + 6y^5(\frac{dy}{dx}) = 0$$

Hence:

$$\frac{dy}{dx} = \frac{-(7x^6 + 15x^2y)}{(5x^3 + 6y^5)} \qquad (1)$$

Setting x = y = 1, we obtain that $(\frac{dy}{dx})_{(1,1)} = \frac{-22}{11} = -2.$

[2.4.3 cont'd]

In other words, we simply want the line whose slope is -2 and which passes through (1,1). This line is given by:

$$\frac{y - 1}{x - 1} = -2$$

or:

$$\underline{2x + y = 3}$$

(Note: It is worth checking that (1,1) actually belongs to the curve. It does by virtue of the fact that $x^7 + 5x^3y + y^6 = 7$ is satisfied when $x = 1$ and $y = 1$. The point is that in Equation (1) we can replace x and y by 1 and get an answer, even if (1,1) weren't on the curve!)

2.4.4(L)

At first glance this might look like a problem that we have already solved. The thing to note is that when n was a positive integer we proved the recipe by the binomial theorem. When n was a negative integer, the binomial theorem did not apply, and we used the quotient rule to derive the desired result.

Neither of these cases applies, however, if n is not an integer. Recall that a rational number is by definition the quotient of two integers; so if n is a rational number there exist integers p and q such that $n = p/q$. Therefore $y = x^n$ means that $y = x^{p/q}$, and this in turn says that $y^q = x^p$. Since p and q are integers we can use the previous results to obtain by implicit differentiation:

[2.4.4(L) cont'd]

$$qy^{q-1}\left(\frac{dy}{dx}\right) = px^{p-1}$$

$$\therefore \frac{dy}{dx} = \frac{p}{q}\,\frac{x^{p-1}}{y^{q-1}} = \frac{p}{q}\,\frac{x^{p-1}}{\left(x^{\frac{p}{q}}\right)^{q-1}} = \frac{p}{q}\; x^{p-1}\Big/x^{p-\frac{p}{q}}$$

$$= \frac{p}{q}\,x^{\frac{p}{q}-1}$$

$$= nx^{n-1} \quad \text{since } n = \frac{p}{q}$$

Aside from giving us another illustration of implicit differentiation, this example gives us an excellent illustration of how the same recipe may require a completely different proof in one case than in another. It is important to note the logical study here of how each case is reduced to the case which has been solved previously.

2.4.5

We now know that if $y = u^{\frac{1}{2}}$ then $\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$. (Until we had the result of Exercise 2.4.4 we did not know this.)

At any rate, we can now solve this problem by our previous techniques. Namely, we can let $u = x^2 + 16$ and $v = \frac{x}{x-1}$. The problem translates into finding $\frac{du^{1/2}}{dv}$ at $x = 3$.

[2.4.5 cont'd]

Thus:

$$\frac{du^{\frac{1}{2}}}{dv} = \frac{du^{\frac{1}{2}}}{dx} \bigg/ \frac{dv}{dx}$$

Now $\frac{d\left(u^{\frac{1}{2}}\right)}{dx} = \frac{1}{2}u^{-\frac{1}{2}}\frac{du}{dx} = \frac{1}{2}u^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{u}} = \frac{x}{\sqrt{x^2 + 16}}$

while $\frac{dv}{dx} = \frac{d(\frac{x}{x-1})}{dx} = \frac{(x-1)\frac{dx}{dx} - x\frac{d}{dx}(x-1)}{(x-1)^2} = \frac{(x-1) - x}{(x-1)^2} = \frac{-1}{(x-1)^2}$

$$\therefore \frac{d(\sqrt{x^2 + 16})}{d(\frac{x}{x-1})} = \left(\frac{x}{\sqrt{x^2 + 16}}\right) \bigg/ \frac{-1}{(x-1)^2}$$

$$= \frac{-x(x-1)^2}{\sqrt{x^2 + 16}}$$

and when x = 3 this becomes

$$\frac{-3(3-1)^2}{\sqrt{3^2 + 16}} = \frac{-3(4)}{\sqrt{25}} = \frac{-12}{5}$$

2.4.6

$$x^3 - y^3 = 1 \quad \rightarrow$$

$$3x^2 - 3y^2 \frac{dy}{dx} = 0 \quad \rightarrow$$

$$\frac{dy}{dx} = \frac{x^2}{y^2} \tag{1}$$

From (1), $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{x^2}{y^2}\right)$

$$= \frac{y^2(2x) - x^2\left(2y\frac{dy}{dx}\right)}{y^4}$$

$$= \frac{2xy - 2x^2\frac{dy}{dx}}{y^3} \tag{2}$$

Replacing $\frac{dy}{dx}$ in (2) by its value in (1), we obtain:

$$\frac{d^2y}{dx^2} = \frac{2xy - 2x^2\left(\frac{x^2}{y^2}\right)}{y^3} = \frac{2xy^3 - 2x^4}{y^5}$$

$$= \frac{-2x(x^3 - y^3)}{y^5}$$

$$= \frac{-2x}{y^5} \quad (\text{since } x^3 - y^3 \equiv 1) \tag{3}$$

[2.4.6 cont'd]

Explicitly $x^3 - y^3 = 1$ implies that

$$y^3 = x^3 - 1$$

or

$$y = (x^3 - 1)^{\frac{1}{3}}$$

whereupon

$$\frac{dy}{dx} = \frac{1}{3}(x^3 - 1)^{-2/3}(3x^2) = \frac{x^2}{(x^3 - 1)^{2/3}} \qquad (4)$$

(Recalling that $x^3 - 1 = y^3$, (4) becomes

$$\frac{dy}{dx} = \frac{x^2}{(y^3)^{2/3}} = \frac{x^2}{y^2}$$

which checks with (1).)

If we now apply the quotient rule to (4), we obtain:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{x^2}{(x^3 - 1)^{2/3}}\right)$$

$$= \frac{(x^3-1)^{2/3}(2x) - x^2\left[\frac{2}{3}(x^3-1)^{-\frac{1}{3}}3x^2\right]}{(x^3-1)^{4/3}}$$

$$= \frac{2x(x^3-1)^{2/3} - \frac{2x^4}{(x^3-1)^{1/3}}}{(x^3-1)^{4/3}} = \frac{2x(x^3-1)-2x^4}{(x^3-1)^{5/3}} = \frac{-2x}{(x^3-1)^{5/3}} \qquad (5)$$

[2.4.6 cont'd]

If we replace $x^3 - 1$ by y^3 in (5) we obtain:

$$\frac{d^2y}{dx^2} = \frac{-2x}{(y^3)^{5/3}} = \frac{-2x}{y^5}$$

which checks with (3).

2.4.7(L)

In the previous exercise it was cumbersome but not too difficult to solve explicitly for y in terms of x. In this exercise it is a bit tougher to do this, so it is to our advantage to rely either on inverse functions or implicit differentiation to come to our aid here.

Since the equation is in the form x = f(y), it is more convenient to find $\frac{dx}{dy}$ than $\frac{dy}{dx}$. In fact,

$$\frac{dx}{dy} = 3y^2 + 1$$

By the chain rule $\frac{dy}{dx} = 1 \Big/ \frac{dx}{dy}$

$$\therefore \quad \boxed{\frac{dy}{dx} = \frac{1}{3y^2 + 1}} \qquad (1)$$

[2.4.7(L) cont'd]

Differentiating both sides of (1) with respect to x, we obtain:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{3y^2 + 1}\right)$$

or:

$$\frac{d^2y}{dx^2} = \left[\frac{(3y^2 + 1)\frac{d(1)}{dy} - 1\frac{d}{dy}(3y^2 + 1)}{(3y^2 + 1)^2}\right]\frac{dy}{dx}$$

(We must be very careful here the derivative of g(y) with respect to x is $g'(y)\frac{dy}{dx}$ by the chain rule. That is, the "prime" here indicates differentiation with respect to y; and $\frac{dg(y)}{dx} = \frac{dg(y)}{dy}\,\frac{dy}{dx}$.)

Notice also that while $\frac{dy}{dx} = 1\Big/\frac{dx}{dy}$ it is not true that

$$\frac{d^2y}{dx^2} = \frac{1}{\frac{d^2x}{dy^2}}$$

(in fact if we "invert" $\frac{d^2x}{dy^2}$ we get the form $\frac{dy^2}{d^2x}$ not $\frac{d^2y}{dx^2}$).

At any rate we obtain:

$$\frac{d^2y}{dx^2} = \frac{-6y}{(3y^2 + 1)^2}\frac{dy}{dx} \qquad (2)$$

From (1) $\frac{dy}{dx} = \frac{1}{(3y^2 + 1)}$ and putting this into (2), we get:

[2.4.7(L) cont'd]

$$\boxed{\frac{d^2y}{dx^2} = \frac{-6y}{(3y^2 + 1)^3}} \qquad (3)$$

Let us also observe that we could have obtained the same result by differentiating implicitly with respect to x. Namely:

$$y^3 + y = x \quad \rightarrow$$

$$3y^2 \frac{dy}{dx} + \frac{dy}{dx} = 1 \quad \rightarrow$$

$$(3y^2 + 1)\frac{dy}{dx} = 1 \quad \rightarrow$$

$\frac{dy}{dx} = \frac{1}{3y^2 + 1}$ <u>which is the same as (1).</u>

Next starting with $(3y^2 + 1)\frac{dy}{dx} = 1$ and again differentiating (this product) implicitly with respect to x, we obtain:

$$\left(6y\,\frac{dy}{dx}\right)\frac{dy}{dx} + (3y^2 + 1)\,\frac{d^2y}{dx^2} = 0$$

(where we observe that $\frac{d}{dx}(\frac{dy}{dx}) = \frac{d^2y}{dx^2}$ and by the chain rule $\frac{d}{dx}(3y^2 + 1) = 6y\,\frac{dy}{dx}$.)

[2.4.7(L) cont'd]

$$\therefore \frac{d^2y}{dx^2} = \frac{-6y\left(\frac{dy}{dx}\right)^2}{(3y^2 + 1)}$$

$$= \frac{-6y}{(3y^2 + 1)^3} \quad \text{since } \frac{dy}{dx} = \frac{1}{3y^2 + 1}$$

Let us conclude with two observations: 1) we never had to solve our equation explicitly for y as a function of x, and 2) there was no unique way of solving the problem.

2.4.8

Here $\frac{dt}{dx} = 5x^4 + 3x^2$. Hence

$$\frac{dx}{dt} = \frac{1}{5x^4 + 3x^2} \qquad (1)$$

when x = 1, (1) yields $\frac{dx}{dt} = \frac{1}{8}$. Thus the velocity of the particle at the given instant is $\frac{1}{8}$ ft/sec.

To find the acceleration we use

$$a = \frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}\left[\frac{1}{5x^4 + 3x^2}\right]$$

$$= \frac{d}{dx}\left[\frac{1}{5x^4 + 3x^2}\right]\frac{dx}{dt}$$

[2.4.8 cont'd]

$$= \frac{(5x^4 + 3x^2)(0) - 1(20x^3 + 6x)}{(5x^4 + 3x^2)^2}\left[\frac{1}{5x^4 + 3x^2}\right]$$

$$= \frac{-(20x^3 + 6x)}{(5x^4 + 3x^2)^3} \qquad (2)$$

Putting x = 1 into (2) we obtain that the required acceleration is:

$$\frac{-(20 + 6)}{(5 + 3)^3} = \frac{-26}{512} = \underline{\frac{-13}{256}\text{ ft/sec}^2}$$

2.4.9(L)

This problem not only affords us a review of several principles of calculus as well as an interesting application of calculus to geometry; but it allows us to introduce a rather important geometric concept which has application to several aspects of physics and engineering.

At any rate, let us begin with the solution to the problem.

If the circle is to be tangent to the curve at (1,2), it means that the circle must pass through the point (1,2) and have slope equal to 2 at that point (since the curve has slope 2 at this point; i.e., $\left.\frac{dy}{dx}\right]_{x=1} = 2x\Big]_{x=1} = 2$).

If we differentiate the equation of the circle implicitly, we obtain:

$$(x - h) + (y - k)dy/dx = 0 \qquad (1)$$

[2.4.9(L) cont'd]

whence:

$$dy/dx = -(x - h)/(y - k) \qquad (2)$$

Using (2) with $x = 1$ and $y = 2$, we see that the slope of the curve at $(1,2)$ is $-(1 - h)/(2 - k)$ and at the same time, we know that the slope must equal 2 at this point. Hence:

$$-(1 - h)/(2 - k) = 2$$

and this in turn leads to the fact that:

$$h + 2k = 5 \qquad (3)$$

Since h denotes the x-coordinate of the center of the circle, and k its y-coordinate, we have from (3) that the line $x + 2y = 5$ is the locus of the centers of all circles which are tangent to the curve $y = x^2 + 1$ at the point $(1,2)$. This is illustrated in Figure 1.

Next we observe that d^2y/dx^2 for the curve is 2; hence, for part (b) we wish the second derivative for the equation of the circle to equal 2 when $x = 1$ and $y = 2$. To find d^2y/dx^2 for the circle, we may differentiate (1) [or for that matter, (2)] implicitly. We obtain:

$$1 + (dy/dx)(dy/dx) + (y - k)d^2y/dx^2 = 0$$

(Observe again that in differentiating $(y - k)dy/dx$ with respect to x, we must apply the product rule and that $(dy/dx)' = d^2y/dx^2$ while $(y - k)' = dy/dx$.)

[2.4.9(L) cont'd]

Thus:

$$\frac{d^2y}{dx^2} = \frac{1 + \left(\frac{dy}{dx}\right)^2}{k - y}$$

$$\therefore \left(\frac{d^2y}{dx^2}\right)_{(1,2)} = \frac{1 + \left[\left(\frac{dy}{dx}\right)_{(1,2)}\right]^2}{k - 2} \qquad (4)$$

But we already know that $\left(\frac{dy}{dx}\right)_{(1,2)} = 2$. Putting this into (4), we obtain

$$\left(\frac{d^2y}{dx^2}\right)_{(1,2)} = \frac{5}{k - 2}$$

Since $\left(\frac{d^2y}{dx^2}\right)_{(1,2)}$ must also equal 2, we have:

$$\frac{5}{k - 2} = 2$$

or:

$$2k - 4 = 5$$

or:

$$\boxed{k = \frac{9}{2}}$$

This value of k put into (3) shows us that

$$h + 2\left(\frac{9}{2}\right) = 5$$

[2.4.9(L) cont'd]

or:

$$\boxed{h = -4}$$

Finally since $(x - h)^2 + (y - k)^2 = r^2$ must be satisfied when $x = 1$ and $y = 2$, we have that h, k, and r are always related by:

$$(1 - h)^2 + (2 - k)^2 = r^2$$

Replacing h by -4 and k by $\frac{9}{2}$, this yields:

$$5^2 + (\frac{-5}{2})^2 = r^2; \qquad \therefore r^2 = 25 + \frac{25}{4} = \frac{125}{4}$$

$$\therefore \boxed{r = \frac{5\sqrt{5}}{2}}$$

(Physically the radius of a circle cannot be negative; so we discarded the negative root.)

Hence our required circle in (b) is centered at (-4,9/2) with radius $\frac{5\sqrt{5}}{2}$ (see Figure 1). Thus the equation of this circle is

$$(x + 4)^2 + (y - \frac{9}{2})^2 = \frac{125}{4}$$

or

$$4(x + 4)^2 + (2y - 9)^2 = 125$$

or

$$4x^2 + 32x + 4y^2 - 36y = -20$$

[2.4.9(L) cont'd]

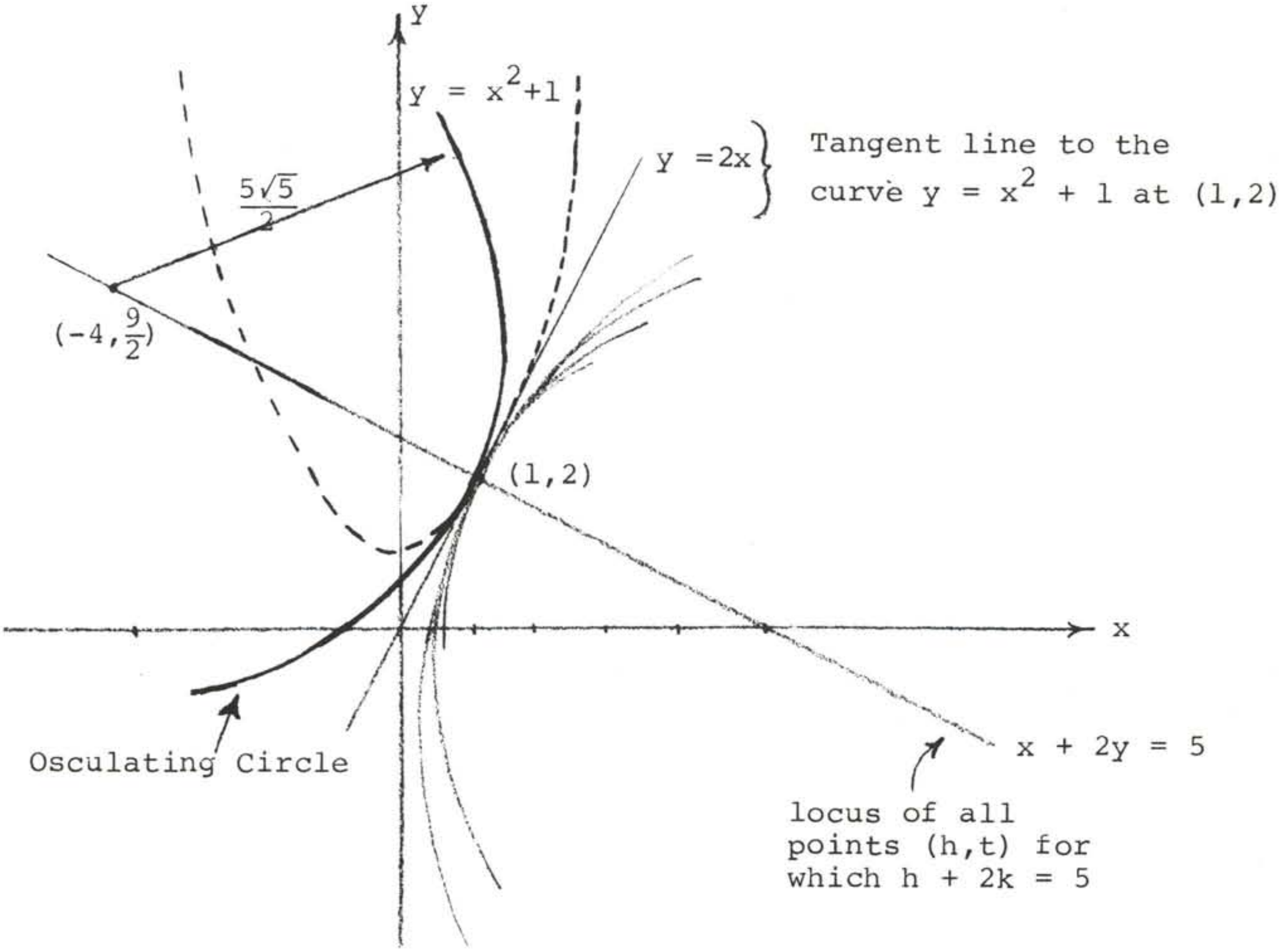


Figure 1

The circle whose first and second derivatives are equal to those of the curve at (1,2) is called the OSCULATING circle at (1,2). In many types of kinematics problems when we wish to study the acceleration and speed of a particle moving along a curve, it turns out that in a neighborhood of the point in question,

[2.4.9(L) cont'd]

we may replace the curve by its osculating circle. Indeed since the velocity and acceleration of a particle at point P depend only on the first two derivatives at P (and since the point P has no way of knowing whether we are viewing it as being part of the curve or as being part of the circle) it "figures" that we can replace the curve by the circle in dealing with instantaneous speed and acceleration.

UNIT 5: Continuity

2.5.1(L)

Our definition of continuity requires that

$$\lim_{x \to c} f(x) = f(c)$$

if f is to be continuous at $x = c$.

In the present exercise $c = 2$, while $f(x) = \frac{x^2 - 5x + 6}{x - 2}$

Now:
$$\frac{x^2 - 5x + 6}{x - 2} = \frac{(x - 2)(x - 3)}{(x - 2)} \qquad (1)$$

Hence, if $x \neq 2$, $f(x) = x - 3$ (since, as long as $x \neq 2$, we can cancel $x - 2$ from numerator and denominator of (1))

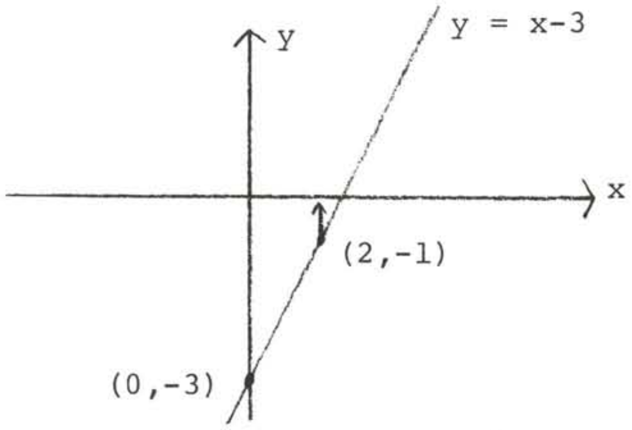
Thus:

$$\lim_{x \to c} f(x) = \lim_{x \to 2} (x - 3) = -1$$

Or the other hand, $f(c) = f(z) = \frac{2^2 - 5(2) + 6}{2 - 2} = \%$, so f(2) is undefined. In other words, in this example, $\lim_{x \to c} f(x)$ exists but f(c) doesn't.

For this reason $\frac{x^2 - 5x + 6}{x - 2}$ is not continuous at $x = 2$. To see what has happened here, observe that the graph of $\frac{x^2 - 5x + 6}{x - 2}$ is the same as that of $x - 3$ when $x \neq 2$. Thus:

[2.5.1(L) cont'd]



y = f(x) is the
curve y = x-3
with the point
(2,-1) deleted.

(Figure 1)

Thus, our curve seems to be missing a point, and this is what prevents the curve from being "unbroken." (Here is a good place to point out the difference between a point and a dot. A point, having no thickness, cannot be detected when it is deleted from a curve; for if we can see the deletion we have taken out more than a point. In other words, "deleting a point" can be done analytically, but not pictorially, and this, then, is one place where the graph can be misleading.)

Of course, the fact that $\lim_{x\to 2} f(x) = -1$ gives us a strong hint as to how f(2) should be defined if we want f to be continuous at x = 2. Namely, f(2) should equal -1. More formally what we do is observe that f(2) is not defined, so we "invent" a new function, say, g such that

$$g(x) = \begin{cases} f(x) & \text{if } x \neq 2 \\ -1 & \text{if } x = 2 \end{cases}$$

[2.5.1(L) cont'd]

The main idea is that $g(x)$ is precisely $f(x)$ when $x \neq 2$, but when $x = 2$ $g(x)$ is defined (that is $g(2) = -1$ by definition of g) while $f(x)$ is not defined at $x = 2$. We shall expand this idea in the next exercise, but for now let us observe that our last remarks are the analytic equivalent of "plugging in" the point $(2,-1)$ to the curve $y = \frac{x^2 - 5x + 6}{x - 2}$ to make it continuous.

Aside from providing us drill with the definition of continuity, this problem is meant to emphasize two things. In the first place, by definition, f cannot be continuous at a point at which it is not defined. In still other words, the criterion that $\lim_{x \to c} f(x) = f(c)$ implies that $f(c)$ must exist.

Secondly, this exercise provides us with an example of a discontinuity which is neither a "jump" nor an "infinity." It is merely a missing point from an otherwise continuous curve.

Such a discontinuity is called a <u>removable singularity</u> since, as the name implies, we can remove the "bad" spot by "plugging in" the appropriate point.

Observe that this problem was with us from the time we defined a derivative. Namely $\frac{f(x + \Delta x) - f(x)}{\Delta x}$ is <u>not</u> defined when $\Delta x = 0$. This is why we introduced deleted neighborhoods of 0 when we talked about $\lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x}\right]$.

In other words if we define $g(\Delta x) = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$, g cannot be continuous at $\Delta x = 0$ since $g(0) = \%$ which is indeterminate.

[2.5.1(L) cont'd]

In this case $\lim\limits_{\Delta x \to 0} g(\Delta x)$ is <u>not</u> $g(0)$. That's why the condition $\lim\limits_{x \to c} f(x) = f(c)$ is not as trivial a condition as what first meets the eye.

(By the way, do not confuse this discussion with the special problem in this exercise. In this exercise $f'(c)$ doesn't exist because $f(c)$ doesn't exist. That is,

$$f'(2) = \lim_{\Delta x \to 0} \left[\frac{f(2 + \Delta x) - f(2)}{\Delta x} \right]$$

but $f(2)$ is undefined.)

2.5.2

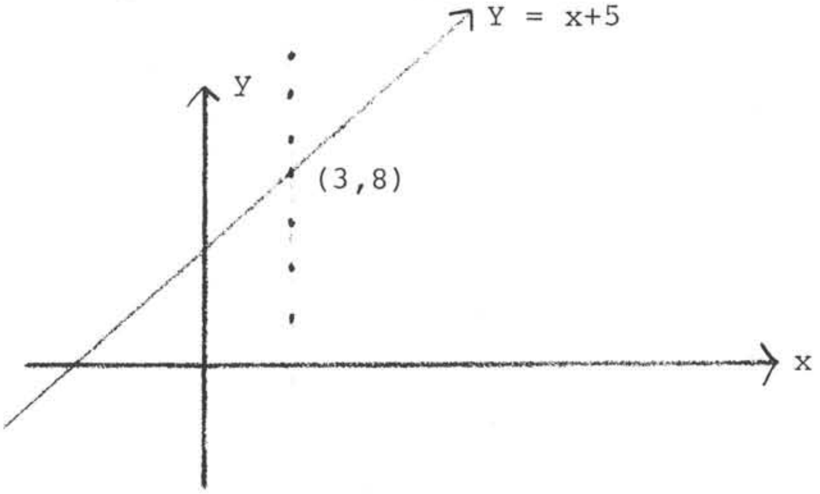
Here we see that $g(3)$ is not defined. On the other hand, however, if $x \neq 3$ then we can cancel $x-3$ from both the numerator and denominator of $g(x)$ to obtain that $g(x) = x + 5$ for all x provided $x \neq 3$. In any event $\lim\limits_{x \to 3} g(x) = 8$, and since for the continuity of g at $x = 3$ we must have $\lim\limits_{x \to 3} g(x) = g(3)$, it follows that $g(3) = 8$.

In still other words, if we define, say, $h(x)$ by

$$h(x) = \begin{cases} g(x) & \text{if } x \neq 3 \\ c & \text{if } x = 3\text{, where } c \text{ is any constant} \end{cases}$$

[2.5.2 cont'd]

then h is defined for all real numbers. The only value of c that can be chosen, however, if we desire that h be continuous at x = 3 is c = 8. If c ≠ 8 then h is discontinuous at x = 3 just as g is, even though h is defined at x = 3 while g isn't.



The graph of h is the line y = x+5 but with the point (3,8) deleted and replaced by the point (3,c) (If c=8, then we get the continuous line, y = x+5)

2.5.3(L)

Before proceeding with this exercise, let us point out from a more intuitive point of view what this exercise is trying to tell us. Geometrically, continuity refers to an unbroken curve. Now it is clear that a curve can be unbroken but still have "sharp edges". In other words, the curve can be unbroken but not "smooth". Differentiability refers to "smoothness". After all the derivative involves the slope of a line tangent to a curve at a point, and if we can draw the tangent line the curve must be "smooth".

[2.5.3(L) cont'd]

We shall substantiate these ideas more rigorously in a few moments. For now, observe that it is fairly clear, geometrically, that a smooth curve must be unbroken but an unbroken curve does not have to be smooth. Translated into the more formal language this says that a differentiable function must be continuous (part (a) of this exercise) but that a continuous function does not have to be differentiable (part (b) of this exercise).

As for the actual details, let us start with part (a). To prove that f is continuous at $x = x_1$ we must show that $\lim_{x \to x_1} f(x) = f(x_1)$, and this is equivalent to showing that $\lim_{x \to x_1} [f(x) - f(x_1)] = 0$. (Again, continue to be alert in distinguishing between $x \to x_1$ and $f(x) \to f(x_1)$ If f is not continuous at $x = x_1$, $f(x)$ need not approach $f(x_1)$ as $x \to x_1$)

Now, we are armed with the knowledge that $f'(x_1)$ exists. This, in turn, means that $\lim_{x \to x_1} \left[\frac{f(x) - f(x_1)}{x - x_1}\right]$ exists and is equal to $f'(x_1)$.

To utilize this information, we write $f(x) - f(x_1)$ in our usual "clever" way. That is:

$$f(x) - f(x_1) = \left[\frac{f(x) - f(x_1)}{x - x_1}\right] (x - x_1)$$

Applying our limit theorems, we obtain:

[2.5.3(L) cont'd]

$$\lim_{x \to x_1} [f(x) - f(x_1)] = \lim_{x \to x_1} \left[\frac{f(x) - f(x_1)}{x - x_1} \right] \lim_{x \to x_1} (x - x_1)$$

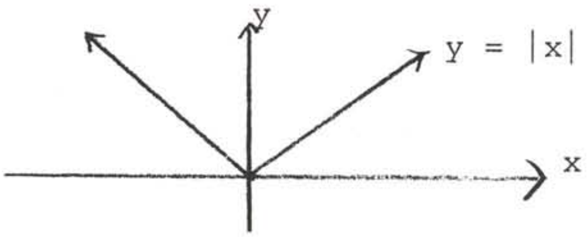
$$= f'(x_1)(0) \text{ where } f'(x_1) \text{ is some real number}$$

or $\lim_{x \to x_1} [f(x) - f(x_1)] = 0$, which establishes part (a).

As for part (b), we need only give a counter-example (that is, one example in which f is continuous at $x = x_1$ but not differentiable at $x = x_1$) to establish that the result need not be true.

To this end, consider, for example, f defined by $f(x) = |x|$. It is easy to see that $\lim_{x \to 0} f(x) = f(0)$.

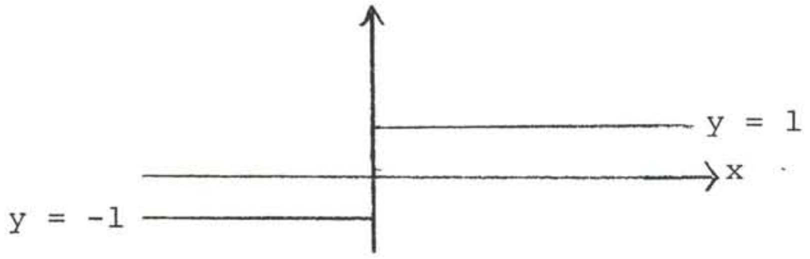
Pictorially:



$$\therefore\ y = \begin{cases} x & x \geqslant 0 \\ -x & x < 0 \end{cases}$$

Now, for $x < 0$ $f(x) = -x$, hence $f'(x) = -1$, while for positive values of x, $f(x) = x$, hence $f'(x) = +1$. Thus, the graph of $y = f'(x)$ is given by:

[2.5.3(L) cont'd]



graph of f'(x)
where f(x) = |x|

and we see that there is a "jump" at x = 0.

More importantly f'(0) doesn't exist.

$$\text{Indeed } f'(0) = \lim_{\Delta x \to 0} \frac{f(\Delta x) - f(0)}{\Delta x}$$

and in this case since f(x) = |x|,

$$f'(0) = \lim_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x}^{*} \tag{1}$$

In (1) $|\Delta x| = \Delta x$ if $\Delta x > 0$ while $|\Delta x| = -\Delta x$ if $\Delta x < 0$

Therefore (1) implies:

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[*] For what it's worth notice that $\frac{|u|}{u}$ is just another way of saying $\begin{cases} 1 & \text{if } u > 0 \\ -1 & \text{if } u < 0 \end{cases}$ since the magnitude of $\frac{|u|}{u}$ $(u \neq 0)$ is always 1, but since $|u| > 0$ the sign is positive if u is positive and negative if u is negative. In any event the strange-looking expression agrees with our previous results.

[2.5.3(L) cont'd]

$$\lim_{\Delta x\to 0^+}\left[\frac{f(\Delta x) - f(0)}{\Delta x}\right] = \lim_{\Delta x\to 0^+}\left[\frac{|\Delta x|}{\Delta x}\right] = \lim_{\Delta x\to 0^+}\left(\frac{\Delta x}{\Delta x}\right) = +1 \qquad (2)$$

while

$$\lim_{\Delta x\to 0^-}\left[\frac{f(\Delta x) - f(0)}{\Delta x}\right] = \lim_{\Delta x\to 0^-}\left[\frac{|\Delta x|}{\Delta x}\right] = \lim_{\Delta x\to 0^-}\left(\frac{-\Delta x}{\Delta x}\right) = -1 \qquad (3)$$

(2) and (3) show that $\lim_{\Delta x\to 0}\left[\frac{f(\Delta x) - f(0)}{\Delta x}\right]$ does not exist

since $\lim_{\Delta x\to 0^+}\left[\frac{f(\Delta x) - f(0)}{\Delta x}\right] \neq \lim_{\Delta x\to 0^-}\left[\frac{f(\Delta x) - f(0)}{\Delta x}\right]$

2.5.4

Let the polynomial be denoted by

$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ where a_0, a_n are constants.

For any number c, we have:

[2.5.4 cont'd]

$$\lim_{x\to c} P(x) = \lim_{x\to c} \left(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0\right)$$

$$= a_n \left(\lim_{x\to c} x\right)^n + a_{n-1}\left(\lim_{x\to c} x\right)^{n-1} + \ldots + a_1\left(\lim_{x\to c} x\right) + a_0$$

$$= a_n c^n + a_{n-1}c^{n-1} + \ldots + a_1 c + a_0$$

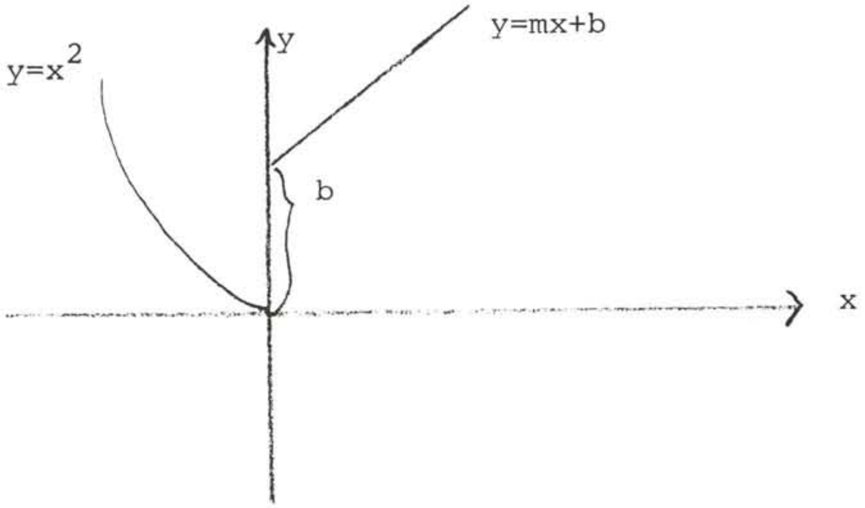
$$= P(c)$$

But $\lim_{x\to c} P(x) = P(c)$ means that P is continuous at $x = c$.

An alternative way which utilizes the result of Exercise 2.5.3(L) is to observe that we have already proven that P is a differentiable function, hence by 2.5.3(L) part (a), P must also be continuous.

2.5.5(L)

Graphically, $y = mx + b$ is the straight line of slope m with b as its y-intercept. Thus:

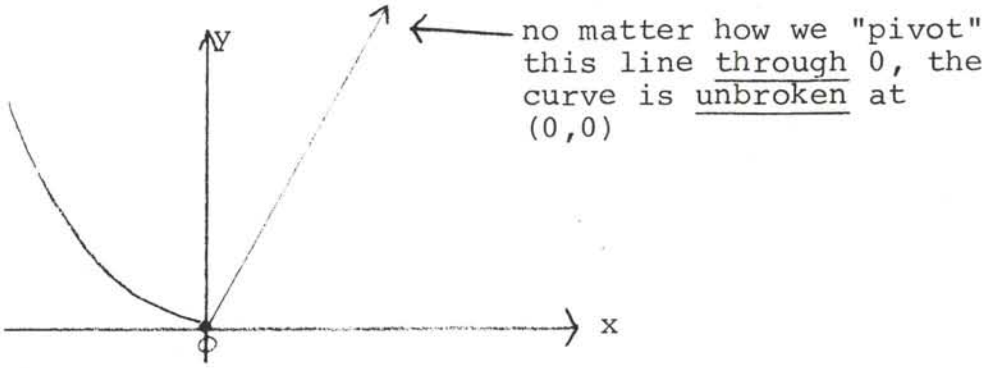


[2.5.5(L) cont'd]

we have the graph of f(x).

From the graph, it is clear that we have a jump discontinuity at x = 0 unless b = 0.

Once b is zero, the graph is continuous for any real value of m. That is



From a more analytic point of view, we observe that $\lim_{x\to 0^+} f(x) = b$, $\lim_{x\to 0^-} f(x) = 0$ and $f(0) = b$. (Notice that we cannot say $f(0) = 0$ because $f(x) = x^2$ for $x < 0$; 0 falls into the $x \geqslant 0$ category, and for $x \geqslant 0$, $f(x) = mx + b \therefore f(0) = b$).

Thus, for f to be continuous at x = 0 we must have $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$ and this says b = 0.

As for part (b), we observe that for $x < 0$, $f'(x) = 2x$ and for $x > 0$, $f'(x) = m$. Thus $\lim_{x\to 0^-} f'(x) = 0$, while $\lim_{x\to 0^+} f'(x) = m$.

[2.5.5(L) cont'd]

Hence, if there is any hope that f'(0) exists, it had better be that m = 0. (What we mean by "any hope" will be discussed at the end of this exercise.)

It should be stressed that as always we can work directly from the basic definition. That is, we want to compute f'(0) and we know that:

$$f'(0) = \lim_{\Delta x \to 0} \left[\frac{f(0 + \Delta x) - f(0)}{\Delta x} \right]$$

From part (a) we have $f(x) = mx$ for $x \geqslant 0$, hence $f(0) = 0$. Thus:

$$f'(0) = \lim_{\Delta x \to 0} \frac{f(\Delta x)}{\Delta x}$$

We next note that $f(\Delta x)$ depends on whether $\Delta x < 0$ or $\Delta x > 0$. If $\Delta x < 0$ then $f(\Delta x) = (\Delta x)^2$*; whereupon

$$\lim_{\Delta x \to 0^-} \frac{f(\Delta x)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{(\Delta x)^2}{\Delta\ x} = \lim_{\Delta x \to 0^-} \Delta x = 0 \qquad (1)$$

If $\Delta x > 0$ then $f(\Delta x) = m\Delta x$; whereupon

---

*Again observe that, for example, $f(x) = x^2$ means $f([\ ]) = [\ ]^2$ (or: output = $(\text{input})^2$). Thus $f(\Delta x) = (\Delta x)^2$.

[2.5.5(L) cont'd]

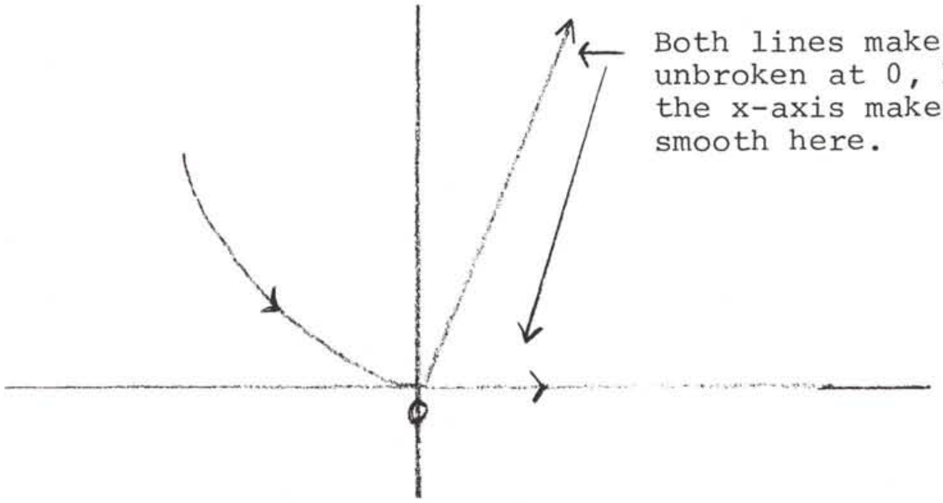
$$\lim_{\Delta x \to 0^+} \frac{f(\Delta x)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{m\Delta x}{\Delta x} = m \qquad (2)$$

In particular, for $\lim_{\Delta x \to 0} \frac{f(\Delta x)}{\Delta x}$ to exist we must have:

$$\lim_{\Delta x \to 0^+} \frac{f(\Delta x)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{f(\Delta x)}{\Delta x}$$

and from (1) and (2), this requires that m = 0.

In terms of the graph, the x-axis is the only straight line by which we can "continue" $y = x^2$ if the resulting curve is to be smooth at the origin. That is:



Both lines make the curve unbroken at 0, but only the x-axis makes the curve smooth here.

[2.5.5(L) cont'd]

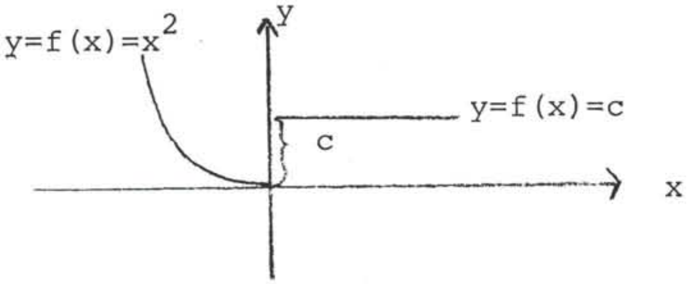
In Summary:

$$\begin{cases} f \text{ is differentiable at } x = 0 \longleftrightarrow m \text{ and } b \text{ are } 0 \\ f \text{ is continuous at } x = 0 \longleftrightarrow b = 0 \end{cases}$$

There is a subtle case that occurs and which can cause great confusion if not properly handled. Let us modify the present exercise by letting g be defined by:

$$g(x) = \begin{cases} x^2, & \text{if } x < 0 \\ c, & \text{if } x \geqslant 0 \end{cases}$$

Our f(x) is the special case of g when c = 0. The graph of g looks pretty much like the graph of f (when m = b = 0) except that the x-axis looks "displaced". That is:



If we look at the graph, we might get the idea that g'(0) exists and is equal to 0. Certainly in terms of the picture, it seems that the slope of the curve is 0 at x = 0, but here we run into the subtle problem of the difference between a point and a dot. More specifically, notice that it is true that for

[2.5.5(L) cont'd]

$x < 0$, $g'(x) = 2x$ and as a result $\lim\limits_{x \to 0^-} g'(x) = 0$. Moreover, since $g'(x) = 0$ for $x > 0$, $\lim\limits_{x \to 0^+} g'(x) = 0$. Thus we see that

$$\lim_{x \to 0^-} g'(x) = \lim_{x \to 0^+} g'(x) = 0$$

and all this tells us for sure is that <u>if</u> $g'(0)$ exists, then it equals 0. However $g'(0)$ need not exist.

In fact, in this case we see that:

$$g'(0) = \lim_{\Delta x \to 0} \frac{g(\Delta x) - g(0)}{\Delta x}$$

From the definition of g, $g(0) = c$ (since 0 belongs to $\{x\colon x \geqslant 0\}$)

$$\therefore g'(0) = \lim_{\Delta x \to 0} \frac{g(\Delta x) - c}{\Delta x}$$

Now if $\Delta x < 0$, $g(\Delta x) = \overline{\Delta x}^2$ while if $\Delta x \geqslant 0$, $g(\Delta x) = c$.

$\therefore \lim\limits_{\Delta x \to 0^+} \left[\dfrac{g(\Delta x) - c}{\Delta x}\right] = \lim\limits_{\Delta x \to 0^+} \left[\dfrac{c - c}{\Delta x}\right] = 0$, as we probably suspected. However

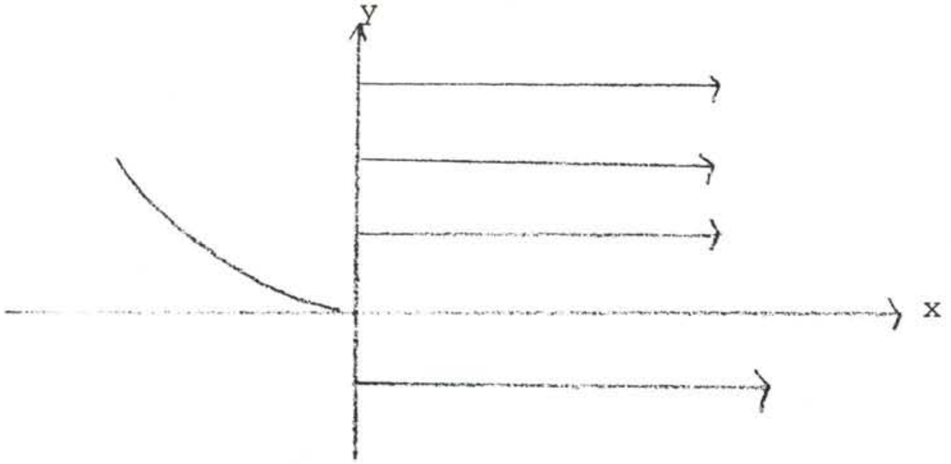
$$\lim_{\Delta x \to 0^-} \left[\frac{g(\Delta x) - c}{\Delta x}\right] = \lim_{\Delta x \to 0^-} \left[\frac{\overline{\Delta x}^2 - c}{\Delta x}\right] = \lim_{\Delta x \to 0} \left[\Delta x - \frac{c}{\Delta x}\right]$$

$$= -\infty \quad \underline{\text{unless } c = 0}$$

[2.5.5(L) cont'd]

That is if $c \neq 0$ $\lim\limits_{\Delta x \to 0^-} \left[\frac{g(\Delta x) - c}{\Delta x}\right]$ doesn't exist.

Pictorially



for any of these horizontal lines it is true that

$$\lim_{x \to 0^+} g'(x) = \lim_{x \to 0^-} g'(x) = 0;$$ but only for the x-axis (c=0) is g differentiable at x = 0.

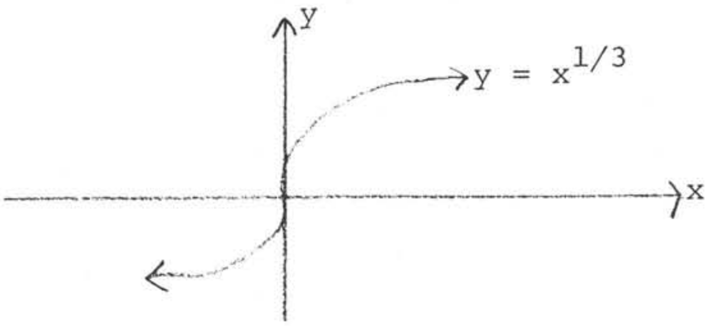
From a practical point of view this means that displacing (i.e., moving it "parallel" to itself) a graph changes its properties only at the point of origin of the displacement.

This last example shows again that if a curve is broken at a point, it is not smooth at that point. In terms of more analytic language, then, this also illustrates that a discontinuous curve is not differentiable at that point, no matter

[2.5.5(L) cont'd]

how nicely it behaves at all other points in the neighborhood.

There is, however, one other special case for which "differentiable" and "smooth" are not the same. Consider for example the curve $y = x^{1/3}$. This curve is smooth. In fact, its graph is given by:



Now, if we look at the picture, we see that the curve has a vertical tangent at (0,0). Since we exclude "infinity" as a limit, we have that $\frac{dy}{dx}$ does not exist at (0,0). In the language of functions, if $f(x) = x^{1/3}$ then $f'(0)$ does <u>not</u> exist. Indeed:

$$f'(0) = \lim_{\Delta x \to 0} \left[\frac{f(0 + \Delta x) - f(0)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{f(\Delta x)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{\Delta x^{1/3}}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{1}{\Delta x^{2/3}} \right] = \frac{"1"}{0} = \infty$$

[2.5.5(L) cont'd]

In any event at this stage it might be to our advantage to summarize various combinations of continuity and discontinuity. Briefly, a curve is continuous, (in which case it can be "smooth" but doesn't have to be) or it is discontinuous.

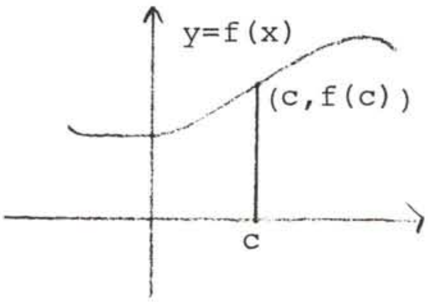
If the function is undefined at x = c, it cannot be continuous at x = c since in this case f(c) doesn't exist, and as a result, it is impossible to satisfy the criterion that $\lim\limits_{x \to c} f(x) = f(c)$.

Hopefully, it is also clear that if f(c) doesn't exist then neither can f'(c) since the definition of f'(c) involves f(c) explicitly. That is:

$$f'(c) = \lim_{x \to c} \left[\frac{f(x) - f(c)}{x - c} \right]$$

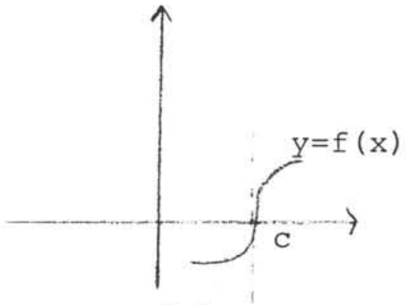
In any event, perhaps the following sketches will be of help:

Continuous:



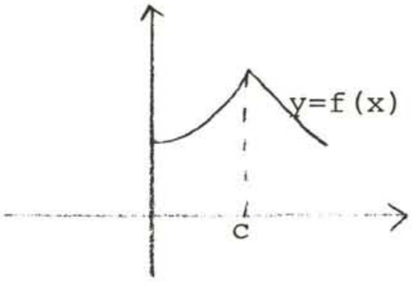
(a)

smooth since f'(c) exists



(b)

curve is smooth at x=c but f'(c) doesn't exist in the sense that f'(c) = ∞

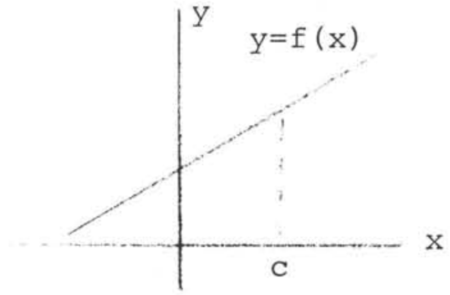


(c)

curve has "sharp" corner at x = c. That is, f is continuous at x=c, but not f' which has a "jump" at x=c

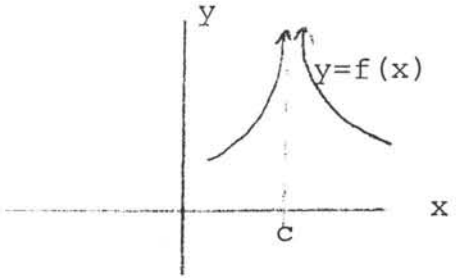
[2.5.5(L) cont'd]

Discontinuous



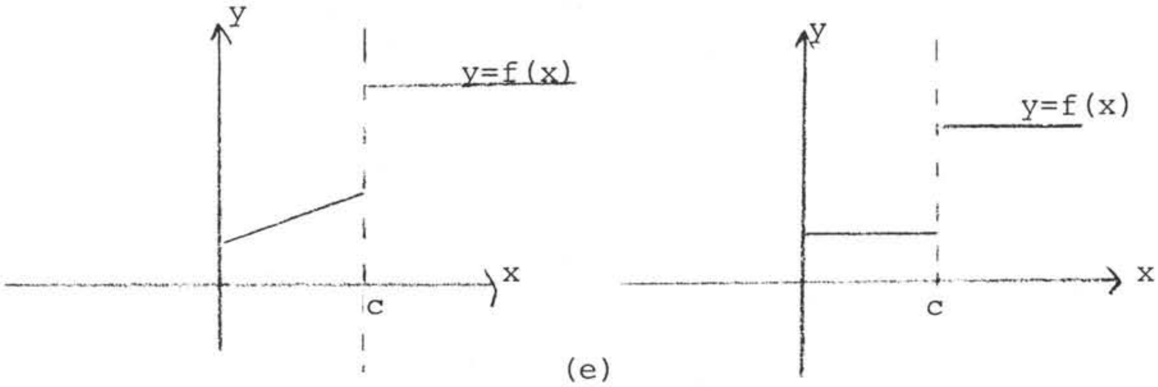
(d)

Removable singularity at x=c but f(c) is not defined. Hence f'(c) is not defined either.



(e)

"Infinite" discontinuity at f(c) doesn't exist. That is, $\lim_{x\to c^-} f(x) = \infty$, $\lim_{x\to c^+} f(x) = \infty$



(i)

(e)

(ii)

Both (i) and (ii) are examples of "jump" discontinuties. In each case $\lim_{x\to c^+} f(x) \neq \lim_{x\to c^-} f(x)$

[2.5.5(L) cont'd]

Also in both (i) and (ii) f'(c) fails to exist. This is not too surprising in (i) since $\lim_{x\to c^+} f'(x) \neq \lim_{x\to c^-} f'(x)$. However in (ii) $\lim_{x\to c^+} f'(x) = \lim_{x\to c^-} f'(x)$ (=0, as we've drawn it) but f'(c) still doesn't exist . (In this case, at all costs, avoid confusing f'(c) with $\lim_{x\to c} f'(x)$; f'(c) is, if it exists,

$$\lim_{x\to c}\left[\frac{f(x) - f(c)}{x - c}\right]\Bigg)$$

2.5.6

For $x \geqslant 0$ we have $f'(x) = 3x^2$ while for $x < 0$, we have $f'(x) = 4x^3$

Hence

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^-} f'(x) = 0 \qquad (1)$$

On the other hand

$$f'(0) = \lim_{\Delta x\to 0} \frac{f(\Delta x) - f(0)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{f(\Delta x) - 0^*}{\Delta x} = \lim_{\Delta x\to 0} \frac{f(\Delta x)}{\Delta x}$$

---

[*] Again f(0) is defined by $f(x) = x^3$ not by $f(x) = x^4+1$. Since 0 belongs to $\{x: x>0\}$ and not to $\{x:x<0\}$.

[2.5.6 cont'd]

Now $$\lim_{\Delta x\to 0^+} \frac{f(\Delta x)}{\Delta x} = \lim_{\Delta x\to 0^+} \frac{(\Delta x)^3}{(\Delta x)} = \lim_{\Delta x\to 0^+} (\Delta x)^2 = 0$$

But $$\lim_{\Delta x\to 0^-} \frac{f(\Delta x)}{\Delta x} = \lim_{\Delta x\to 0^-} \frac{(\Delta x)^4 + 1}{\Delta x} = \lim_{\Delta x\to 0^-} (\Delta x)^3 + \frac{1}{\lim\limits_{\Delta x\to 0^-} \Delta x} = \text{"}\infty\text{"}$$

$\therefore$ f'(0) does not exist since

$$\lim_{\Delta x\to 0^+} \frac{f(\Delta x)-f(0)}{\Delta x} \neq \lim_{\Delta x\to 0^-} \frac{f(\Delta x)-f(0)}{\Delta x} \qquad (2)$$

(Notice that we are talking about different limits in (1) and (2) and it's critical that we see the difference.)

2.5.7

From an intuitive point of view, this exercise is an example of trying to show that "combinations" of "smooth" functions yield "smooth" functions. Slightly, more formally, this exercise is specifically concerned with showing that the sum of continuous functions is also continuous (or at least a finite sum - we must deal with infinite sums later in the course). Similar results hold for differences, products, and quotients (except, of course, for dividing by zero).

[2.5.7 cont'd]

As for part (a) the fact that f and g are continuous at c means that $\lim_{x\to c} f(x) = f(c)$ and $\lim_{x\to c} g(x) = g(c)$. If we now define h by $h(x) = f(x) + g(x)$, we are being asked to show that h is continuous at $x = c$. The key point here is that everything follows directly from our previously-derived limit theorems. In this case:

$$\begin{aligned}\lim_{x\to c} h(x) &= \lim_{x\to c} [f(x) + g(x)] \\ &= \lim_{x\to c} f(x) + \lim_{x\to c} g(x) \\ &= f(c) + g(c) \quad \text{(by definition of f and g being continuous at c)} \\ &= h(c)\end{aligned}$$

but $\lim_{x\to c} h(x) = h(c)$ is precisely what we needed to show in order to prove that $h = f + g$ is continuous at $x = c$.

As for part (b) we can show that if $f_1, \ldots, f_n$ are each continuous at $x = c$ then so also is h where h is defined by $h(x) = f_1(x) + \ldots + f_n(x)$.

Moreover, we do not even have to invoke mathematical induction here since the key steps have already been proven by induction (such as the fact that the limit of a sum is the sum of the limits).

[2.5.7 cont'd]

We have:

$$\lim_{x\to c} h(x) = \lim_{x\to c} [f_1(x) + \dots + f_n(x)]$$

$$= \lim_{x\to c} f_1(x) + \dots + \lim_{x\to c} f_n(x)$$

Since $f_1, \dots, f_n$ are each continuous at $x = c$, we have that:

$$\lim_{x\to c} f_1(x) = f_1(c), \dots, \lim_{x\to c} f_n(x) = f_n(c)$$

Hence:

$$\lim_{x\to c} h(x) = f_1(c) + \dots + f_n(c)$$

$$= h(c)$$

and the proof is complete.

2.5.8(L)

While one major aim of this exercise is to emphasize some important properties of continuous functions, the exercise also seems to illustrate how topics which were once considered under the heading of "advanced algebra" are better off if left to the calculus. For example, in pre-calculus

[2.5.8(L) cont'd]

algebra courses, we were often concerned with finding roots of equations such as $f(x) = 0$ where $f(x)$ might have been a polynomial or a trigonometric function or a logarithmic function etc.

A common technique for solving such problems came under the name of "Horner's Method." In this method we tried to find two numbers, say m and n (without loss of generality assume $m < n$) such that $f(m)$ and $f(n)$ had different signs. We could then conclude that $f(x) = 0$ had a root for some number x between m and n.

To understand why this method was valid, let us first observe that equations and curves are related by the fact that the roots of $f(x) = 0$ are precisely the x-coordinates of the points at which the curve $y = f(x)$ crosses the x-axis. The method relies on the fact that f is a continuous function, for if the curve is continuous it cannot get from above the x-axis to below the x-axis without crossing the x-axis. (Pictorially this is simple to see. In the language of mathematical analysis we say that if f is continuous on $[m,n]$ then for every number y between $f(m)$ and $f(n)$ there exists a number $x \varepsilon [m,n]$ such that $f(x) = y$. That is, a continuous function assumes all intermediary values.)

With respect to this particular problem, since f is continuous on $[0,1]$ and $f(0)$ is negative while $f(1)$ is positive, we have that $f(0) < 0 < f(1)$ and by the result concerning intermediary values there must be <u>at least</u> one number x between 0 and 1 such that $f(x) = 0$. More intuitively,

[2.5.8(L) cont'd]

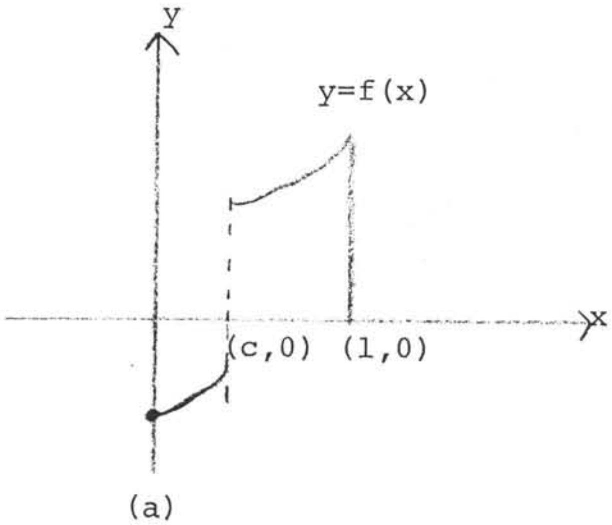
since the curve is below the x-axis when x = 0 and above the x-axis when x = 1, the fact that the curve is unbroken means that it crosses the axis somewhere between 0 and 1.

Again notice the fact that the interval is from 0 to 1 has no bearing on the theory here. Quite in general if f is continuous between a and b and f(a) and f(b) have different signs then we know there is at least one number x between a and b such that f(x) = 0.

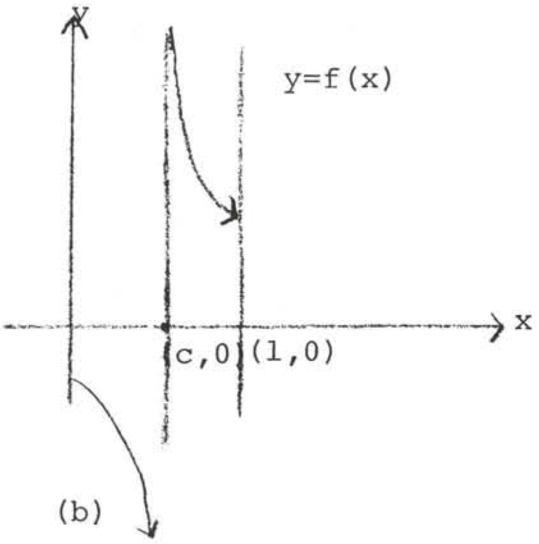
Of even more importance, we must be careful to avoid pitfalls and not read more into this result than what is really there. We shall illustrate our remaining remarks pictorially:

(1) If f is not continuous on [0,1] the conclusion need not be true (although it can be true).

For example:



(a)

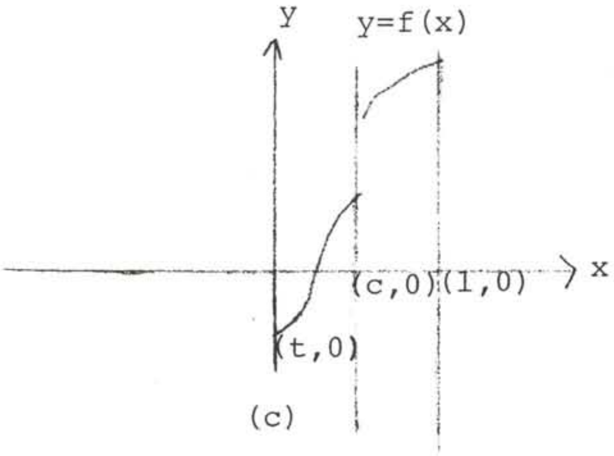


(b)

[2.5.8(L) cont'd]

In (a) we have a finite jump discontinuity at c where $0 < c < 1$ while in (b) we have an infinite jump at c where $0 < c < 1$. In both cases $f(0) < 0$, $f(1) > 0$, yet there is no $x \in [0,1]$ for which $f(x) = 0$.

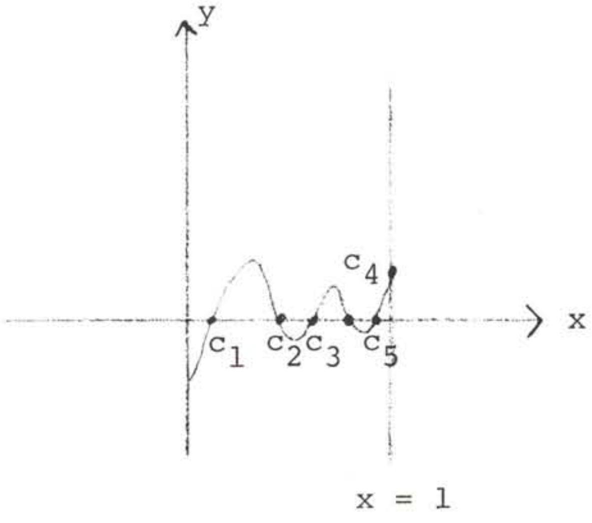
Of course the jump could occur after we cross the axis in which case the result would hold sort of by "luck." For example:



(c)

Here $0<t<1$ and $f(t)=0$ even though f is still discontinuous at $x=c$.

(2) We are guaranteed that there is <u>at least one</u> root between 0 and 1, but Figure (d) indicates that there can be more:



(d)

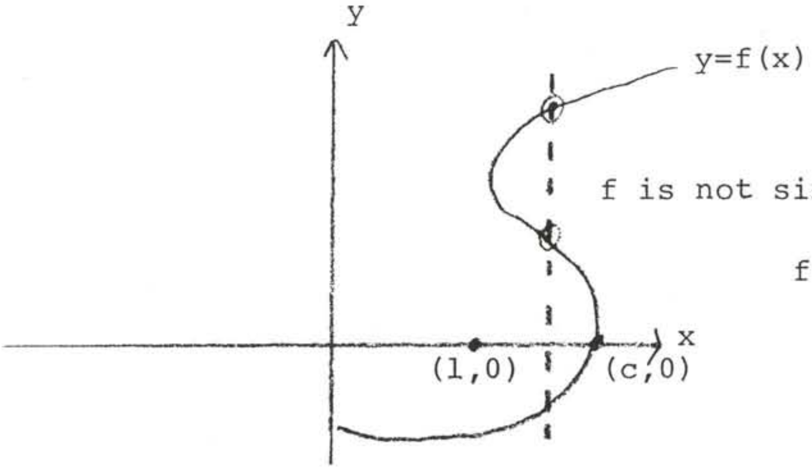
[2.5.8(L) cont'd]

In (d) $f(0) < 0$, $f(1) > 0$ and

$$f(c_1) = f(c_2) = f(c_3) = f(c_4) = f(c_5) = 0$$

where $0 < c_1 < c_2 < c_3 < c_4 < c_5 < 1$.

(3) The result of this exercise also depends on the fact that f is single-valued. If f were multivalued the curve, if continuous, would still have to cross the x-axis, but not necessarily between 0 and 1. For example:



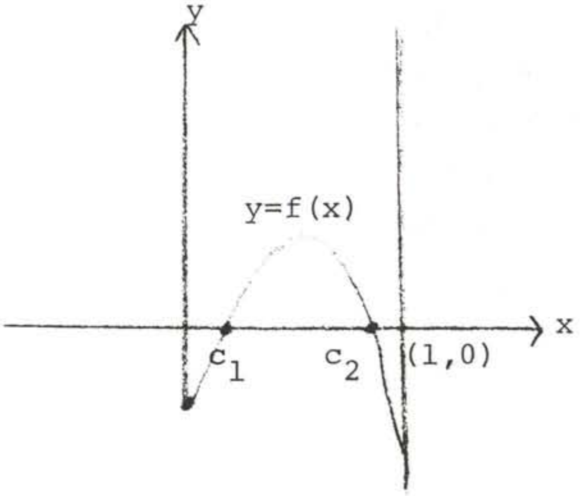
f is not single-valued

$f(c)=0$ but $c \notin [0,1]$

(e)

(4) The fact that f changes sign is sufficient to guarantee a root but it's not necessary. For example:

[2.5.8(L) cont'd]

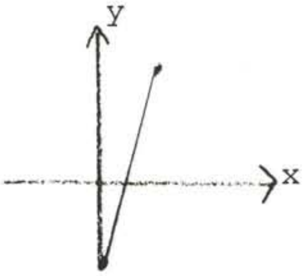


(f)

In (f) $f(c_1)=f(c_2)=0$ where $0<c_1<c_2<1$ yet f(0) and f(1) are both negative and certainly f is continuous on [0,1]

(Figure (f) indicates the weakness of Horner's Method. Namely, if we don't pick our "inputs" sufficiently close, we can "skip over" candidates for roots. In (f) notice that there are neighborhoods of both c_1 and c_2 in which f(x) changes sign but [0,1] is too "crude" to pick these neighborhoods up!)

(5) Finally, if we add our knowledge of differentiability to that of continuity we can make stronger statements. For example, if $f'(x) > 0$ for all $x \varepsilon [0,1]$ and $f(0) < 0$ while $f(1) > 0$ then there exists <u>exactly one</u> number c where $0 < c < 1$ for which $f(c) = 0$. Namely



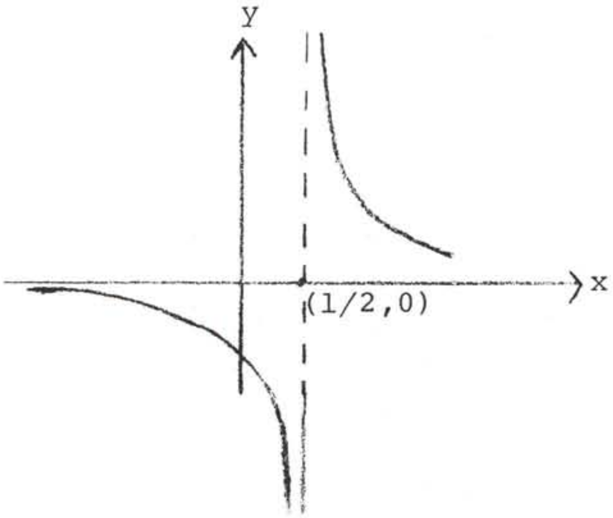
(g)

The fact that $f'(x)>0$ means the curve is always rising, hence once it gets above the x-axis it can't get back down!

2.5.9

(a) Since a fraction cannot be zero unless the numerator is zero, it is impossible for f(x) = 0 for any real number x since the numerator of f(x) is always 1, and hence, in particular, never equal to zero. Since f(x) can never equal zero for any real value of x, there are no values of x between 0 and 1 for which f(x) can equal 0.

(b) To obtain the result of Exercise 2.5.8(L) it was necessary that f be continuous for all x between 0 and 1. However, in this problem f isn't defined (let alone, continuous) at x = 1/2. That is, f(1/2) = $\frac{1}{1-1}$ which looks like "infinity." That is f(x) increases without bound as x approaches 1/2 from the right and it decreases without bound as x approaches 1/2 from the left. In other words, f has an "infinite-type" of discontinuity at x = 1/2. In terms of a picture:



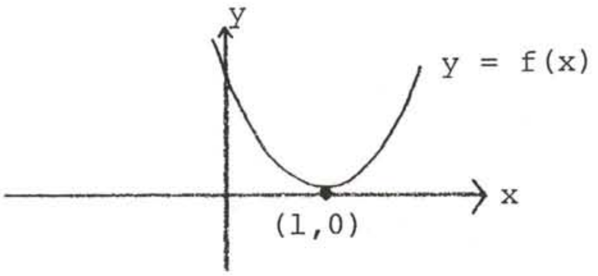
UNIT 6: Applications of the Derivative I

2.6.1(L)

Notice that these two parts are identical except for the fact that one deals with f' while the other deals with f". Thus, these two parts, side by side, serve as a good vehicle for emphasizing the difference between the first and second derivatives as they apply to curve plotting.

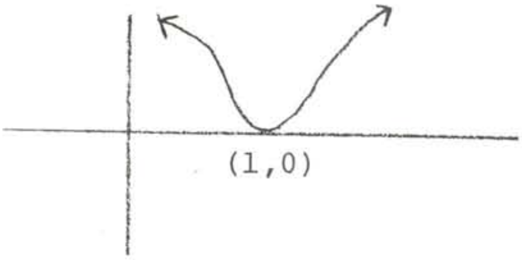
As for (a), we have that our curve must pass through the point (1,0) and the curve falls to the left of the line x = 1 while it rises to the right of this line.

Thus a suitable curve would be:



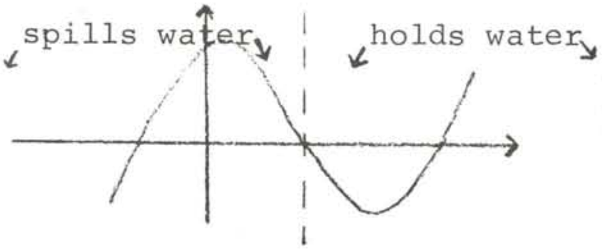
Observe that in the above diagram we drew the curve as if it always "held water," that is, as if f"(x) > 0 for all x. Nothing in the requirements of (a) makes this mandatory. For example, the curve below also exhibits the required properties.

[2.6.1(L) cont'd]



f" in the same domain is negative here but f' is still positive

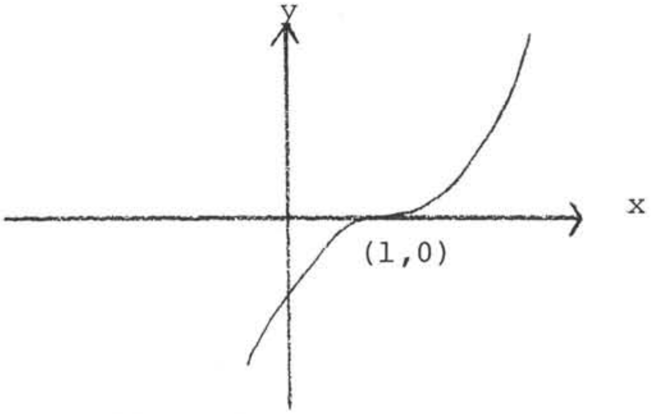
Just as (a) emphasizes rising and falling without regard to concavity, (b) emphasizes concavity without regard to rising and falling. That is, in (b) we are told that the curve passes through the point (1,0) and that the curve "spills water" to the left of x = 1 while it "holds water" to the right of x = 1. Thus, for example, we could have



(even though the curve rises part of the time and falls at other times)

Notice here that the tangent line to the curve at (1,0) need not be horizontal. All that must happen is that the tangent line appear to "cross" the curve at (1,0) [because the concavity reverses as we pass through (1,0)]. Of course the tangent line could be horizontal at (1,0). For example,

[2.6.1(L) cont'd]



Notice, in this case, that while the sign of f" changes as we pass through (1,0), the sign of f' is always positive (the curve is never falling) except at (1,0) where it is stationary.

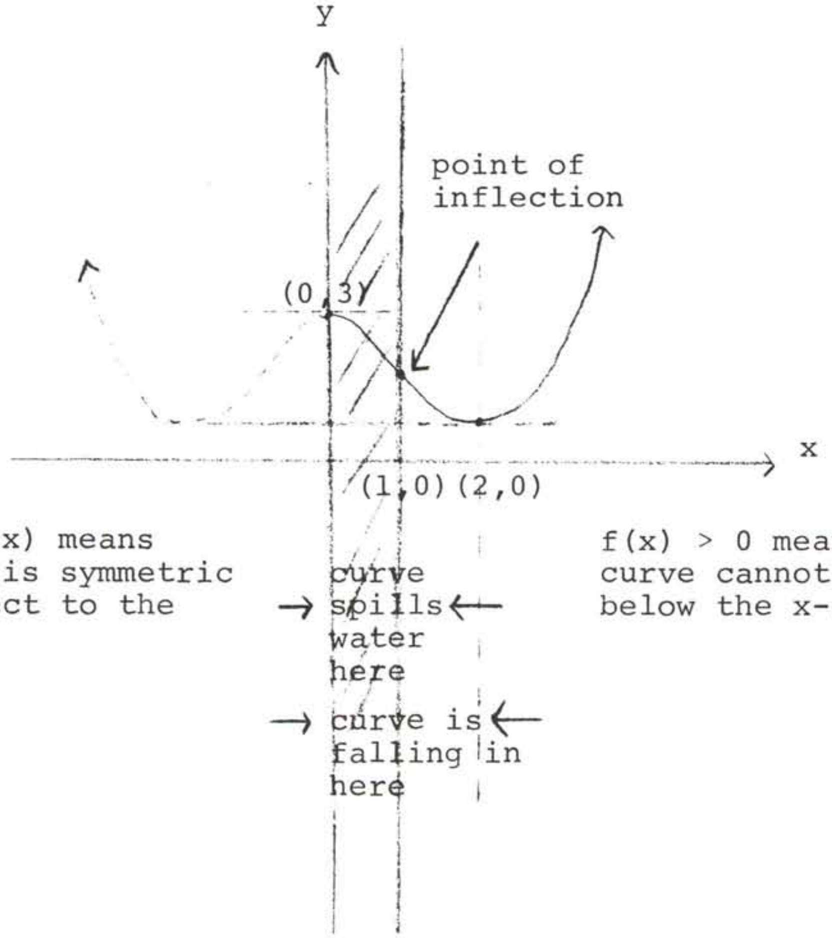
As a final point in the treatment of this exercise let us call attention to the notion of SMOOTHNESS. Suppose that we wanted to find the curve y = f(x) such that the conditions imposed in (a) and (b) are simultaneously obeyed. Then to the left of x = 1 the curve must be falling and spilling water. That is:  To the right of x = 1 the curve must rise and hold water. That is: 

If we put these two pictures together we obtain:

[2.6.1(L) cont'd]

This curve is CONTINUOUS but not smooth. It "abruptly" goes from falling to rising as it passes through (1,0). (that is, the slope goes from negative to positive without ever equalling 0).

2.6.2



f(x) = f(-x) means the curve is symmetric with respect to the y-axis

f(x) > 0 means the curve cannot be on or below the x-axis

2.6.3(L)

While, on the one hand, this is a typical exercise in the use of the first and second derivatives in curve plotting, on the other hand, it also affords us an excellent review of the material on ways of combining function.

To begin with, let us find y' and y". Applying the quotient rule to $y = x/(x + 1)$, we obtain:

$$y' = \frac{1}{(x + 1)^2} \qquad (1)$$

and from (1), we obtain:

$$y'' = \frac{-2}{(x + 1)^3} \qquad (2)$$

Analyzing (1), we see that y' cannot be defined for $x = -1$. [More formally, y' increases without bound as x "nears" -1]. When $x \neq -1$, y' is positive since $(x + 1)^2$ is positive. Thus, in the sketch of the curve we will find that it is rising wherever it is defined.

From (2) we see that y" also is undefined at $x = -1$. We also see that when $x < -1$, $(x + 1)^3$ is negative; hence, $-2/(x + 1)^3$ is positive. That is, when $x < -1$, y" is positive, and similarly if $x > -1$, then y" is negative. Thus, the curve holds water for $x < -1$ and spills water for $x > -1$.

We can now couple this information with a few facts about $y = x/(x + 1)$ and sketch the curve quite accurately.

[2.6.3(L) cont'd]

For one thing, we again see that x = -1 is a "bad" value, since y is undefined there. We can also see that when x is "near" -1, (x + 1) is small in magnitude, and y very large in magnitude. The sign of (x + 1) is positive if x is greater than -1, so if x is "a little more" than -1 our numerator is a negative number nearly equal to -1 while our denominator is a positive number very nearly equal to 0. Thus the quotient is negative and large in magnitude. In more formal language:

$$\lim_{x \to -1^+} [x/(x + 1)] = - \infty$$

In terms of the sketch of the curve this means that for values of x immediately to the right of x = -1 the points on the curve are very "low." A similar analysis shows that to the left of x = -1 the points on the curve are very high. That is:

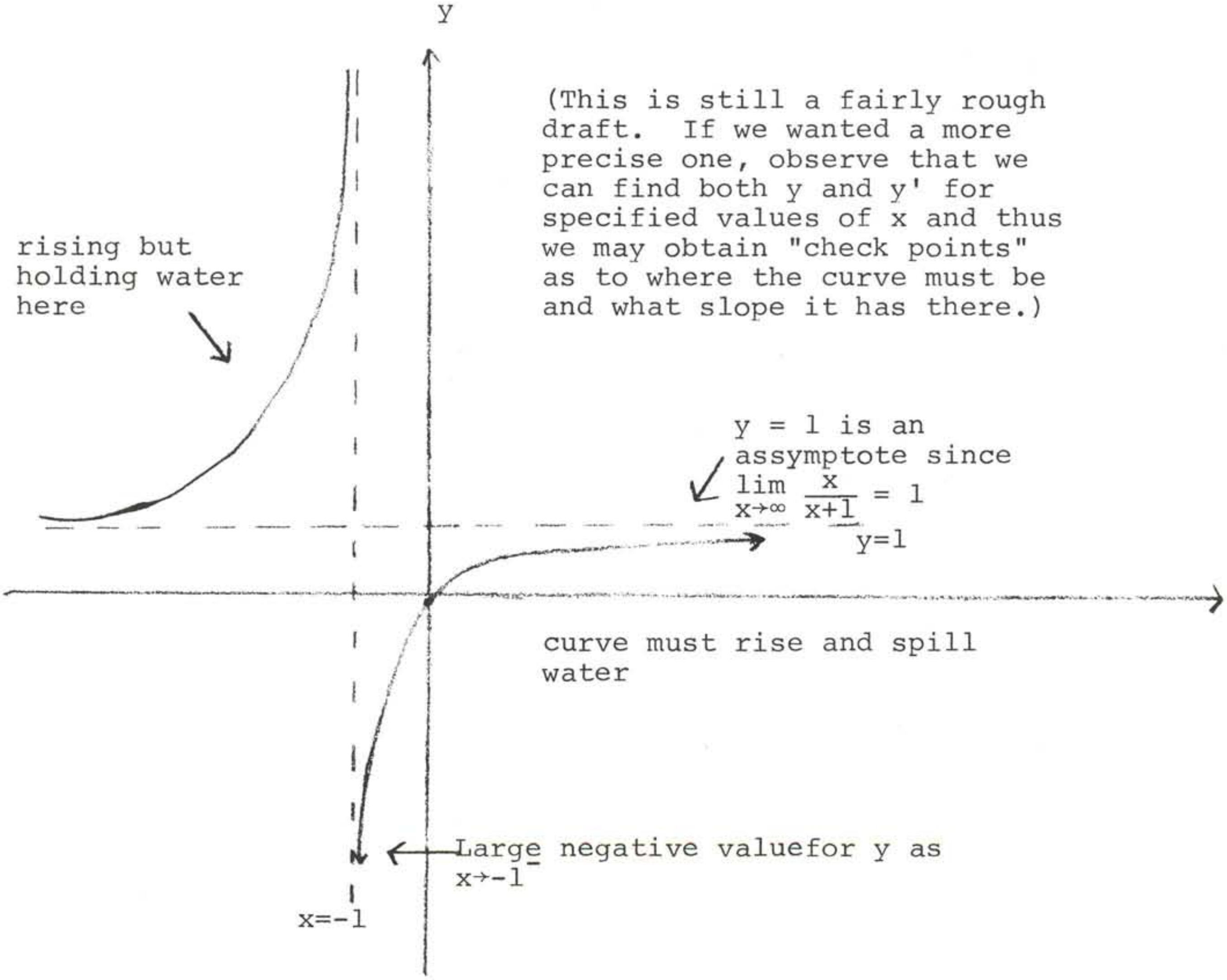
$$\lim_{x \to -1^-} [x/(x + 1) = \infty$$

We also see that when x = 0, y = 0 [so (0,0) is a point on the graph]. A final observation might be that for large values of x, x and x + 1 have approximately the same magnitude [that is, in the sense that $\lim_{x \to \infty} [x/(x + 1)] = 1$].

The fact that the curve is defined for all values of x except for x = -1 is the strong hint that x = -1 will be an assymptote.

[2.6.3(L) cont'd]

We now sketch the curve based on the above information:



(This is still a fairly rough draft. If we wanted a more precise one, observe that we can find both y and y' for specified values of x and thus we may obtain "check points" as to where the curve must be and what slope it has there.)

rising but holding water here

y = 1 is an assymptote since $\lim_{x\to\infty} \frac{x}{x+1} = 1$

curve must rise and spill water

Large negative valuefor y as $x \to -1^-$

(Figure 1)

[2.6.3(L) cont'd]

While it is always a dangerous habit to try to determine the equation of a curve merely by looking at the graph, the fact does remain that Figure 1 suggests that the curve is a hyperbola. Let us now see how we can handle this notion from the point of view of combining functions. To begin with, let

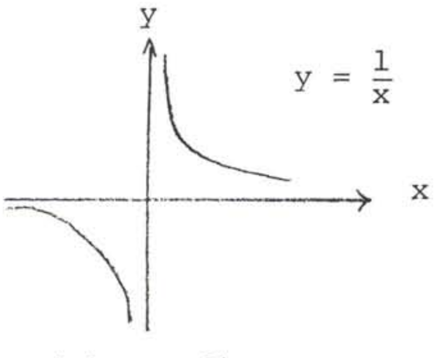
$$g(x) = x/(x + 1)$$

Now $x/(x + 1) = 1 - \frac{1}{x + 1}$

(This result can be obtained by long division or by the "clever" ruse of writing $\frac{x}{x + 1}$ as $\frac{(x + 1) - 1}{(x + 1)}$ which in turn can be written as:

$$\frac{(x + 1)}{(x + 1)} - \frac{1}{x + 1} \qquad)$$

If we now let $f(x) = 1/x$ (and this is a function we have graphed many times and which, hopefully, we know by now is a hyperbola)



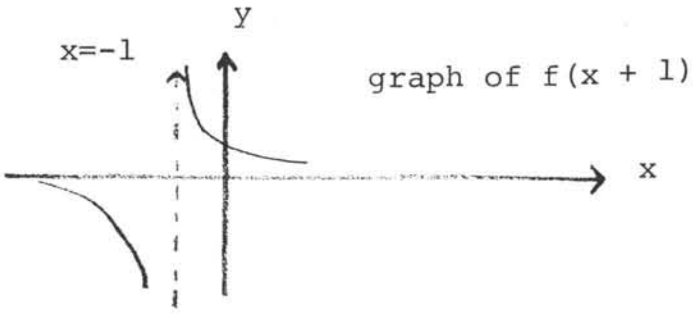
(Figure 2)

[2.6.3(L) cont'd]

we see that

$$\boxed{g(x) = 1 - f(x + 1)}$$

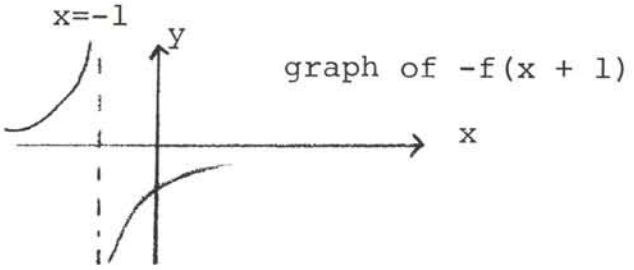
Now for any function f, the graph of f(x + 1) is the same as the graph of f(x) EXCEPT it is displaced to the LEFT by one unit. Applying this information to Figure 2, we see that



(Figure 3)

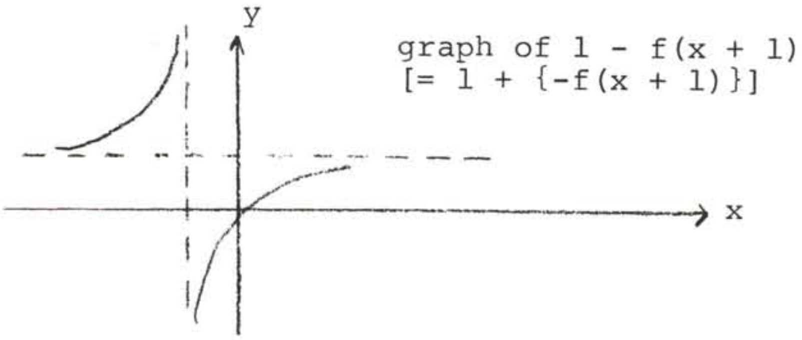
We also know that the graph of f(x + 1) and the graph of -f(x + 1) are mirror images of one another with respect to the x-axis. If we apply this knowledge to Figure 3, we obtain

[2.6.3(L) cont'd]



(Figure 4)

Finally, if we think of -f(x + 1) as being k(x) we recall that the graphs of k(x) and 1 + k(x) are the same EXCEPT that 1 + k(x) is superpositioned one unit ABOVE the graph of k(x). If we apply this to Figure 4, we obtain:



(Figure 5)

[2.6.3(L) cont'd]

Figure 5 represents the graph of $1 - f(x + 1) = g(x)$, and, hence, (just as it should be), we see that Figures 5 and 1 represent the same picture. Moreover, in going through Figures 2, 3, 4, and 5, we not only develop the proper graph of $g(x) = x/(x + 1)$ but we also see just how it is generated from the more basic function $f(x) = 1/x$.

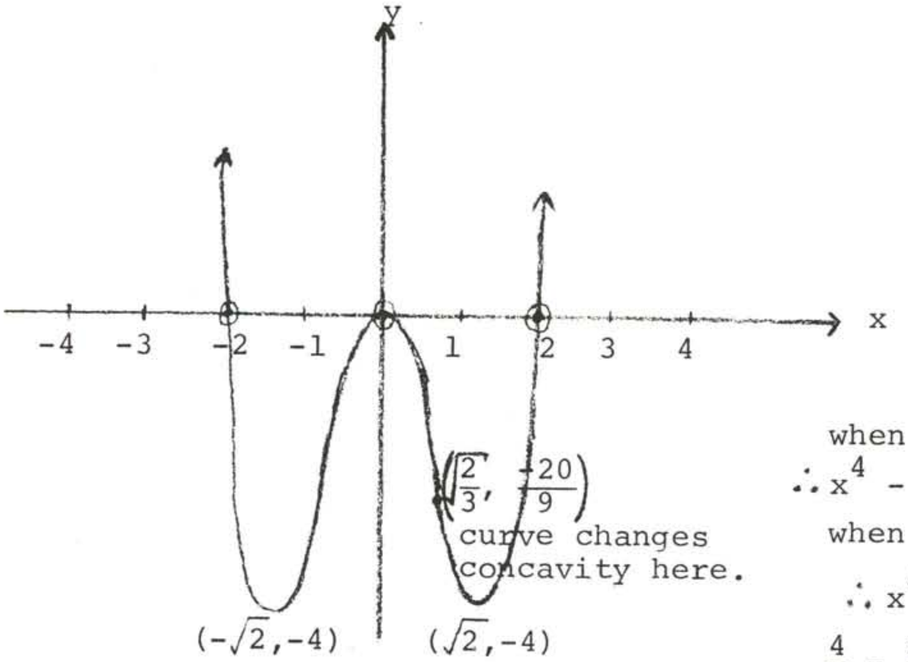
2.6.4

$y = f(x) = x^4 - 4x^2 = x^2(x^2 - 4)$ $\therefore$ curve crosses x-axis at 0, -2, 2

$f'(x) = 4x^3 - 8x = 4x(x^2 - 2)$ $\therefore$ Horizontal tangent at points for which $x = 0, -\sqrt{2}, \sqrt{2}$

$f''(x) = 12x^2 - 8$ $\therefore$ Points of inflection when $x^2 = \frac{2}{3}$ or $x = \pm \frac{1}{3}\sqrt{6} \approx \pm 0.8$

[2.6.4 cont'd]



when $x^2 = 2$, $x^4 = 4$

$\therefore x^4 - 4x^2 = 4 - 8 = -4$

when $x^2 = \frac{2}{3}$, $x^4 = \frac{4}{9}$

$\therefore x^4 - 4x^2 =$

$\frac{4}{9} - \frac{8}{3} = \frac{4}{9} - \frac{24}{9} = -\frac{20}{9}$

$f(x) = f(-x)$ → the curve is symmetric with respect to the y-axis

2.6.5(L)

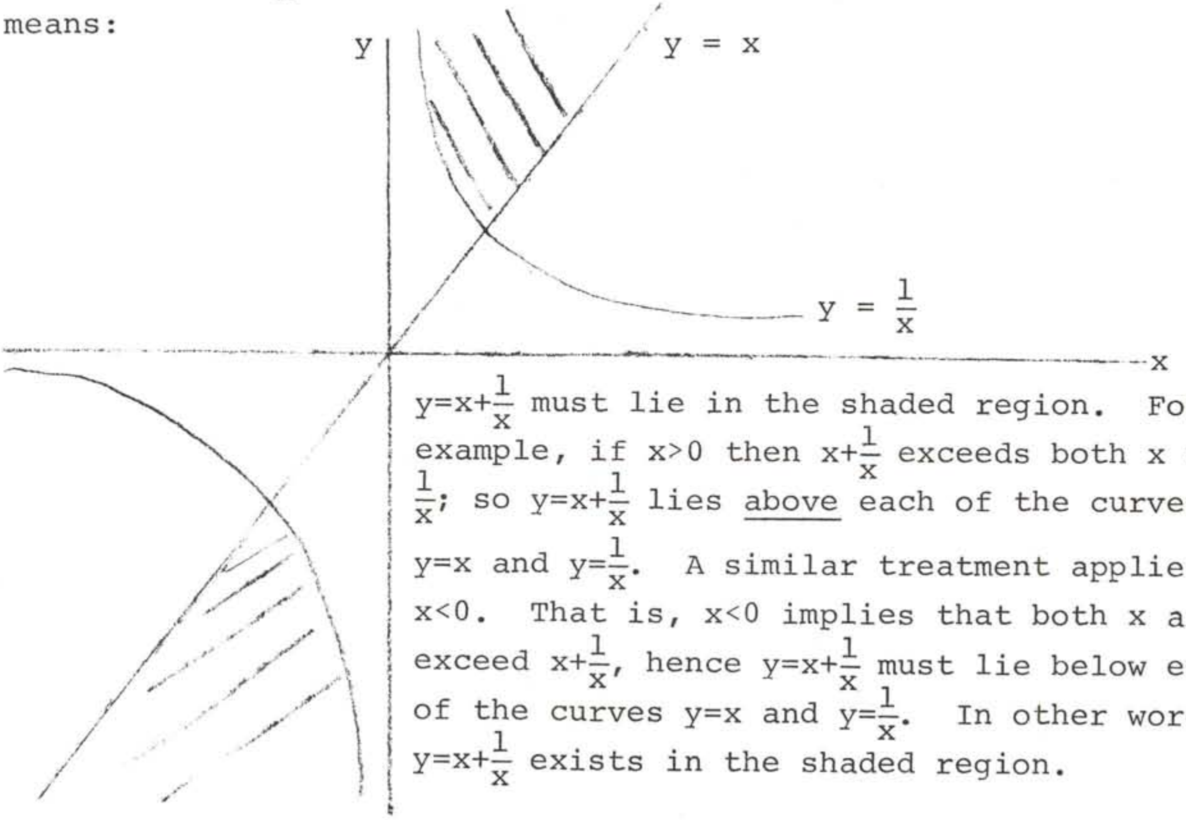
This exercise affords us the opportunity to see how curve plotting is related to analytic problems, and it also allows us to reinforce an idea presented in the text concerning a short-cut method for sketching certain types of curves.

[2.6.5(L) cont'd]

To begin with, if we let x denote any real number, then the reciprocal of x is denoted by 1/x (except, of course, if x = 0 in which case there is no reciprocal). Then, (a) we wish to examine $x + \frac{1}{x}$ for positive values of x and (b) see for what value of x this expression is minimized.

In terms of curve plotting, this means that we want to look at the curve $y = x + \frac{1}{x}$ and find where the lowest point of the curve is in the region to the right of the y-axis.

To sketch the curve we first observe that if x is small in magnitude then $x + \frac{1}{x}$ "behaves like" $\frac{1}{x}$; while if x is large in magnitude $x + \frac{1}{x}$ "behaves like" x. In terms of a picture, this means:



$y=x+\frac{1}{x}$ must lie in the shaded region. For example, if x>0 then $x+\frac{1}{x}$ exceeds both x and $\frac{1}{x}$; so $y=x+\frac{1}{x}$ lies above each of the curves y=x and $y=\frac{1}{x}$. A similar treatment applies to x<0. That is, x<0 implies that both x and $\frac{1}{x}$ exceed $x+\frac{1}{x}$, hence $y=x+\frac{1}{x}$ must lie below each of the curves y=x and $y=\frac{1}{x}$. In other words, $y=x+\frac{1}{x}$ exists in the shaded region.

[2.6.5(L) cont'd]

The above analysis gives us a pretty good idea about the graph of $y = x + \frac{1}{x}$, but to be more precise we can compute y' and y". This leads to:

$$y = x + \frac{1}{x} \qquad (1)$$

$$y' = 1 - \frac{1}{x^2} \qquad (2)$$

$$y'' = \frac{2}{x^3} \qquad (3)$$

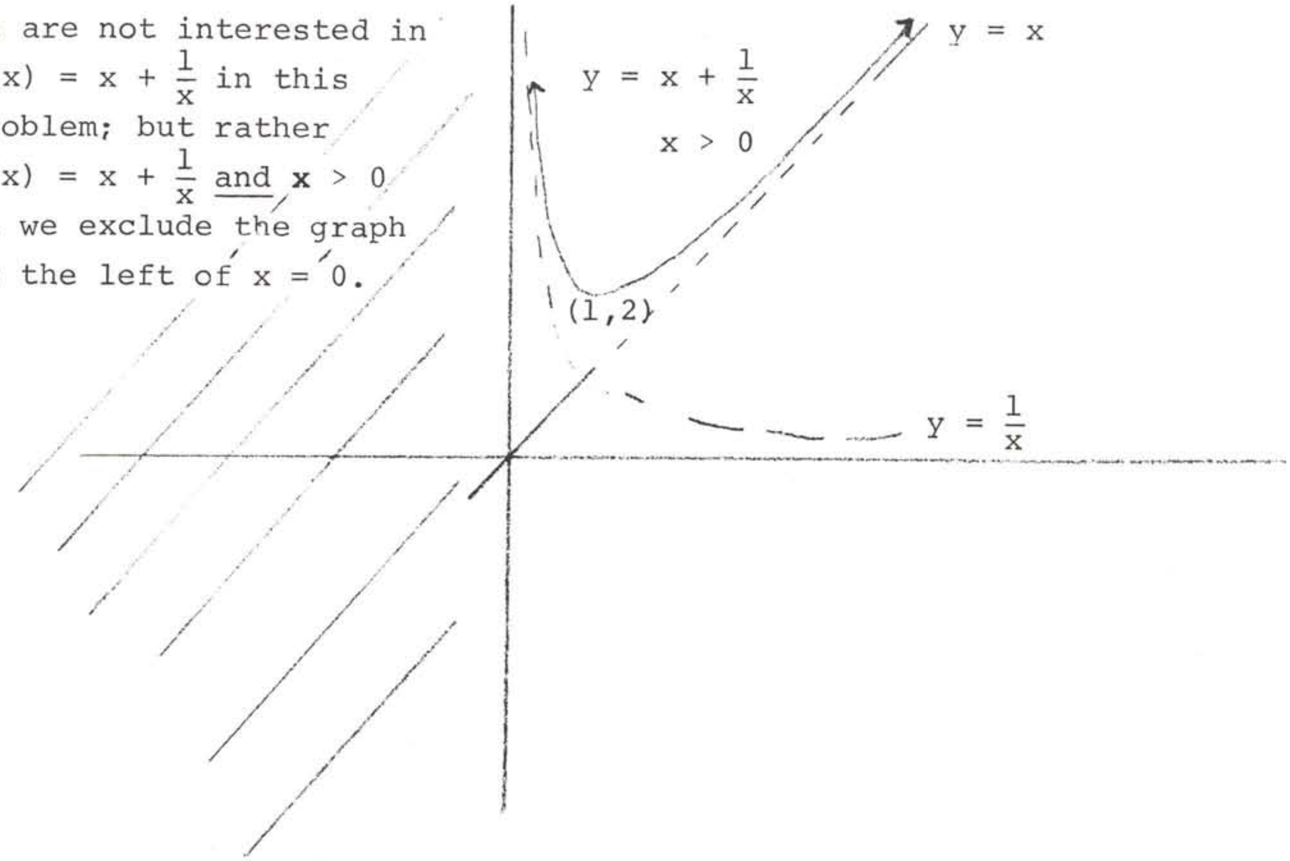
From (2) we see that the places on the curve at which the slope is 0 are the points corresponding to x = 1 and x = -1; and from (1) we see that these are the points (1,2) and (-1,-2).

Moreover (3) tells us that the curve is holding water at (1,2) and spilling water at (-1,-2). Putting all this together, we obtain:

[2.6.5(L) cont'd]

we are not interested in $f(x) = x + \frac{1}{x}$ in this problem; but rather $f(x) = x + \frac{1}{x}$ <u>and</u> **x** > 0

$\therefore$ we exclude the graph to the left of $x = 0$.



From the above diagram it is clear that for positive values of x, $x + \frac{1}{x}$ is at least as great as 2.

2.6.6

Let $f(x) = x + \frac{4}{x^2}$

Then, our problem is equivalent to finding the lowest point on the curve $y = f(x) = x + \frac{4}{x^2}$ and $x > 0$

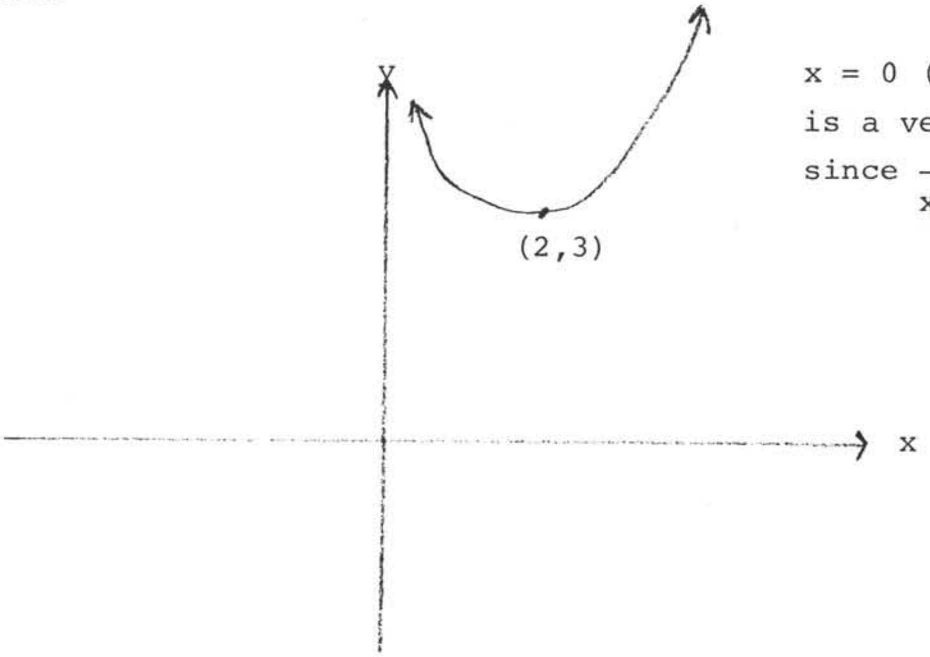
$$f(x) = x + \frac{4}{x^2}$$

$$f'(x) = 1 - \frac{8}{x^3} \therefore \left\{ f'(x) \text{ is} \begin{cases} \text{negative if } 0 < x < 2 \\ \text{zero if } x = 2 \\ \text{positive if } x > 2 \end{cases} \right.$$

$$f''(x) = \frac{24}{x^4} \therefore f''(x) > 0 \text{ for all } x$$

$\therefore$ Lowest point on curve occurs when $x = 2$, at which point $y = 3$, since $y = x + \frac{4}{x^2}$

Hence:



$x = 0$ (the y-axis) is a vertical assymptote since $\frac{4}{x^2} \to \infty$ as $x \to 0$.

[2.6.6 cont'd]

Thus, if x is positive, $x + \frac{4}{x^2}$ is at least 3 and this occurs when $x = 2$.

2.6.7(L)

This exercise actually highlights the use of calculus in solving algebraic equations. The point is that the roots of the equation $2x^3 + 2x^2 - 2x - 1 = 0$ are precisely the x-coordinates of the points at which the curve $y = 2x^3 + 2x^2 - 2x - 1$ crosses the x-axis, i.e. when $y = 0$.

To find where the curve meets the x-axis, we proceed as usual and determine y' and y". This leads to:

$$y = 2x^3 + 2x^2 - 2x - 1 \qquad (1)$$

$$\begin{aligned} y' &= 6x^2 + 4x - 2 \\ &= 2(3x - 1)(x + 1) \qquad (2) \end{aligned}$$

$$y'' = 12x + 4 \qquad (3)$$

From (1) we know that when $x = 0$, $y = -1$; and from (2) we know that when $x = 0$, $y' = -2$.

Thus the curve passes through $(0,-1)$ with slope equal to -2.

(2) also tells us that the slope of the curve is 0 when $x = 1/3$ or $x = -1$. From (3) we see that y" is positive when $x = 1/3$ and negative when $x = -1$. We also see from (1)

[2.6.7(L) cont'd]

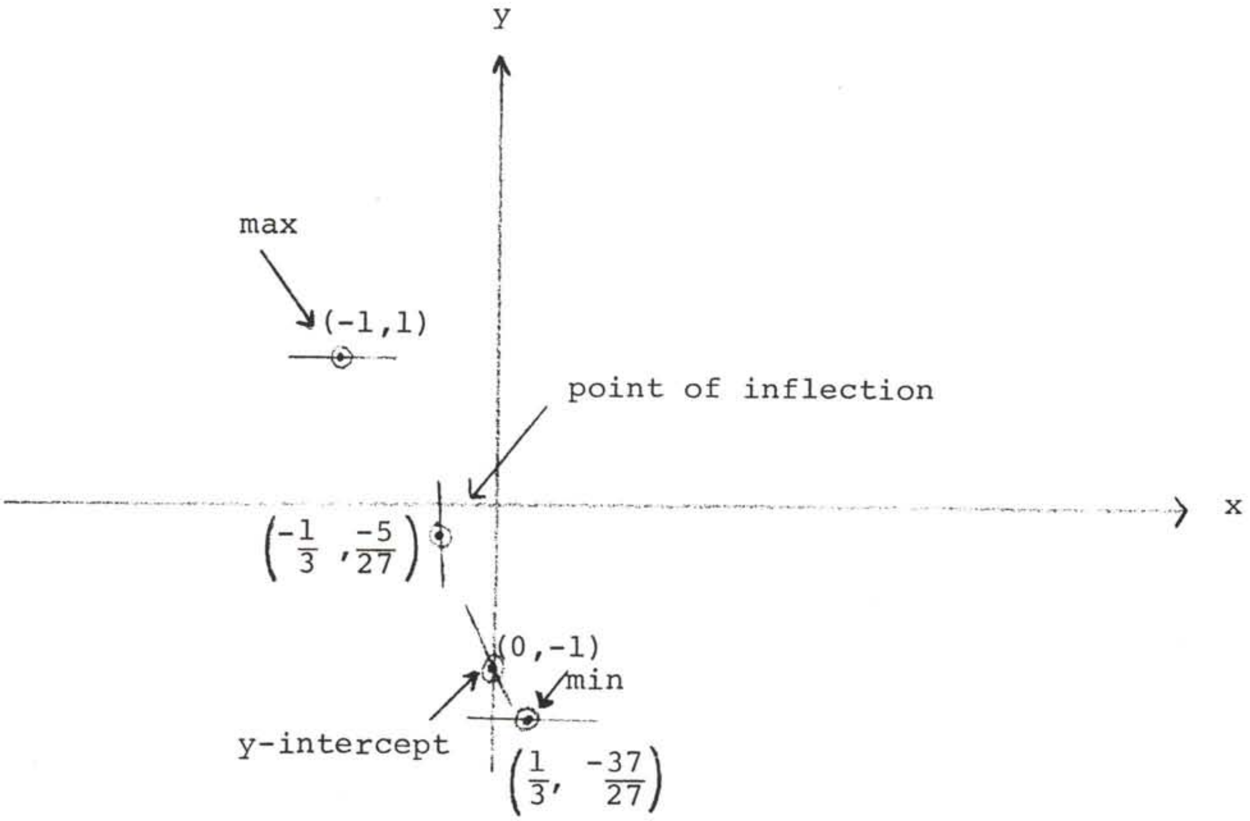
that when x = 1/3, y = -37/27; and when x = -1, y = 1.

Thus we also know that the curve has a minimum point at (1/3,-37/27) and a maximum point at (-1,1).

Fianlly, (3) tells us that y" = 0 at x = -1/3. From (1) we see that when x = -1/3, y = -5/27; and from (2) we see that when x = -1/3, y' = -8/3.

This means that the point of inflection is (-1/3,-5/27) and that the slope of the curve at this point is -8/3.

Putting all this into a sketch, we obtain:



[2.6.7(L) cont'd]

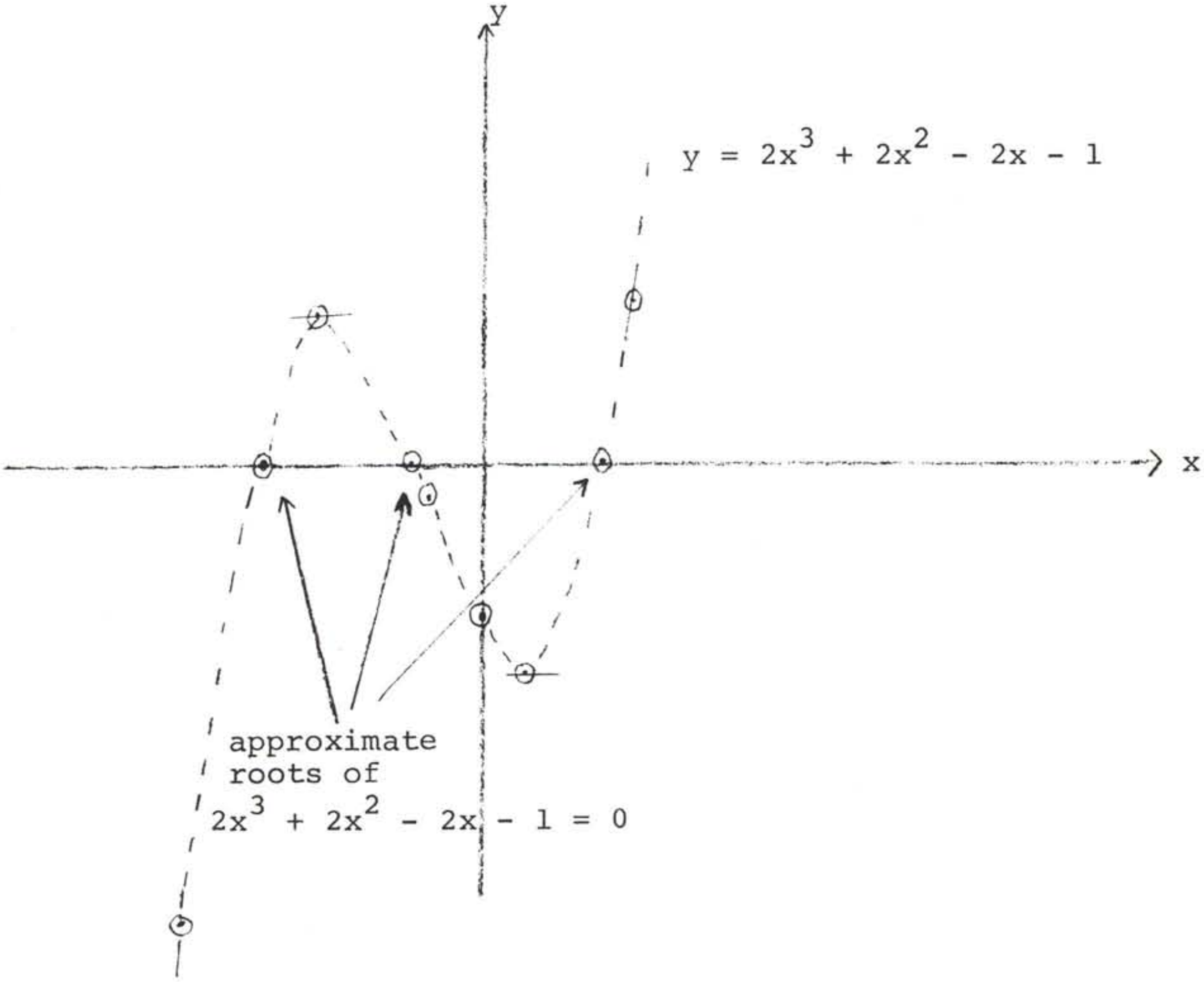
From the sketch so far we can get additional "checks." For example, our sketch seems to indicate that the curve will fall at all points whose x-coordinate is between -1 and +1/3; while it will rise elsewhere. This is verified from the factored form of (2). We also see from (3) that the curve spills water to the left of the line $x = -1/3$ and holds water to the right of this line. This is also indicated by our plotted data.

A glance at the sketch also seems to indicate that the curve will cut the x-axis in three places, one of which is to the right of the y-axis and the other two to the left of the y-axis. As for the x-intercept of the curve to the right of the y-axis, we sense that it is close to 1. To see how close, we let $x = 1$ in (1) and obtain that when $x = 1$, $y = 1$. This tells us that (1,1)is on the curve. In particular, the curve lies above the x-axis when $x = 1$; hence, it must have crossed the x-axis at a point whose x-coordinate was less than 1. From (2) we see that $y' = 8$ when $x = 1$, so that the curve is rising quite quickly at the point (1,1).

Our graph also indicates that another x-intercept must be slightly to the left of $x = -1/3$, while the third x-intercept must lie to the left of $x = -1$ (why?). To help us get a better perspective, we look at (1) and (2) when $x = -2$. (1) shows us that when $x = -2$, $y = -3$ [thus the x-intercept is between -2 and -1 (why?)]; while (2) shows us that at (-2,-3) the slope of the curve is 14.

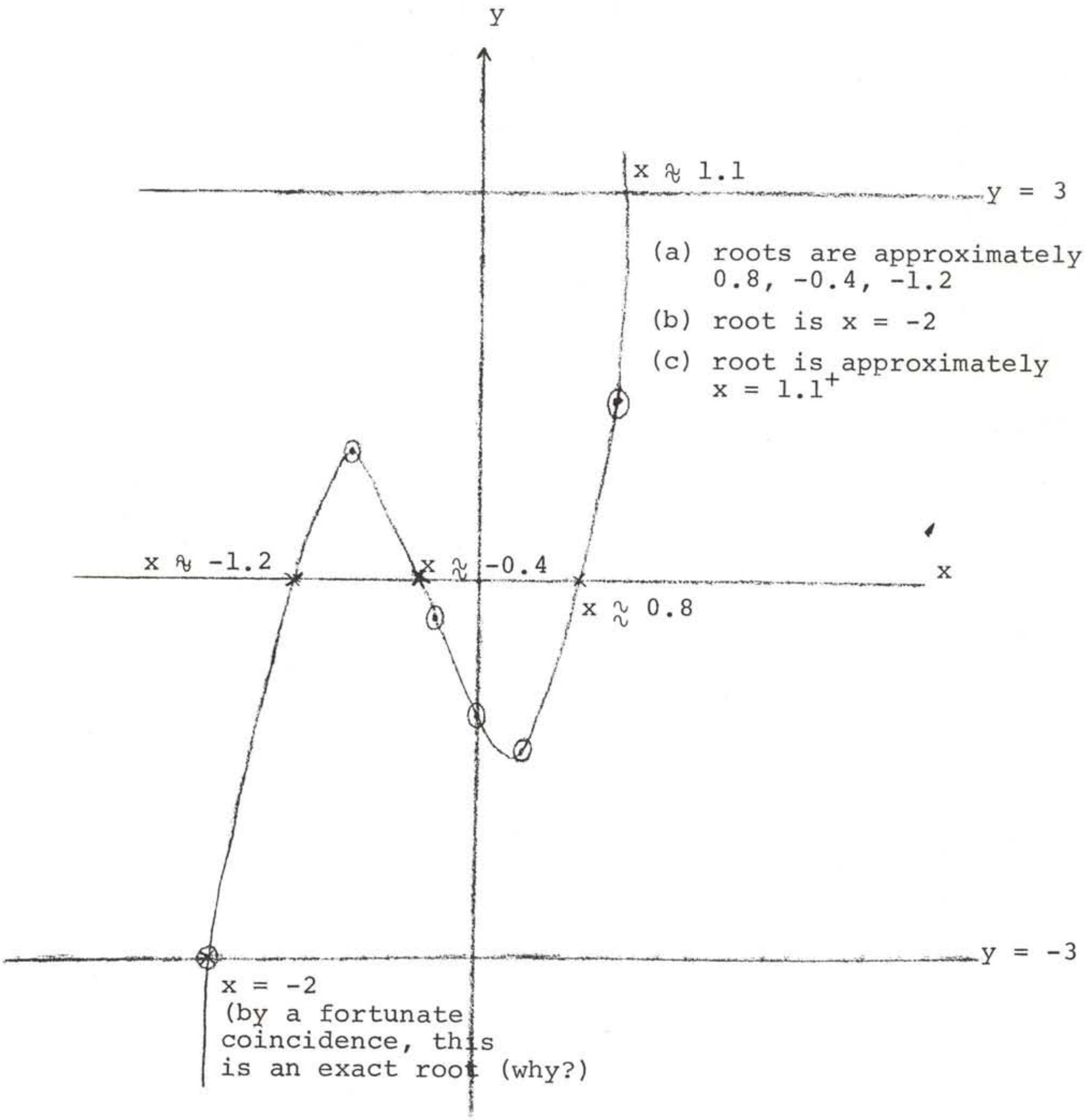
[2.6.7(L) cont'd]

Adding this data to our sketch, we obtain:



(b) Next we observe that adding 3 to each y value raises the curve by 3 units; and this is equivalent to replacing the x-axis in the above diagram by the line $y = -3$. In a similar way, subtracting 3 from each y value means that we replace the x-axis in the above diagram by the line $y = 3$. Putting all this into one picture, we obtain:

[2.6.7(L) cont'd]



(a) roots are approximately 0.8, -0.4, -1.2

(b) root is x = -2

(c) root is approximately x = 1.1$^+$

[2.6.7(L) cont'd]

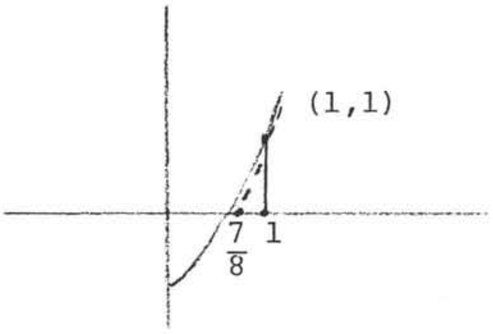
As a final observation in this exercise let us notice that we could locate the roots by analytic methods rather than by geometric methods.

For example, with $f(x) = 2x^3 + 2x^2 - 2x - 1$ we observe:

$f(-2) = -16 + 8 + 4 - 1$	< 0	root between -2 and -1 by "Horner's Method" (see solution to Ex. 2.5.8)
$f(-1) = -2 + 2 + 2 - 1$	> 0	root between -1 and 0 by "Horner's Method"
$f(0) = -1$	< 0	
$f(1) = 2 + 2 - 2 - 1$	> 0	root between 0 and 1 by "Horner's Method"

(Recall "Horner's Method" refers to continuous functions which change sign).

Now, to get a better approximation of the root of $f(x) = 0$ which lies between 0 and 1, we can use "Newton's Method."



[2.6.7(L) cont'd]

This amounts, for example, to picking x = 1 as a first approximation and drawing the tangent to the curve at (1,1) and seeing where it crosses the x-axis. When x = 1,

$$f'(x)\Big]_{x=1} = 6x^2 + 4x - 2\Big]_{x=1} = 8$$

$\therefore$ Equation of tangent line is $\frac{y-1}{x-1} = 8$ or $y = 8x - 7$

$\therefore$ $(0, \frac{7}{8})$ is where this line crosses the x-axis (y = 0) $\frac{7}{8}$ is a bit greater than the desired root (see above figure), and we can repeat the process starting with $x = \frac{7}{8}$ and in this way obtain better approximations.

2.6.8(L)

(a) The slope of $y = \frac{c}{x+1}$ at (x_1, y_1) is given by

$$\frac{dy}{dx} = \frac{-c}{(x+1)^2}$$

Hence when x = 2, $\frac{dy}{dx} = \frac{-c}{(2+1)^2} = \frac{-c}{9}$

Since we desire $\frac{dy}{dx}\Big]_{x=2} = 1$, it follows that $1 = \frac{-c}{9}$

$c = -9$

(b) Again (b) is the part that requires a less mechanical approach. The key difference between (a) and (b) is that in (a) we were given the point on the curve that was of interest to us. In (b) we do not know where x + y = 3 is tangent to $y = \frac{c}{x+1}$.

[2.6.8(L) cont'd]

It might also be interesting to observe that except for helping us visualize the graph $y = \frac{c}{x + 1}$ (and nothing in this problem says we have to be able to draw the graph), we can solve this problem without recourse to calculus. Even though we can avoid the use of calculus, part (b) has much to offer us in terms of enhancing the analytic approach to geometry problems.

At any rate if:

$$y = \frac{c}{x + 1} \tag{1}$$

$$\frac{dy}{dx} = \frac{-c}{(x + 1)^2} \tag{2}$$

$$\frac{d^2y}{dx^2} = \frac{2c}{(x + 1)^3} \tag{3}$$

Let us assume that c is positive (If c is negative all that happens is that y changes sign. Hence, if c is negative, our graph is the mirror image with respect to the x-axis of the graph of $y = \frac{|c|}{x + 1}$).

Since $c \neq 0$. $\frac{c}{x + 1} \neq 0$, hence $y \neq 0$ (from (1))

$\therefore$ <u>the curve never crosses the x=axis</u> (a)

[2.6.8(L) cont'd]

Moreover from (1) we also see that y is undefined when $x + 1 = 0$

$\therefore$ the curve never crosses the line $x = -1$. (b)

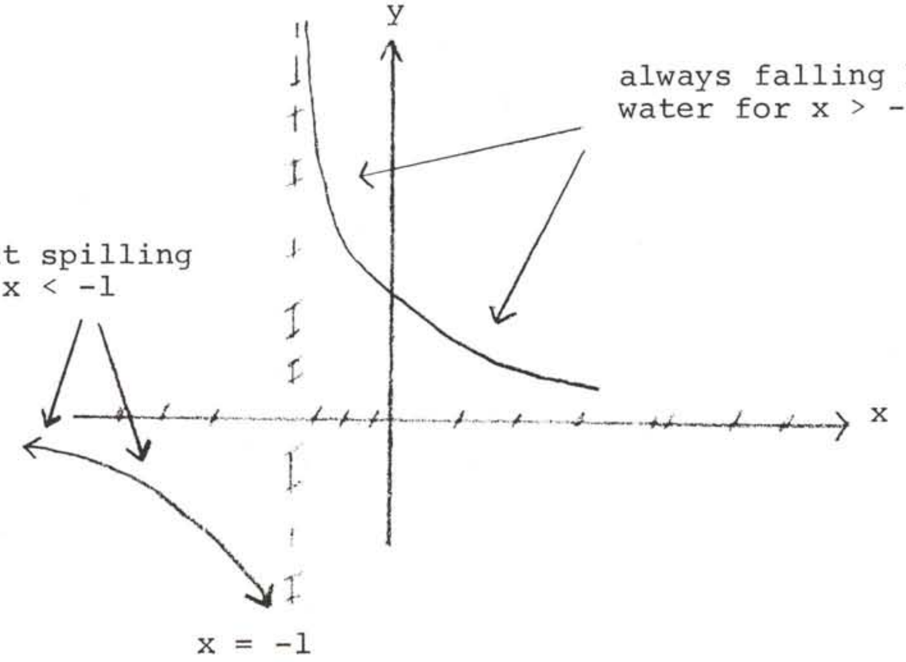
From (2), since $(x + 1)^2$ can never be negative and since c is positive, $\frac{dy}{dx}$ is always negative (if c were negative then $\frac{dy}{dx}$ would be positive).

$\therefore$ the curve is always falling. (c)

Finally (3) tells us that $\frac{d^2y}{dx^2}$ has the same sign as $x + 1$ [since a number and its cube always have the same sign]

$$\therefore \left.\begin{array}{l} \text{the curve holds water for } x > -1 \\ \text{the curve spills water for } x < -1 \end{array}\right\} \quad (d)$$

Putting (a), (b), (c), and (d) together, we obtain



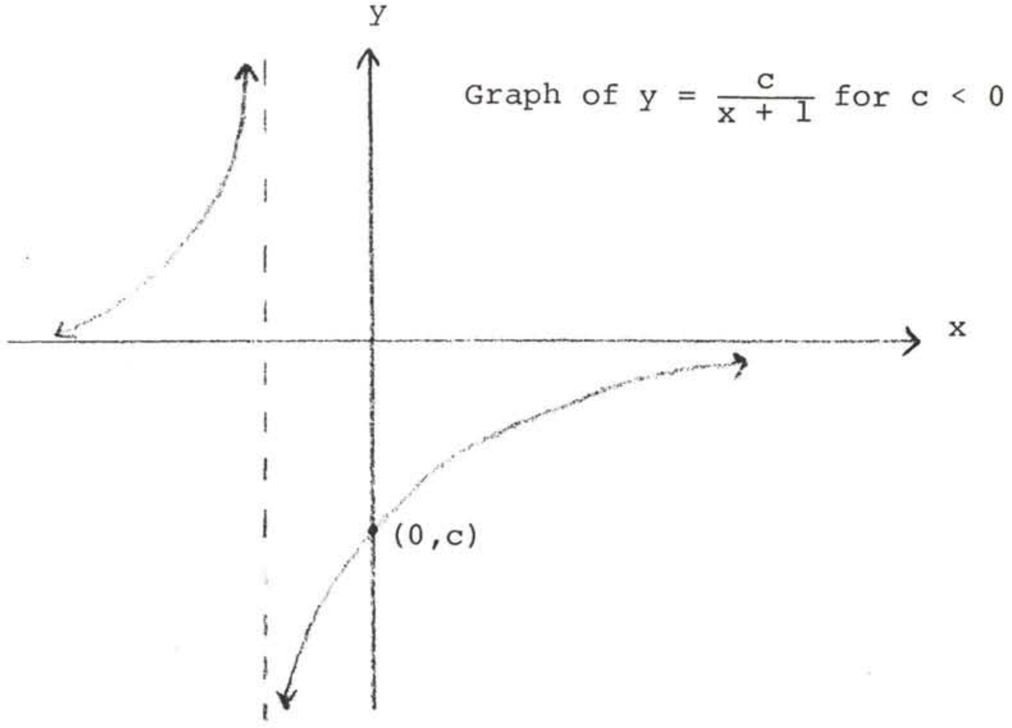
always falling but holding water for $x > -1$

falling but spilling water for $x < -1$

(Figure 1)

[2.6.8(L) cont'd]

If c is negative, the graph has the form

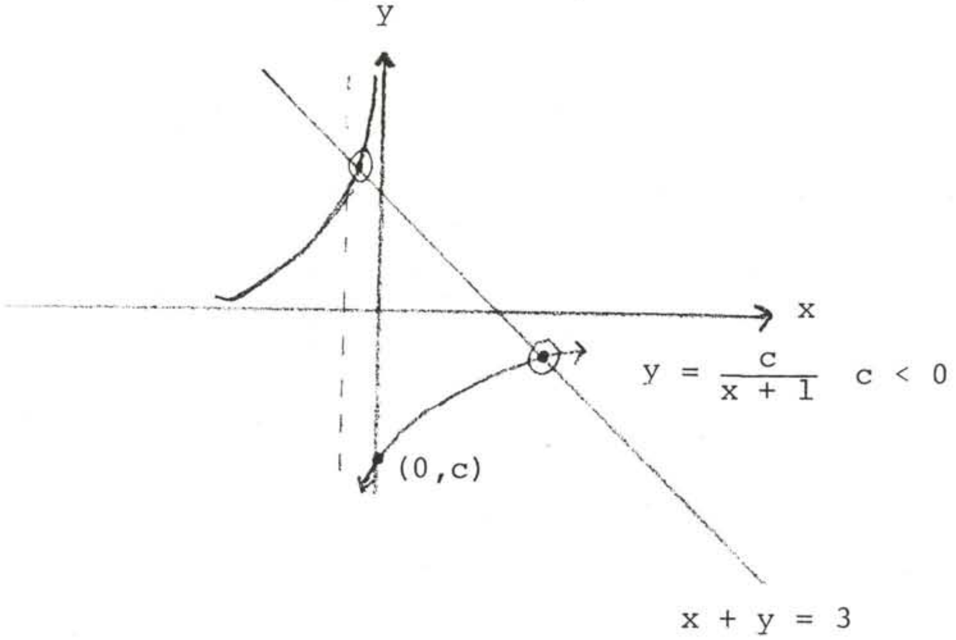


(Figure 2)

A glance at Figure 2 shows us that there can be no solution to our problem if c is negative, for in this case the curve $y = c/(x + 1)$ is always rising, while the line $x + y = 3$ has a negative slope $(m = -1)$. Thus, the line $x + y = 3$ can never be tangent to the curve $y = c/(x + 1)$ if c is negative, since the line is falling while the curve is rising. Again, this is seen at a glance from the diagram since it is visually clear that the line, for any negative choice of c, intersects the curve twice,

[2.6.8(L) cont'd]

once on each branch. That is,

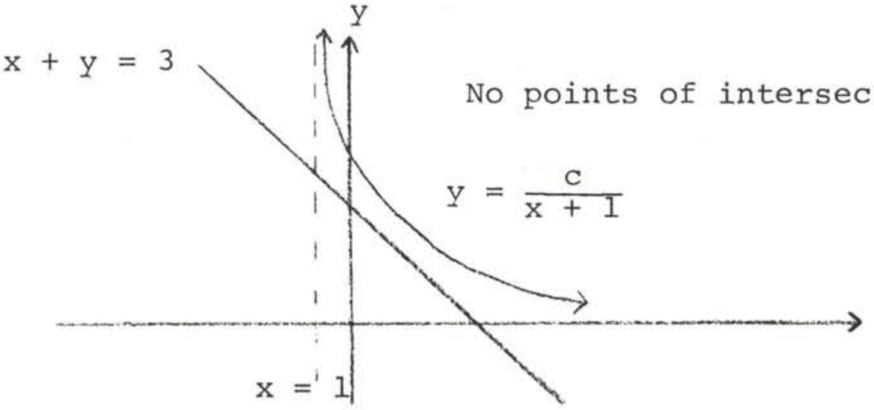


(Figure 3)

Thus, it becomes clear that if our problem is to have a solution, it must occur only for positive values of c. Therefore, let us turn our attention to the case in which c is positive. Again our graph seems to indicate two general cases: namely, the line intersects the curve in two places or not at all. (By the way, when there are points of intersection, they must be on the upper branch of the curve, for the lower branch of the curve is in the 3rd quadrant while the line x + y = 3 is never in the third quadrant.)

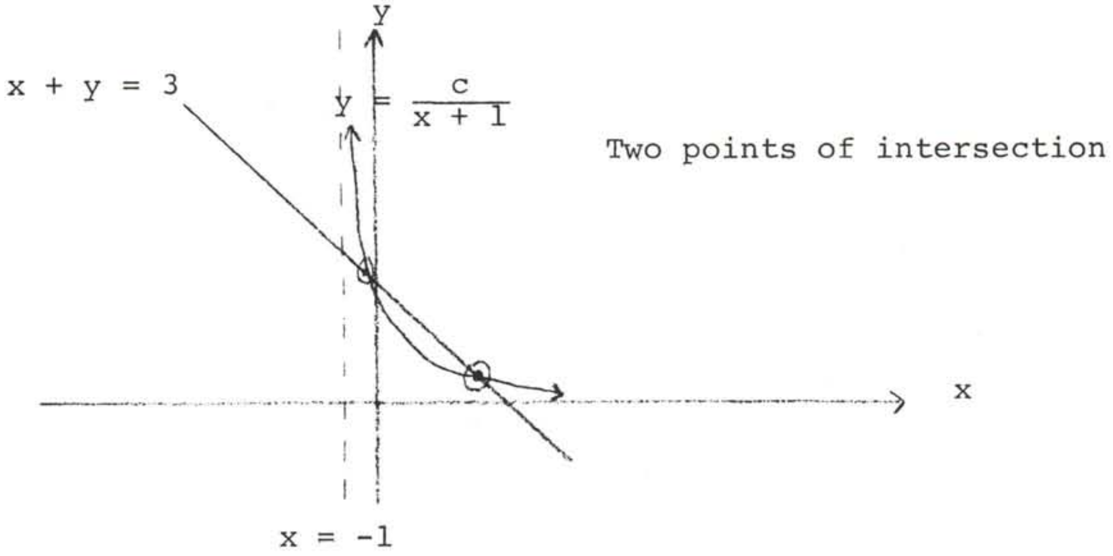
[2.6.8(L) cont'd]

Pictorially, we have



(Figure 4)

or:



(Figure 5)

[2.6.8(L) cont'd]

How can c be located? There are a few nice ways, and we shall discuss one of them here. For one thing, if we wish to find where the two curves intersect, we may find the simultaneous solution of their equations. Thus we may try to solve the equations:

$$y = 3 - x$$

and

$$y = c/(x + 1)$$

This leads to:

$$3 - x = c/(x + 1)$$

or:

$$(3 - x)(x + 1) = c$$

$$\therefore\ 3x - x^2 + 3 - x = c$$

or:

$$x^2 - 2x + (c - 3) = 0 \qquad (4)$$

We can now use the quadratic formula in (4) to find the x-coordinate of the point of intersection. Namely

$$x = \frac{x \pm \sqrt{4-4(c-3)}}{2} = \frac{2 \pm 2\sqrt{1-(c-3)}}{2}$$

Hence:

$$\boxed{x = 1 \pm \sqrt{4 - c}} \qquad (5)$$

[2.6.8(L) cont'd]

Equation (5) shows that if $4 - c < 0$ (i.e., if $c > 4$) then x cannot be real. In terms of our previous geometric considerations, (5) shows that Figure 4 applies when $c > 4$.

Secondly, if $c < 0$, (5) shows that one value of x is less than -1 while the other is greater than 3. (That is, $c < 0 \rightarrow \sqrt{4 - c} > 2 \rightarrow 1 - \sqrt{4 - c} < -1$.)

Therefore, as suspected earlier, (5) confirms that Figure 3 applies when c is negative.

Finally if the required solution is a transition between the case of two intersections and the case of no intersections, (5) indicates that the solution occurs when $\sqrt{4 - c} = 0$ (since then $1 \pm \sqrt{4 - c}$ is the same regardless of our choice of sign.)

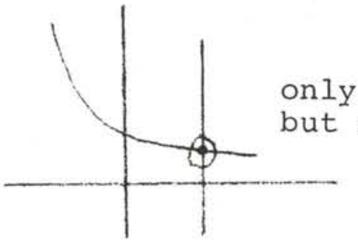
Thus $c = 4$ is the only candidate for a solution.

Now, when $c = 4$, $x = 1$. Therefore our curve is $y = \frac{4}{x + 1}$ and the point at which $y = 3 - x$ is tangent to the curve occurs when $x = 1$. Therefore the point is (1,2). As a check, when $y = \frac{4}{x + 1}$, $\frac{dy}{dx} = \frac{-4}{(x + 1)^2} \therefore \left(\frac{dy}{dx}\right)_{x=1} = -1 \therefore$ Equation of tangent line is $\frac{y - 2}{x - 1} = -1$ or $\underline{y = 3 - x}$.

The main reason for checking is that there are some general problems which happen not to occur in this example, but which can occur in other cases.

[2.6.8(L) cont'd]

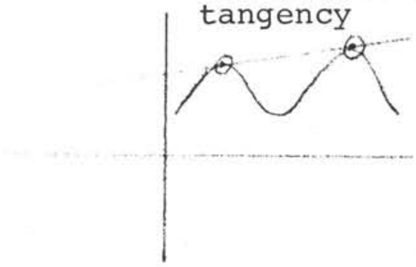
For example, the mere fact that there is only one value of x does not guarantee that we have found a point of tangency. The graph might have looked like:

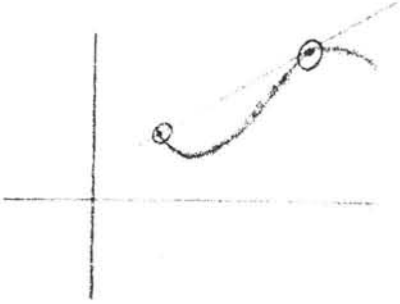


only one point of intersection but not a tangent

Secondly, the fact that a line meets a curve more than once does not exclude it from being tangent to the curve as the following configurations indicate:

Two points of tangency





One point of tangency; one other point of intersection.

What is important is that a combination of analysis and geometry has shown us precisely what is happening in this problem.

UNIT 7: Applications of the Derivative II

2.7.1(L)

We first observe that the solution of this exercise does not depend on whether we understand why the "recipe" $pv^{1.4} = c$ works.* All that is important in terms of related rates is that we have managed to relate p and v. We merely assume that both p and v are differentiable functions of time and we then differentiate $pv^{1.4} = c$ implicitly with respect to time (notice here that we have a new use for implicit differentiation; that is, we assume that somehow or other p and v are functions of time even though we do not know explicitly how they vary with time). In any event we obtain

$$\frac{dp}{dt}v^{1.4} + 1.4pv^{0.4}\frac{dv}{dt} = 0 \tag{1}$$

If we next observe that $v^{0.4} \neq 0$, and since for physical reasons we can assume that $v \neq 0$, we may divide through by $v^{0.4}$ in (1) to obtain

---

*Just as a little aside for those who may be more familiar with the relation pv = constant, let us observe that pv = c pertains to an isothermal situation (that is, where the temperature is held constant). In this problem we are assuming that we are studying a process in which there is no flow of heat. Quite in general, in an isothermal situation we must add or subtract heat. For example, this is done in a temperature bath when we hold temperature constant by either heating or cooling the bath as conditions change.

[2.7.1(L) cont'd]

$$v\frac{dp}{dt} + 1.4p\frac{dv}{dt} = 0 \qquad (2)$$

The beauty of (2) is that it tells us how v, p, $\frac{dp}{dt}$, and $\frac{dv}{dt}$ are related at any given instant and thus we can find any one of these quantities once the other three are known. Notice that all we have to know is what the values are at that instant and not how they got that way. That is, equation (2) utilizes nothing more than the fact that p and v are differentiable functions of time. It does not matter, for example, what was going on to make $p = 40$ when $v = 16$.

Now to solve Exercise 1.7.1(L) explicitly with respect to the given data, all we need do is return to (2) and let $p = 40$, $v = 16$ and $\frac{dv}{dt} = -2$. (Recall that $\frac{dv}{dt}$ is just the calculus way for expressing the change of volume with respect to time and we use the minus sign to indicate that the volume is decreasing.) We obtain

$$16\frac{dp}{dt} + (1.4)(40)(-2) = 0$$

or:

$$\frac{dp}{dt} = 7$$

That is, at the given instant the pressure is <u>increasing</u> (since $\frac{dp}{dt}$ is positive) at the rate of 7 pounds per square inch per second.

More importantly, notice that (2) was the most crucial part of this problem since once (2) is obtained we can find out what's

[2.7.1(L) cont'd]

happening at any particular instant PROVIDED ONLY THAT p AND v ARE DIFFERENTIABLE FUNCTIONS OF t.

2.7.2

Here we again assume that p and v are differentiable functions of t and we differentiate pv = c implicitly to obtain

$$p\frac{dv}{dt} + v\frac{dp}{dt} = 0$$

Now letting p = 40, v = 16 and $\frac{dv}{dt} = -2$, we obtain:

$$(40)(-2) + 16\frac{dp}{dt} = 0$$

whence

$$\frac{dp}{dt} = 5$$

Thus, at the given instant the pressure is increasing at a rate of 5 pounds per square inch per second.

2.7.3

At any instant, $v = \frac{4}{3}\pi r^3$. So, if we only assume that v and r are differentiable functions of time, we obtain

$$\frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt} \qquad (1)$$

[2.7.3 cont'd]

Equation (1) shows how $\frac{dv}{dt}$, $\frac{dr}{dt}$, and r are always related.

In particular, if we now also know that $\frac{dv}{dt} = k(4\pi r^2)$, for some constant k (which is exactly what is meant by the fact that the change in volume is proportional to the surface area), (1) becomes

$$k4\pi r^2 = 4\pi r^2 \frac{dr}{dt} \qquad (2)$$

From (2), we see that

$$\frac{dr}{dt} = k = \text{constant}$$

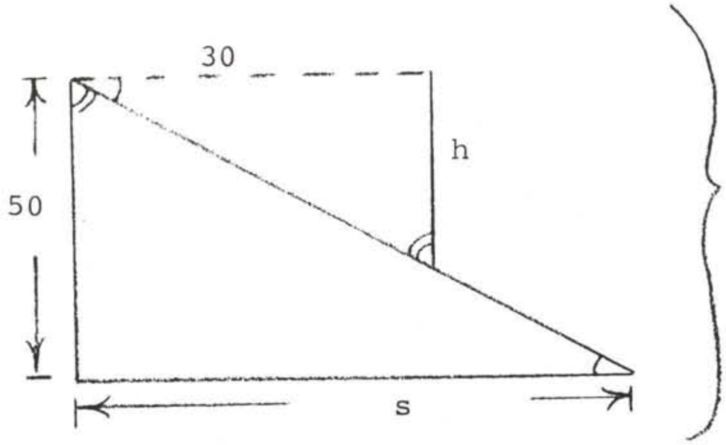
which is the desired result.

2.7.4(L)

In principle, this exercise is the same as the others, but it is the first one we've tackled in which we must derive the relationship between the variables rather than having the relationship given to us at the outset. Moreover, this exercise also affords us the opportunity to do the same problem both implicitly and explicitly and thus be able to compare two different computational techniques.

Pictorially, we have

[2.7.4(L) cont'd]



In the language of this diagram, we want to find $\frac{ds}{dt}$ when, by similar triangles, we know that:

$$\frac{s}{50} = \frac{30}{h}$$

$$s = 1500h^{-1}$$

$$\frac{ds}{dt} = -1500h^{-2}\frac{dh}{dt} = \frac{-1500}{h^2}\frac{dh}{dt} \qquad (1)$$

Now (1) can further be refined by the information that $h = 16t^2$. Hence, $\frac{dh}{dt} = 32t$, and putting this into (1), we obtain

$$\frac{ds}{dt} = \frac{-1500(32t)}{h^2} \qquad (2)$$

Thus, the solution to this exercise is obtained from (2) with $t = 1/2$ (which means that $h = 4$). That is, at $t = 1/2$,

$$\frac{ds}{dt} = \frac{-1500(16)}{(4)^2} = -1500 \text{ ft/sec}$$

[2.7.4(L) cont'd]

(If the magnitude of this answer seems too large from an intuitive point of view, merely observe that if we were to draw the diagram to scale, after 1/2 second the distance to the shadow is small compared with what it was just 1/2 second earlier. Namely, at the instant t = 0 the shadow is "infinitely far away." Thus, the rate of change is extremely great during the first few "instants" of the fall.)

Observe that we could have, in this particular example, found $\frac{ds}{dt}$ more explicitly. In fact, using the same diagram as before, we can replace h by $16t^2$ BEFORE we differentiate, and obtain at once that

$$s = \frac{1500}{16t^2}$$

$$\frac{ds}{dt} = \frac{-1500}{8t^3} \qquad (3)$$

(3) yields at once the result that $\frac{ds}{dt} = -1500$ when $t = 1/2$.

It is again of interest to observe that (3) yields much more than the correct answer to the exercise. It shows us how ds/dt depends on t for all values of t during the fall of the ball. In particular, (3) tells us that the rate of change of the shadow is inversely proportional to the cube of the time. (Thus, for example, if the time of fall is doubled, the rate of change of the shadow decreases by a factor of 8.) As sort of a check, we observe that the relation $50 = 16t^2$ shows that the ball is in

[2.7.4(L) cont'd]

flight for approximately 1.75 seconds which is $3\frac{1}{2}$ times greater than 1/2. Now $(3.5)^3 = 42.875$; hence at the instant the ball hits the ground the rate of change of its shadow should be about 1/43 of 1500 or about 35 ft/sec. Replacing t in (3) by $\sqrt{50/16} = \frac{5\sqrt{2}}{4}$, we obtain that $ds/dt = -24\sqrt{2}$, which agrees nicely with our approximation. Our main point is that results such as (3) are often more important than the specific value at which the exercise asks us to evaluate (3).

2.7.5(L)

Here again we need only assume that x and y are differentiable functions of t. Then the fact that x and y are related by $x^2 + y^2 = 1$, leads to:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \qquad (1)$$

Since we are interested in finding $\frac{dy}{dt}$, we rewrite (1) as:

$$\frac{dy}{dt} = \frac{-x\frac{dx}{dt}}{y} \qquad (2)$$

We then observe that (2) tells us how x, y, $\frac{dx}{dt}$, and $\frac{dy}{dt}$ are related at any time t except that we must be very careful when y = 0 since we cannot divide by 0. From a philosophical point of view, since the circle has a vertical tangent at y = 0 we do

[2.7.5(L) cont'd]

not talk about a y-component of speed at $y = 0$ since at this instant y is changing while x remains 0. However, we can look at the limit of $\frac{dy}{dt}$ as y approaches zero in (2).

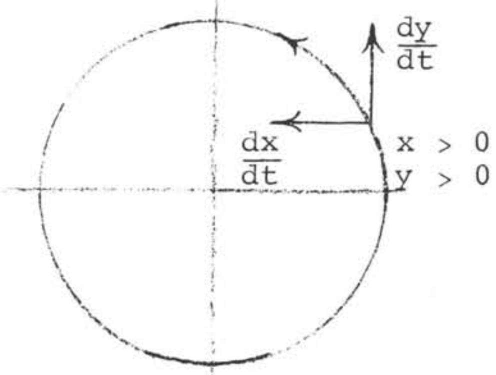
In any event, in our given exercise, we know the additional information that $\frac{dx}{dt} = -y$, hence if $y \neq 0$, (2) yields the result that:

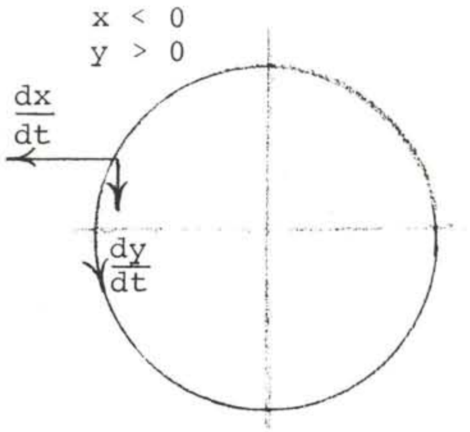
$$\frac{dy}{dt} = x \tag{3}$$

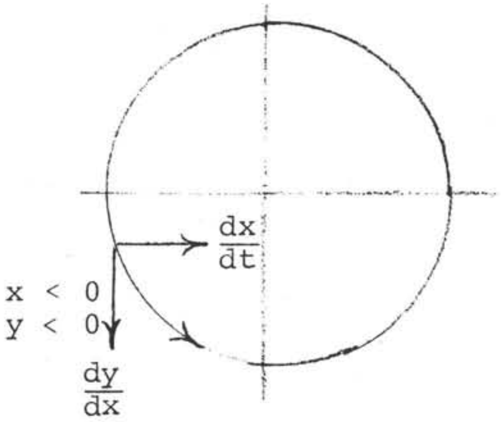
(Referring to our previous parenthetical remark, we may define the speed at $y = 0$ to be the limit in (3) as y approaches 0. In this case x is either 1 or -1; hence, when $y = 0$, if the speed is continuous, it must be that $\frac{dy}{dt}$ is either 1 or -1 depending on the orientation of the motion.)

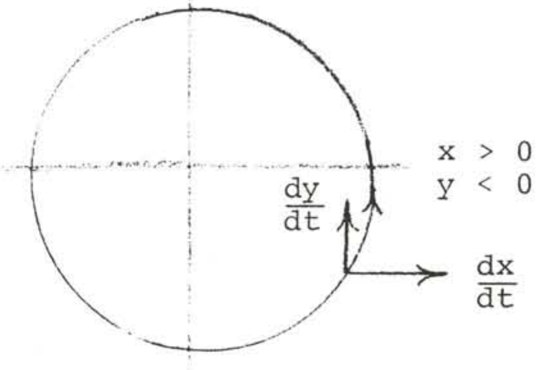
With regard to the orientation, we have from our given information together with (3) that $\frac{dx}{dt} = -y$ while $\frac{dy}{dt} = x$. Thus, in the first quadrant where x and y are both positive we have that $\frac{dx}{dt}$ is negative while $\frac{dy}{dt}$ is positive. This means that in the first quadrant the particle is moving up and to the left and this means the motion is counter-clockwise. A similar check verifies this result in all quadrants. For, example, in the second quadrant x is negative while y is positive; hence $\frac{dx}{dt}$ is negative while $\frac{dy}{dt}$ is also negative. This means that in the second quadrant the particle is moving down and to the left, and this is also the counter-clockwise direction. Pictorially,

[2.7.5(L) cont'd]









We may also check the result (3) by means of the chain rule. Namely, by implicitly differentiating $x^2 + y^2 = 1$ with respect to x, we obtain that $2x + 2y\frac{dy}{dx} = 0$ or that $\frac{dy}{dx} = \frac{-x}{y}$. (4)

On the other hand, by the chain rule, $\frac{dy}{dx}$ is the quotient of $\frac{dy}{dt}$ divided by $\frac{dx}{dt}$ which is $\frac{x}{-y}$ or $\frac{-x}{y}$ and this checks with (4).

[2.7.5(L) cont'd]

NOTE:

If we define the instantaneous speed of the particle in the "usual" physical sense, we notice some interesting special results in this case. Namely, let us define the speed as $\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}$. Then, since $\frac{dx}{dt} = -y$ and $\frac{dy}{dt} = x$ we obtain that the speed is given by $\sqrt{x^2 + y^2}$. We next note that x and y are related by $x^2 + y^2 = 1$. Hence, in this problem, the particle is moving along the circle with a constant speed of 1 (and this clearly checks with our earlier parenthetical remarks that if the speed is continuous it must be 1 or -1 when y is 0).

Now, we have also learned in physics that, in such a case, the acceleration is always directed toward the center of the circle. An interesting point is that we are now in the position of checking such facts by a purely mathematical approach, and we no longer have to rely exclusively on physical intuition. For example, we can compute both d^2x/dt^2 and d^2y/dt^2 to find the horizontal and vertical components of the acceleration. We obtain

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}(-y) = \frac{-dy}{dt} = -x$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}(x) = -y$$

$\therefore$ The slope of the acceleration "vector" is $\frac{-y}{-x}$ or $\frac{y}{x}$.

[2.7.5(L) cont'd]

On the other hand, the slope of the circle $(\frac{dy}{dx})$ is $\frac{-x}{y}$.

Since $\frac{y}{x}$ and $\frac{-x}{y}$ are negative reciprocals, the acceleration is at right angles (normal) to the circle; hence, the acceleration is always directed along the radius and a simple check of signs shows that this direction is always toward the center.

2.7.6

We know that $\frac{dy}{dt} = x$ and that $4x^2 + 9y^2 = 36$. Thus, if we assume, as usual, that x and y are both differentiable functions of t (and please note that unless such assumptions are made, we cannot use the results of differential calculus), we obtain by implicit differentiation

$$8x\frac{dx}{dt} + 18y\frac{dy}{dt} = 0$$

Hence,

$$8x\frac{dx}{dt} + 18yx = 0$$

Thus, if we assume that $x \neq 0$ (which, in particular, is true in the present exercise), we obtain that:

$$8\frac{dx}{dt} + 18y = 0$$

or:

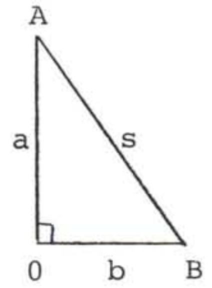
$$\frac{dx}{dt} = \frac{-9y}{4}$$

[2.7.6 cont'd]

In particular, then, when $y = \frac{4\sqrt{2}}{9}$, $\frac{dx}{dt} = -\sqrt{2}$, and this, of course, is the x-component of the speed.

2.7.7

In terms of a diagram we have:



What the problem asks us to do is find $\frac{ds}{dt}$ at the instant the given conditions apply. At the given instant a = 40 and b = 30 so s = 50 by the Pythagorean Theorem, but note that we do not know at what instant this occurs. Moreover, we cannot assume that the ships are moving at a constant rate for, in particular, the ship which has gone the furthest has AT THE GIVEN INSTANT a lesser rate of speed than the ship that has not gone as far.

However, once we assume that a and b are differentiable functions of t, the Pythagorean theorem yields the following general result:

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2s\frac{ds}{dt} \quad (\text{since } a^2 + b^2 = s^2)$$

[2.7.7 cont'd]

In the given exercise a = 40, b = 30, $\frac{da}{dt} = 20$, $\frac{db}{dt} = 25$ and s = 50. Thus:

$$2(40)(20) + 2(30)(25) = 2(50)\frac{ds}{dt}$$

or:

$$\frac{ds}{dt} = 31 \text{ miles per hour.}$$

Unit 8: Applications of the Derivative III

2.8.1 (L)

In most problems of this type the usual procedure is to find the values for which $\frac{dy}{dx}$ is zero (in this exercise there are no points at which the derivative does not exist, nor need we check the endpoints of a closed interval since we are assuming that f is defined for all real numbers). Thus we easily arrive at $x = 1$, $x = 2$, $x = 3$, and $x = 4$ as our CANDIDATES for high-low points.

One then looks at $f''(x)$ for these candidates and unless $f''(x) = 0$, we can tell from the concavity whether we have a high or a low. If $f''(x) = 0$, however, we gain no information about high or low points.

In this particular problem, notice that it is first of all quite a chore to compute the second derivative since we have a product of four factors. Even worse is the fact that once we do compute $f''(x)$ we find that $f''(1) = f''(2) = f''(3) = f''(4) = 0$. Namely, if we recall the recipe that we differentiate a different factor each time, we obtain:

$$\begin{aligned} f''(x) &= 4(x-1)^3(x-2)^3(x-3)^6(x-4)^8 + 3(x-1)^4(x-2)^2 \\ &\quad (x-3)^6(x-4)^8 + 6(x-1)^4(x-2)^3(x-3)^5(x-4)^8 \\ &\quad + 8(x-1)^4(x-2)^3(x-3)^6(x-4)^7 \\ &= (x-1)^3(x-2)^2(x-3)^5(x-4)^7 g(x) \end{aligned}$$

$$\text{where } g(x) = 4(x-2)(x-3)(x-4) + 3(x-1)(x-3)(x-4) + 6(x-1)(x-2)(x-4) + 8(x-1)(x-2)(x-3)$$

[2.8.1 (L) cont'd]

So, in this exercise, we must make use of the first derivative more directly. Recall that

$$f'(x) = \frac{dy}{dx} = (x - 1)^4(x - 2)^3(x - 3)^6(x - 4)^8 \qquad (1)$$

Since every factor in (1) except $(x - 2)^3$ appears to an even power, only $(x - 2)^3$ can change sign as x varies.

In any event, let us investigate what happens to f'(x) in a neighborhood of x = 1. That is, we shall look at f'(x) when x is "a little less" than 1 and also when it is "a little greater" than 1. We then see from (1) that:

$$f'(1^-) = (+)(-)(+)(+)^* = -$$

and

$$f'(1^+) = (+)(-)(+)(+) = -$$

Thus, the curve is falling "just before" and "just after" x = 1. In other words, x = 1 is neither a high nor a low. In a similar way:

---

*We are focusing our attention only on the sign of the factors in a deleted neighborhood of 1. All factors except $(x - 2)^3$ are automatically non-negative because of even exponents. $(x - 2)^3$ changes sign in a neighborhood of 2; however, since we are interested in what is happening near 1, we may assume that our neighborhood of 1 is chosen sufficiently small so that $(x - 2)^3$ will always be negative.

[2.8.1 (L) cont'd]

$$f'(2^-) = (+)(-)(+)(+) = -$$

$$f'(2^+) = (+)(+)(+)(+) = +$$

Hence the curve is falling just before $x = 2$ and rising just after $x = 2$; that is, $x = 2$ denotes a low point, even though $f''(2) = 0$*.

Continuing, we find:

$$f'(3^-) = (+)(+)(+)(+) = +$$

$$f'(3^+) = (+)(+)(+)(+) = +$$

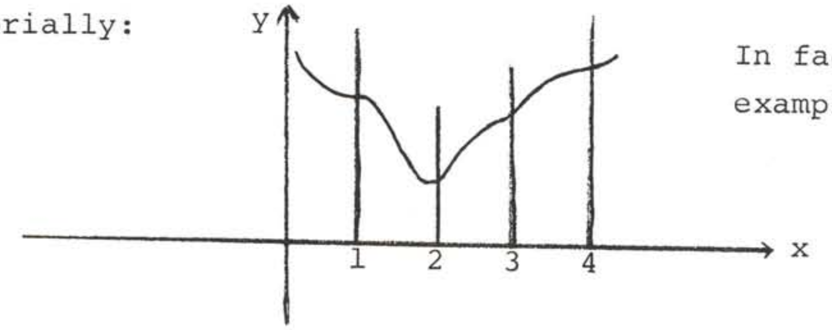
and again we see that $f(3)$ is neither a high nor a low; rather the curve rises before and after $x = 3$.

Finally:

$$f'(4^-) = (+)(+)(+)(+) = +$$

$$f'(4^+) = (+)(+)(+)(+) = +$$

Pictorially:



In fact in this particular example $\frac{dy}{dx} \leqslant 0$ if $x < 2$
$\geqslant 0$ if $x > 2$

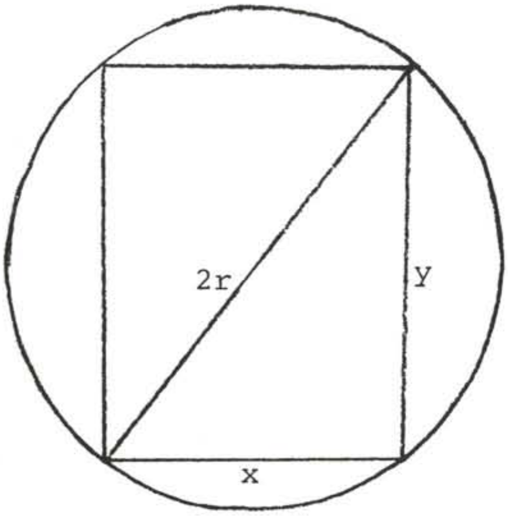
---

*If this seems unnatural to accept, sketch, for example, $y = (x - 2)^4$ which has a minimum at $x=2$, yet $\left.\frac{d^2y}{dx^2}\right|_{x=2} = 0$.

[2.8.1 (L) cont'd]

The important point is that when the second derivative is zero we can find out all we need about high and low points by considering the first derivative (slope) directly. We emphasize this point mainly because of the overwhelming desire to compute f"(c) once f'(c) = 0. It's fine when this works, but we must remember the basic concept of slope when it doesn't work.

2.8.2



$$A = xy \tag{1}$$

$$x^2 + y^2 = 4r^2 \tag{2}$$

Method #1 (Expressing A explicitly in terms of x)

From (2), $y = (+)\sqrt{4r^2 - x^2}$

$\therefore$ (1) becomes:

$$A = x\sqrt{4r^2 - x^2}$$

$$= x(4r^2 - x^2)^{1/2} \tag{3}$$

$$\therefore \frac{dA}{dx} = x[\tfrac{1}{2}(4r^2 - x^2)^{-1/2}(-2x)] + (4r^2 - x^2)^{1/2}$$

$$= \frac{-x^2}{\sqrt{4r^2 - x^2}} + \sqrt{4r^2 - x^2} \qquad = \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}} \tag{4}$$

[2.8.2 cont'd]

$\therefore \frac{dA}{dx} = 0 \longleftrightarrow x^2 = 2r^2$

But $x^2 + y^2 = 4r^2$

$\therefore x^2 = 2r^2 \longleftrightarrow y^2 = 2r^2$

$\therefore \frac{dA}{dx} = 0 \longleftrightarrow x^2 = y^2$

$\longleftrightarrow x = y$ (Recall physically that $x,y > 0$)

$\therefore$ Rectangle is a square.

(It is also intuitively clear that these values yield a maximum for A rather than a minimum, for we can make A as small as we like merely by choosing x as small as we like.)

To be sure, we could find A" from (4) and see that A" is negative when $x = \sqrt{2}r$. This would tell us that the curve spills water at this point and hence that we must have a maximum. A less tedious way involves the principle of Exercise 2.8.1. Namely, in (4) we observe that when x is slightly greater than $\sqrt{2}r$, $\frac{dA}{dx}$ is negative. Thus the curve is rising before we get to $x = \sqrt{2}r$ and falling afterwards. This also indicates that we have a high point on the curve when $x = \sqrt{2}r$.

<u>Method #2</u> (Implicitly)

From (1)

$$\frac{dA}{dx} = x\frac{dy}{dx} + y \qquad (5)$$

From (2)

$$2x + 2y\frac{dy}{dx} = 0$$

or

$$\frac{dy}{dx} = -\frac{x}{y} \qquad (6)$$

[2.8.2 cont'd]

Putting (6) into (5):

$$\frac{dA}{dx} = x(-\frac{x}{y}) + y$$

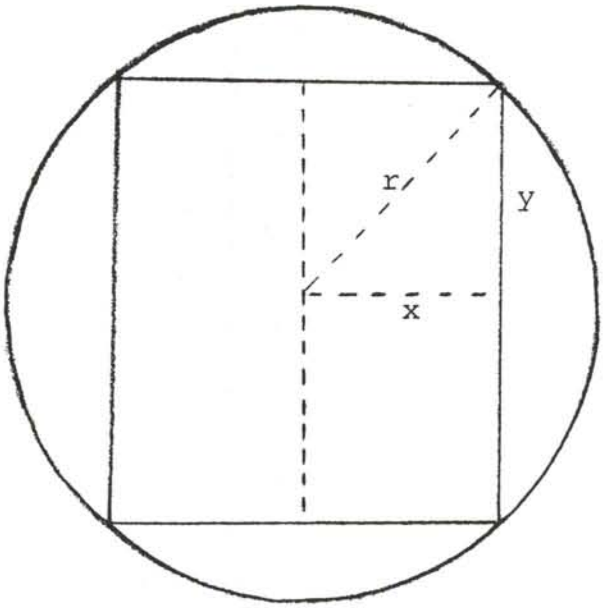
$$= \frac{y^2 - x^2}{y} \qquad (7)$$

$$\therefore \frac{dA}{dx} = 0 \longleftrightarrow y^2 = x^2 \quad \left\{ \text{Again: } x,y > 0 \right.$$

$$\longleftrightarrow y = x$$

2.8.3 (L)

Sketching the situation, we have the following central cross-section:



x denotes the radius of the base of the cylinder. y denotes half the height of the cylinder.

Since the height of the cylinder is 2y and the radius of its base is x, its volume V is given by:

$$V = 2\pi x^2 y \qquad (1)$$

[2.8.3 (L) cont'd]

where the variables x and y are related by:

$$x^2 + y^2 = r^2 \qquad (2)$$

We can now find $\frac{dV}{dx}$ either by solving for x as a function of y [from (2)] and putting this into (1), whereupon we differentiate explicitly, or we may differentiate both (1) and (2) implicitly with respect to x. That is,

Method #1

From (2) $x^2 = r^2 - y^2$

Putting this into (1), we obtain

$$\left.\begin{aligned} V &= 2\pi(r^2 - y^2)y \\ &= 2\pi r^2 y - 2\pi y^3 \end{aligned}\right\} \underline{0 \leqslant y \leqslant r}$$

We then obtain

$$\frac{dV}{dy} = 2\pi r^2 - 6\pi y^2 \qquad (3)$$

$$\frac{d^2V}{dy^2} = -12\pi y \qquad (4)$$

From (3), we see that

$$\frac{dV}{dy} = 0 \longleftrightarrow 2\pi r^2 = 6\pi y^2$$

Method #2

From (1),

$$\frac{dV}{dx} = 4\pi xy + 2\pi x^2 \frac{dy}{dx} \qquad (9)$$

and from (2)

$$2x + 2y\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{x}{y} \qquad (10)$$

Putting (10) into (9),

$$\begin{aligned} \frac{dV}{dx} &= 4\pi xy + 2\pi x^2\left(-\frac{x}{y}\right) \\ &= \frac{4\pi xy^2 - 2\pi x^3}{y} \qquad (11) \end{aligned}$$

[2.8.3 (L) cont'd]

$$\longleftrightarrow y^2 = \frac{r^2}{3} \qquad (5)$$

$$\longleftrightarrow \underline{y = \frac{r}{\sqrt{3}}} \qquad (6)$$

Putting (5) into (2), we find

$$x^2 = \frac{2}{3}r^2 \qquad (7)$$

Then, putting (6) and (7) into (1),

$$V = 2\pi\left(\frac{2}{3}r^2\right)\left(\frac{r}{\sqrt{3}}\right)$$

$$= \frac{4\pi r^3}{3\sqrt{3}} = \boxed{\frac{4\sqrt{3}\pi r^3}{9}} \qquad (8)$$

Finally, since (4) shows us that $\frac{d^2V}{dy^2} < 0$ for all $y > 0$, (8) must denote a <u>maximum</u>.

From (11), $\frac{dV}{dx} = 0 \longleftrightarrow$

$$4\pi xy^2 = 2\pi x^3$$

$$\longleftrightarrow \quad x^2 = 2y^2 \qquad (12)$$

Putting (12) into (2),

$$2y^2 + y^2 = r^2$$

$$y^2 = \frac{r^2}{3} \qquad (13)$$

[Notice that (13) and (5) say the same thing so that the two methods yield the same result.]

We shall not quibble as to whether Method #1 or Method #2 is to be preferred. What is interesting is that while the two methods ultimately yield the same result, they seem to emphasize different aspects. For example, in Method #1, we find y and x in terms of r directly, while in Method #2, we find the relationship between x and y which yields the maximum result.

We would now like to make a few observations based on this exercise:

[2.8.3 (L) cont'd]

(1) As we saw in Exercise 2.8.2 the largest rectangle that can be inscribed in a circle of radius r is a square. Thus, it might be natural to assume that the greatest area generates the greatest volume. However, the solution to this problem shows that as natural as it may seem, the assumption is false. The reason for this can be best explained from Equation (1). Namely, x appears to the second power while y appears only linearly. Thus a small change in x can produce a more drastic change than a similar change in y. We shall say more about this in a separate note at the end of this exercise.

In particular the inscribed square has the length of its side equal to $\sqrt{2}r$. Noticing that in this problem x and y denote <u>half</u> the length of a side, we see that $x = y = \frac{1}{2}\sqrt{2}r = \frac{r}{\sqrt{2}}$ in the case of the square. Putting these results into (1) we obtain:

$$V = 2\pi\left(\frac{r}{\sqrt{2}}\right)^2 \frac{r}{\sqrt{2}} = \frac{\pi}{\sqrt{2}}r^3 \qquad (14)$$

The point is that the inscribed square has a larger area than the inscribed rectangle for which $y = \frac{r}{\sqrt{3}}$ (see equation (6)). Yet that inscribed rectangle generates a larger volume than that generated by rotating the square. The proof lies in the fact that the volume generated by the rectangle is $\frac{4\sqrt{3}}{9}\pi r^3$ (see equation (8)) while from (14) the volume generated by the square is $\frac{\pi}{\sqrt{2}}r^3$. Since $\frac{4\sqrt{3}}{9} > \frac{1}{\sqrt{2}}$* the result follows.

---

*A convenient way of comparing radicals in inequalities is to raise both sides to a power which eliminates the radical. In this case

$$\frac{4\sqrt{3}}{9} > \frac{1}{\sqrt{2}} \longleftrightarrow \left(\frac{4\sqrt{3}}{9}\right)^2 > \left(\frac{1}{\sqrt{2}}\right)^2 \longleftrightarrow \frac{48}{81} > \frac{1}{2}.$$

But it is easy to see that $\frac{48}{81} > \frac{1}{2}$.

[2.8.3 (L) cont'd]

(2) In our solution of this exercise, we must not lose sight of the independence of equations (1) and (2). No matter what function of x and y we were given, the fact remains in this problem that x and y are <u>always</u> related by (2). That is, from our initial diagram, we must have that $x^2 + y^2 = r^2$.

By way of illustration, suppose we wanted to find the cylinder which had the maximum surface area. Letting S denote the surface area, we have that our cylinder consists of two faces which are circles of radius x (hence of area πx^2) and a "side" of area 2y times the circumference of one of the circles, that is, $(2y)(2\pi x)$. (This is called the lateral area of the cylinder.)

$$S = 2\pi x^2 + 4\pi xy \tag{16}$$

We now want to maximize S. The point is that (16) replaces (1) but (2) still remains in effect. For example, we may differentiate (16) implicitly with respect to x to obtain:

$$\frac{dS}{dx} = 4\pi x + 4\pi x\,\frac{dy}{dx} + 4\pi y \tag{17}$$

Since $\frac{dy}{dx} = -\frac{x}{y}$ (as before),

$$\frac{dS}{dx} = 4\pi x + 4\pi x\left(-\frac{x}{y}\right) + 4\pi y = \frac{4\pi xy - 4\pi x^2 + 4\pi y^2}{y}$$

$$\therefore \frac{dS}{dx} = 0 \longleftrightarrow 4\pi xy - 4\pi x^2 + 4\pi y^2 = 0$$

$$\longleftrightarrow y^2 + xy - x^2 = 0$$

[2.8.3 (L) cont'd]

$$\longleftrightarrow y = \frac{-x \pm \sqrt{x^2 + 4x^2}}{2}$$

$$\longleftrightarrow y = \frac{-x + \sqrt{5}x}{2}$$ (We discard the minus root since y must be positive.)

∴ The maximum surface area occurs when:

$$y = \frac{(\sqrt{5} - 1)\ x}{2} \approx \frac{(2.25 - 1)x}{2}$$

$$\approx \frac{5}{8}x \qquad (18)$$

Summed up: Y = x yields the square (rectangle y maximum area).
$y \approx .7x$ [from (12)] yields the maximum volume.
$y \approx 0.63x$ [from (18)] yields the maximum surface area.

2.8.4 (L)

(a) First of all let us point out why x is defined as it is. In averaging numbers, there is a chance that a large negative number can cancel the effect of a large positive number. In this sense, then, it might be advisible to measure only the magnitudes of the errors without regard to signs. Observe that squaring the error makes each error positive. Thus, the recipe for x involves the sum of the squares of errors and this technique is known as the method of LEAST SQUARES.

The solution of the problem is perhaps the easiest part of the whole thing. Letting $e = (c_1 - x)^2 + \ldots + (c_n - x)^2$, we obtain:

[2.8.4 (L) cont'd]

$$\frac{de}{dx} = -2(c_1 - x) + \ldots + [-2(c_n - x)]$$

$$= -2(c_1 + \ldots + c_n - nx)$$

Therefore:

$$\frac{de}{dx} = 0 \text{ if and only if } x = \frac{c_1 + \ldots + c_n}{n} \qquad (1)$$

Notice that the right hand side of (1) is precisely the AVERAGE of c_1, ..., and c_n. Thus we have shown that the usual notion of arithmetical average turns out also to be the best least square approximation.

A generalization of this idea is presented in the next part in which we try to find the best least square approximation in terms of a straight line to estimate a collection of points in the plane.

(b) Notice that $mx_1 + 1$ is the y-coordinate of the point on the line $y = mx + 1$ whose x-coordinate is x_1. Therefore, $y_1 - (mx_1+1)$ [$= y_1 - mx_1 - 1$] is the difference between the actual y-value and the y-value of the corresponding point on the line, and $(y_1 - mx_1 - 1)^2$ is the square of this difference. In other words, once we find the value of m that minimizes the sum given above, we find the line of the form $y = xm + 1$ that has the property that if we square the verticle distance of each point from the line, the sum of these squares is as small as possible.

Again the actual solution is straight-forward. Namely,

[2.8.4 (L) cont'd]

$$w = (y_1 - mx_1 - 1)^2 + (y_2 - mx_2 - 1)^2 + (y_3 - mx_3 - 1)^2 + (Y_4 - mx_4 - 1)^2$$

$$\frac{dw}{dm} = -2x_1(y_1 - mx_1 - 1) - 2x_2(y_2 - mx_2 - 1) -2x_3(y_3 - mx_3 - 1) - 2x_4(y_4 - mx_4 - 1)$$

$$\therefore \frac{dw}{dm} = 0 \longleftrightarrow \sum_{k=1}^{4} x_k(y_k - mx_k - 1) = 0$$

$$\longleftrightarrow \sum_{k=1}^{4} x_k y_k - m \sum_{k=1}^{4} x_k^2 - \sum_{k=1}^{4} x_k = 0$$

$$\longleftrightarrow m = \frac{\sum_{k=1}^{4} x_k y_k - \sum_{k=1}^{4} x_k}{\sum_{k=1}^{4} x_k^2}$$

$$\longleftrightarrow m = \frac{\sum_{k=1}^{4} x_k(y_k - 1)}{\sum_{k=1}^{4} x_k^2}$$

$$\longleftrightarrow m = \frac{x_1(y_1 - 1) + x_2(y_2 - 1) + x_3(y_3 - 1) + x_4(y_4 - 1)}{x_1^2 + x_2^2 + x_3^2 + x_4^2} \qquad (1)$$

[2.8.4 (L) cont'd]

(1) now tells us how to find m for <u>any</u> four points. In particular in our problem:

	x_k	y_k	y_k-1	x_k^2	$x_k(y_k-1)$
k=1:	-2	$-\frac{1}{2}$	$-\frac{3}{2}$	4	3
k=2:	0	1	0	0	0
k=3:	1	2	1	1	1
k=4:	3	3	2	9	6

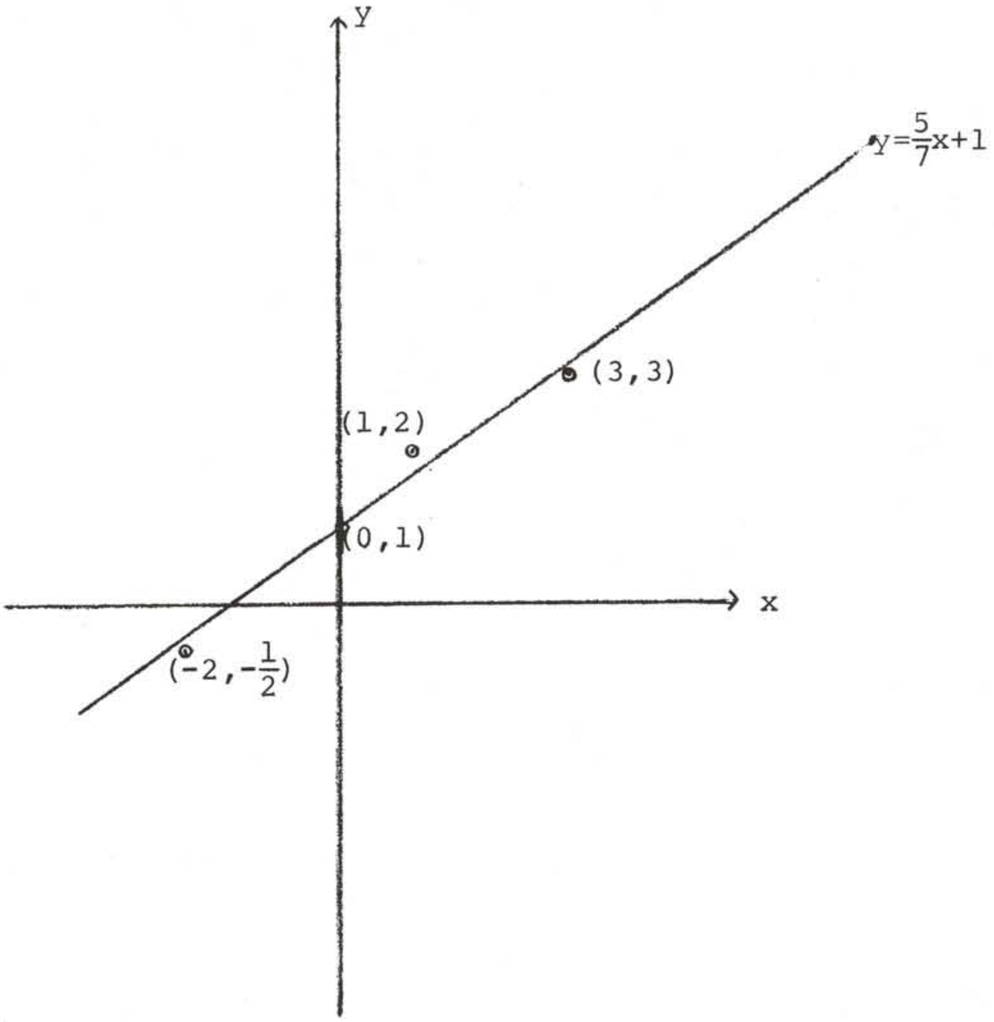
$$\sum_{k=1}^{4} x_k(y_k-1)=3+0+1+6=10$$

$$\sum_{k=1}^{4} x_k^2=4+0+1+9=14$$

$$\therefore m=\frac{10}{14}=\frac{5}{7} \quad \text{ans.}$$

x_k	$\frac{5}{7}x_k$	$\frac{5}{7}x_k+1$	y_k	$\lvert y_k-\frac{5}{7}x_k-1\rvert$
-2	$-\frac{10}{7}$	$-\frac{3}{7}$	$-\frac{1}{2}$	$-\frac{1}{14}$
0	0	1	1	0
1	$\frac{5}{7}$	$\frac{12}{7}$	2	$\frac{2}{7}$
3	$\frac{15}{7}$	$\frac{22}{7}$	3	$-\frac{1}{7}$

[2.8.4 (L) cont'd]

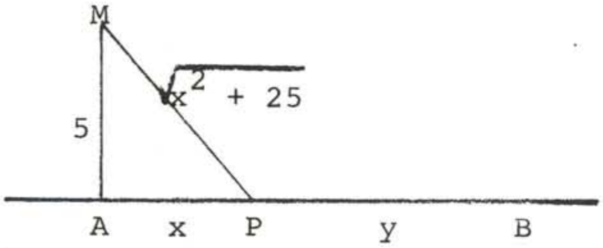


2.8.5 (L)

Here we have an example wherein the shortest distance is not the "quickest" path. On the one hand, we can head straight for A which will give him the shortest distance at the slower speed but which will leave him furthest from B once he reaches the road. On the other hand, he could head straight for B which would give him the longest path at the slower speed but which leaves him no distance to travel once he reaches the road. Of course, if for some reason he could travel faster in the desert than on the road, it is trivial to see in this case that he should head straight for B.

[2.8.5 (L) cont'd]

In any event, let us tackle this problem directly as a max-min problem. We start with the diagram



For part (a),

$$x + y = 5 \qquad (1)$$

Now, to get from M to P takes $\frac{\sqrt{x^2 + 25}}{15}$ hours, while to get from P to B takes $\frac{y}{39}$ hours. Thus, if t denotes the time of the trip, we have:

$$\boxed{t = \frac{\sqrt{x^2 + 25}}{15} + \frac{y}{39}} \qquad (2)$$

Differentiating (2) with respect to x, we have

$$\frac{dt}{dx} = \frac{x}{15\sqrt{x^2 + 25}} + \frac{1}{39}\frac{dy}{dx} \qquad (3)$$

From (1), $1 + \frac{dy}{dx} = 0$

$$\therefore \boxed{\frac{dy}{dx} = -1} \qquad (4)$$

Putting (4) into (3) yields

$$\frac{dt}{dx} = \frac{x}{15\sqrt{x^2 + 25}} - \frac{1}{39}$$

[2.8.5 (L) cont'd]

$$\therefore \frac{dt}{dx} = 0 \longleftrightarrow \frac{x}{15\sqrt{x^2 + 25}} = \frac{1}{39} \longleftrightarrow \frac{x}{5\sqrt{x^2 + 25}} = \frac{1}{13}$$

$$\longleftrightarrow \frac{x^2}{25(x^2 + 25)} = \frac{1}{169}$$

$$\longleftrightarrow 169x^2 = 25x^2 + 625$$

$$\longleftrightarrow 144x^2 = 625$$

$$\longleftrightarrow 12x = 25$$ (We discard the negative root since, physically, x is non-negative.)

$$\longleftrightarrow x = 2\frac{1}{12}$$

This answer is very interesting for two reasons:

(1) In arriving at (4) from (1), we didn't really use $x + y = 5$ as much as we used $x + y =$ constant. That is, if we started with $x + y = c$, we would have still obtained that $dy/dx = -1$. Thus, our answer to (a) would be the same for any value of B PROVIDED ONLY THAT THE DISTANCE FROM A TO B WAS AT LEAST 25/12 miles.

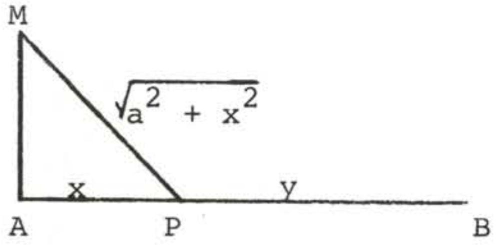
In particular, then, $x = 25/12$ miles would also be the correct answer to (b).

(2) If B is less that 25/12 miles from A, the point P in our diagram would be beyond B. Hence, in this case, it would be quicker to head straight for B. Since in (c) the distance from A to B is only 1 mile, he should head for B directly.

This problem seems to have a rather straightforward generalization. Let us assume that the motorist is <u>a</u> miles from A and that B is c miles from A. Let us also assume that his rate on the desert (or

[2.8.5 (L) cont'd]

any slower medium is v_1 mi/hr while his rate on the road (or any faster medium) is v_2 mi/hr. Then we have



$$x + y = c$$

$$\therefore \frac{dy}{dx} = -1$$

Now,

$$t = \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{y}{v_2}$$

$$\therefore \frac{dt}{dx} = \frac{x}{v_1\sqrt{a^2 + x^2}} + \frac{1}{v_2}\frac{dy}{dx}$$

$$= \frac{x}{v_1\sqrt{a^2 + x^2}} - \frac{1}{v_2}$$

$$\therefore \frac{dt}{dx} = 0 \longleftrightarrow \frac{x}{v_1\sqrt{a^2 + x^2}} = \frac{1}{v_2} \longleftrightarrow \frac{x^2}{v_1^2(a^2 + x^2)} = \frac{1}{v_2^2}$$

$$\longleftrightarrow (v_2^2 - v_1^2)x^2 = v_1^2\, a^2$$

{ Notice that if $v_2 < v_1$ there would be no real roots.

[2.8.5 (L) cont'd]

$\therefore$ t is minimum when

$$x = \frac{v_1 a}{\sqrt{v_2^2 - v_1^2}} \qquad (5)$$

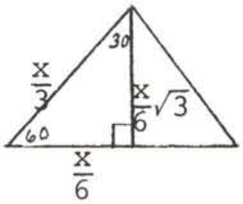
A check on the plausibility of (5) comes from taking a = 5, $v_1 = 15$, and $v_2 = 39$; since we then get the conditions of this exercise. A trivial check then shows that with these values (5) yields x = 25/12 as before.

(5) also tells us in general that if B is within $\frac{v_1 a}{\sqrt{v_2^2 - v_1^2}}$ miles of A, then the quickest path is the straight line that takes the motorist to B.

2.8.6 (L)

We let x denote the amount of wire to be used in the triangle. Hence L - x is the amount to be used in the square. (If we wished we could let y denote the amount used in the square and later we could make use of the fact that x + y = L.)

In any event, one side of the square would have length equal to $\frac{L-x}{4}$, so the area of the square is $(\frac{L-x}{4})^2$. As for the triangle, one side has length x/3; hence, its altitude is $x\sqrt{3}/6$. (Recall, that in a 30°-60°-90° triangle the sides are in the ratio 1, 1/2, $\sqrt{3}/2$.)



[2.8.6 (L) cont'd]

Thus, the area, of the whole triangle is $\frac{x^2\sqrt{3}}{36}$. If we let S denote the sum of the two areas, we have

$$S = \frac{x^2\sqrt{3}}{36} + \left(\frac{L-x}{4}\right)^2 \qquad (1)$$

from which it follows that:

$$S' = \frac{x\sqrt{3}}{18} - \frac{(L-x)}{8}^{*} \qquad (2)$$

(With respect to an earlier remark, had we let y denote the amount of wire in the square, then the area of the square would have been $(y/4)^2 = y^2/16$. The derivative of this with respect to x would have been $(y/8)(dy/dx)$. Differentiating $x + y = L$ implicitly and setting $\frac{dL}{dx} = 0$, follows that $1 + dy/dx = 0$. Thus, $y/8(dy/dx) = -y/8 = -(L-x)/8$ which agrees with the second term of (2) and may have been computationally simpler to obtain that by the first method.)

At any rate, from (2), we see that

$$S'' = \frac{\sqrt{3}}{18} + \frac{1}{8} \qquad (3)$$

Now, from (2) we see that $S' = 0$ if and only if

$$\frac{x\sqrt{3}}{18} - \frac{L}{8} + \frac{x}{8} = 0 \longleftrightarrow$$

$$x\left(\frac{\sqrt{3}}{18} + \frac{1}{8}\right) = \frac{L}{8} \longleftrightarrow$$

$$x\left(\frac{4\sqrt{3} + 9}{72}\right) = \frac{L}{8} \longleftrightarrow$$

$$x = \frac{9L}{4\sqrt{3} + 9} \qquad (4)$$

---

*Remember the chain rule $\frac{d}{dx}\left[\left(\frac{L-x}{4}\right)^2\right] = 2\left(\frac{L-x}{4}\right)\frac{d}{dx}\left(\frac{L}{4} - \frac{x}{4}\right) = 2\left(\frac{L-x}{4}\right)\left(-\frac{1}{4}\right)$.

[2.8.6 (L) cont'd]

Since (3) indicates that S" is always positive (holds water) then the value of x given by 4 must be the one which corresponds to a minimum. Recalling that x denotes the amount of wire in the triangle, we see that the sum of the two areas is a minimum when the amount of wire in the triangle is equal to:

$$\frac{9L}{4\sqrt{3} + 9} \tag{5}$$

and the amount of wire in the square is then found merely by subtracting the number in (5) from L.

As a final note to part (a), we are aware that in engineering considerations we prefer decimals to radicals. To this end, we note that $\sqrt{3}$ is approximately 1.73, so that $4\sqrt{3}$ is approximately 6.92. Thus, $4\sqrt{3} + 9$ is pretty nearly equal to 16. Putting this information into (5), we see that the sum of the areas is a minimum when about 9/16 of the wire is used for the triangle and 7/16 is used for the square. While this result may not seem at all self-evident, it does show us the beauty of the logical approach through the calculus; namely, we can obtain the correct answer even if it does not seem intuitively logical.

Perhaps the greatest significance of this exercise comes in (b). While we have stressed that there are THREE tests for candidates for max-min values, the fact remains that in most exercises the correct answers are all obtained by setting dy/dx equal to 0. However, (3) shows us that we can NEVER find a maximum for S this way since the curve is always holding water.

[2.8.6. cont'd]

Well, since a continuous function on a closed interval (recall that the domain of S is restricted physically to the interval $0 \leqslant x \leqslant L$) must take on its maximum somewhere and since we have seen that it cannot occur at an interior point, then it must occur at an endpoint. In essence, then, we need merely compute S from (1) once with x = 0 and again with x = L; and the larger of these two values will be the maximum sum.

With x = 0, S becomes $L^2/16$ and with x = L, S becomes $L^2\sqrt{3}/36$. It is clear that $L^2\sqrt{3}/36 < L^2/16$ (see note). Hence the sum is maximum when all the wire is used to make the square.

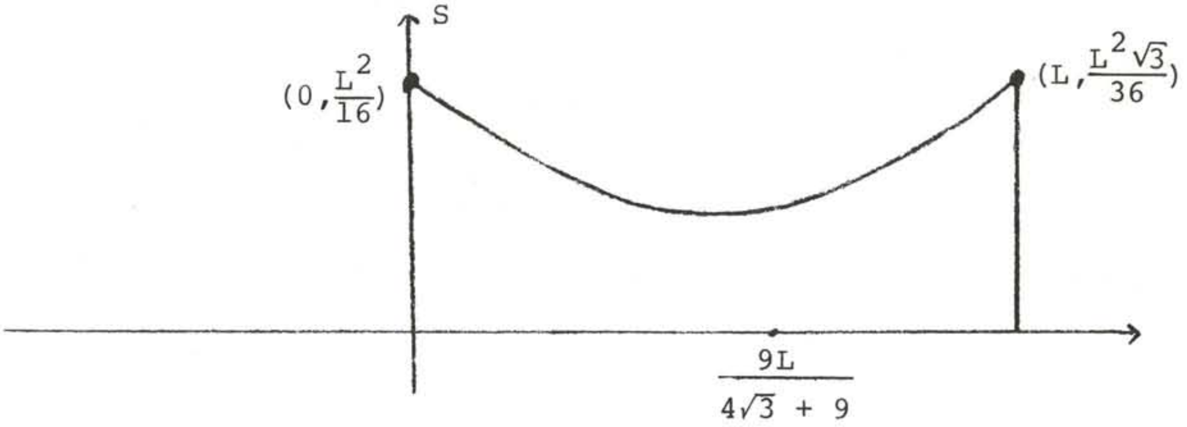
Note:

In general to compare the sizes of positive numbers which involve radicals, we raise all numbers to a common power which eliminates all radicals. The idea is that for positive numbers, the greater number yields the greater power. In the particular problem here, it would suffice to square each number. This would yield

$$\left.\begin{aligned} \left(\frac{L^2}{16}\right)^2 &= \frac{L^4}{256} \\ \left(\frac{L^2\sqrt{3}}{36}\right)^2 &= \frac{3L^4}{1296} \end{aligned}\right\} \frac{3L^4}{1296} < \frac{L^4}{256} \rightarrow \frac{L^2\sqrt{3}}{36} < \frac{L^2}{16}$$

As a final note, do not be confused by radicals, etc. When all is said and done (1) represents a parabola. In fact the graph looks like

[2.8.6 (L) cont'd]



2.8.7

$$s = \lambda t - (1 + \lambda^4)t^2$$

$\therefore \frac{ds}{dt} = \lambda - 2(1 + \lambda^4)t$ (= λ when $t = 0$. Therefore, $\lambda > 0$ $\rightarrow \frac{ds}{dt} > 0$ at $t = 0$.)

$\frac{d^2s}{dt^2} = -2(1 + \lambda^4) < 0$ ($\therefore$ acceleration is negative so particle must reverse direction eventually.)

$$\therefore s \text{ is maximum} \longleftrightarrow \lambda - 2(1 + \lambda^4)t = 0$$

$$\longleftrightarrow t = \frac{\lambda}{2(1 + \lambda^4)}$$

$$\therefore s_{max} = \lambda\left[\frac{\lambda}{2(1 + \lambda^4)}\right] - (1 + \lambda^4)\left[\frac{\lambda}{2(1 + \lambda^4)}\right]^2$$

$$= \frac{\lambda^2}{2(1 + \lambda^4)} - \frac{\lambda^2}{4(1 + \lambda^4)}$$

$$s_{max} = \frac{\lambda^2}{4(1 + \lambda^4)} \qquad (1)$$

[2.8.7 cont'd]

Equation (1) yields the maximum value of s for a given value of λ.

Thus, we may view s_{max} as a function of λ and compute the maximum value that s_{max} assumes as λ is allowed to vary.

Thus, from (1) we obtain:

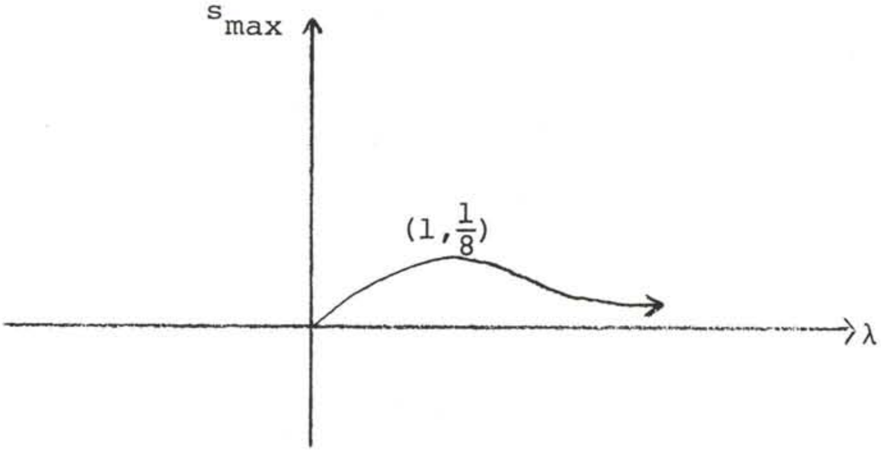
$$\frac{ds_{max}}{d\lambda} = \frac{4(1 + \lambda^4)(2\lambda) - \lambda^2[4(4\lambda^3)]}{[4(1 + \lambda^4)]^2} = \frac{8\lambda(1 + \lambda^4) - 16\lambda^5}{[4(1 + \lambda^4)]^2} = \frac{\lambda(1 - \lambda^4)}{2(1 + \lambda^4)^2}$$

$$= 0 \longleftrightarrow \lambda = 0 \quad \text{or} \quad \lambda = \pm 1$$

$$\lambda > 0 \longrightarrow \frac{ds_{max}}{d\lambda} = 0 \longleftrightarrow \lambda = 1$$

From (1), when $\lambda = 1$,

$$s_{max} = \frac{1}{8}$$



UNIT 9: Rolle's Theorem and its Consequences

2.9.1 (L)

Before we actually solve this problem, let us observe that $(x_1 + x_2)/2$ is the average of x_1 and x_2. Geometrically, this places $(x_3,0)$ midway between $(x_1,0)$ and $(x_2,0)$. This, in turn, says that the point at which the tangent is parallel to the chord is midway (with respect to x) between the other two points.

The proof is not difficult. We let

$$f(x) = ax^2 + bx + c. \tag{1}$$

Then,

$$f'(x) = 2ax + b \tag{2}$$

Now, from (1), we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{(ax_2^2 + bx_2 + c) - (ax_1^2 + bx_1 + c)}{x_2 - x_1}$$

$$= \frac{a(x_2^2 - x_1^2) + b(x_2 - x_1)}{x_2 - x_1} \quad ; \quad x_2^2 - x_1^2 = (x_2 + x_1)(x_2 - x_1)$$

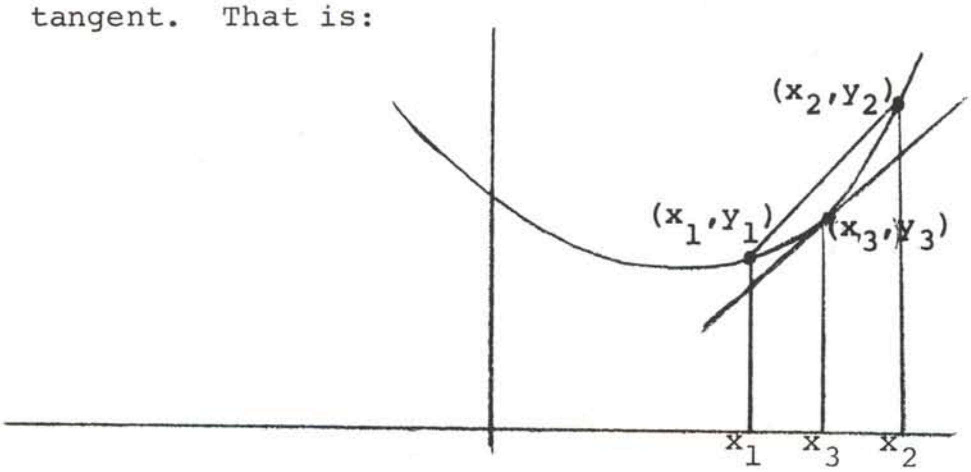
$$= a(x_2 + x_1) + b$$

$$= 2a(\frac{x_2 + x_1}{2}) + b$$

$$= f'(\frac{x_2 + x_1}{2}) \qquad \text{[from (2)]} \tag{3}$$

[2.9.1 (L) cont'd]

An interesting aside to this result is that we now have a construction for drawing the tangent line to the parabola at any point. Recall that in Plane Geometry courses we have only learned to construct tangents to straight lines (since these are self-tangent) and circles. What the solution to this problem implies is the following. Suppose we want the line tangent to the parabola $y = ax^2 + bx + c$ at the point (x_3,y_3). We locate the points (x_1,y_1) and (x_2,y_2) on the parabola by choosing $(x_1,0)$ and $(x_2,0)$ to be symmetrically located with respect to $(x_3,0)$. We then draw the line which joins (x_1,y_1) and (x_2,y_2). At (x_3,x_3) we then draw a line parallel to this one; and this is the required tangent. That is:



x_3 is by construction

$$\frac{x_1 + x_2}{2}$$

So we obtain the required tangent by the result of this exercise.

2.9.2

$$f(x) = \sqrt{x} = x^{1/2}$$

$$\therefore \quad f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\therefore \quad f'(c) = \frac{1}{2\sqrt{c}}$$

[2.9.2 cont'd]

On the other hand

$$\frac{f(4) - f(2)}{4 - 2} = \frac{\sqrt{4} - \sqrt{2}}{2} = \frac{2 - \sqrt{2}}{2}$$

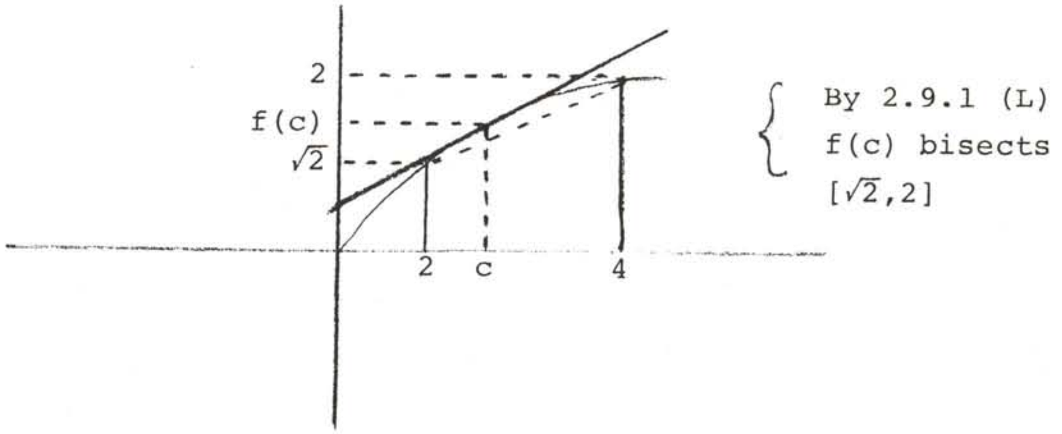
Hence, by the mean value theorem,

$$\frac{1}{2\sqrt{c}} = \frac{2 - \sqrt{2}}{2}$$

$$\therefore \quad \sqrt{c} = \frac{1}{2 - \sqrt{2}} = \frac{2 + \sqrt{2}}{(2 - \sqrt{2})(2 + \sqrt{2})} = \frac{2 + \sqrt{2}}{2}$$

$$\therefore \quad c = [\frac{2 + \sqrt{2}}{2}]^2 = \frac{4 + 4\sqrt{2} + 2}{4} = \frac{6 + 4\sqrt{2}}{4} = \frac{3 + 2\sqrt{2}}{2}$$

Note: $f(c) = \sqrt{c} = \frac{2 + \sqrt{2}}{2}$ is the arithmetical average of f(2) and f(4). That is, the average of $\sqrt{2}$ and $\sqrt{4}$ is $\frac{2 + \sqrt{2}}{2}$. This checks with our result in Exercise 2.9.1 (L) with the roles of y and x reversed. Pictorially,



2.9.3

The corollaries to the mean value theorem tell us that if $f'(x) \equiv g'(x)$ then $f(x) \equiv g(x) + c$.

Hence, in this exercise,

$$h(x) = x^3 + 7x^2 + (2x^2 + 1)^5 + c \tag{1}$$

From (1), it follows that

$$h(0) = 0^3 + 7(0)^2 + (0 + 1)^5 + c = 1 + c \tag{2}$$

Equation (2), coupled with the given fact that $h(0) = 3$, means that:

$$3 = 1 + c$$

or $c = 2$

Since c is a constant, it has the same value for all values of x. Putting this into (1), we finally obtain

$$h(x) = x^3 + 7x^2 + (2x^2 + 1)^5 + 2$$

2.9.4 (L)

By the mean value theorem, we know that

$$f(b) - f(a) = (b - a)f'(c) \text{ where } a < c < b$$

Hence,

$$\begin{aligned} |f(b) - f(a)| &= |(b - a)\ f'(c)| \\ &= |b - a|\,|\ f'(c)| \end{aligned} \tag{1}$$

[2.9.4 (L) cont'd]

Since $|f'(c)| \leqslant 1$, $|b - a||f'(c)| \leqslant |b - a|(1)$

$\therefore \quad |f(b) - f(a)| \leqslant |b - a|$, as asserted.

Now, as it stands, this amount of discussion hardly qualifies as a learning exercise. However, it does give us a chance to mention how the mean value theorem does analytically what we sense to be true intuitively from a geometrical point of view. For example, the fact that the magnitude of $f'(x)$ cannot exceed 1 means that the "rise" of the curve $y = f(x)$ can never exceed the "run". In other words, since $f'(x)$ denotes $\frac{\Delta y}{\Delta x}$, we are saying that $|\Delta y| \leqslant |\Delta x|$. So the result in this exercise is quite clear from a geometric view point. The main idea is that the mean value theorem allows us to obtain the same result without reference to a picture. This is particularly helpful later in the course when we deal with functions of several variables and pictures are either difficult or impossible to come by.

2.9.5

(a) By the mean value theorem, we have:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c), \text{ for some } c \text{ such that } x_1 < c < x_2 \quad (1)$$

Now we are given that $f'(x) > 0$ for all x. Hence, in particular, $f'(c) > 0$.

Combining this with (1), we obtain

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0 \quad (2)$$

[2.9.5 cont'd]

Since $x_1 < x_2$ we know that $x_2 - x_1 > 0$. Then since a quotient is positive if and only if the numerator and denominator have the same sign, and since the denominator in (2) is positive, it follows that the numerator is also positive. That is:

$$f(x_2) - f(x_1) > 0$$

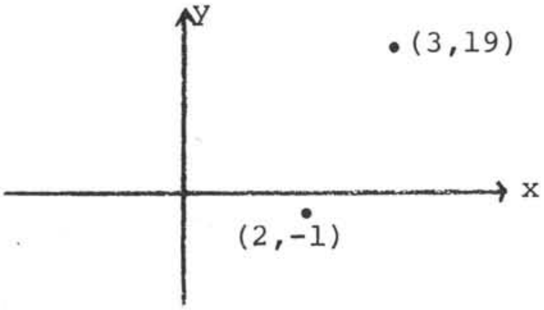
or $f(x_1) < f(x_2)$, as asserted.

(b) Let $P(x)$ denote $x^3 + x - 11 = 0$

Then $P(2) = 2^3 + 2 - 11 = -1 < 0$

$P(3) = 3^3 + 3 - 11 = 19 > 0$

Pictorially:



Since $P(x)$ is continuous (Horner's method), there must be at least one number r such that $P(r) = 0$, $2 < r < 3$ (i.e., $y = P(x)$ cuts the x-axis at least once between $x = 2$ and $x = 3$).

Now:

$$P'(x) = 3x^2 + 1 > 0 \text{ for all } x$$

By part (a) we know that if $f'(x) > 0$, $P(x)$ is an increasing function. In particular, then,

if $x < r$, $P(x) < P(r)$ or $P(x) < 0$

and if $x > r$, $P(x) > P(r)$ or $P(x) > 0$

$\therefore \quad x \neq r \rightarrow P(x) \neq P(r)$ or $P(x) \neq 0$

$\therefore \quad r$ is the only root of $x^3 + x - 11 = 0$

UNIT 10: Anti-derivatives or the Indefinite Integral

2.10.1 (L)

(a) In one sense since differentiation and integration are inverses of one another, the study of one can be explained in terms of the other. Here, we are interested in finding out what we have to differentiate to wind up with $(2x + 1)^3$. Our first guess might be $\frac{(2x + 1)^4}{4}$, but we must keep the chain rule in mind and recall that somehow we must take into account the fact that $\frac{d(2x + 1)}{dx} = 2$. One way of doing this is simply to differentiate $(2x + 1)^4$ with respect to x and obtain by use of the chain rule $8(2x + 1)^3$. Comparing this answer with the "correct" one, that is, with $(2x + 1)^3$ we see that we are "off" by a factor of 8. We then adjust for this by starting with $\frac{(2x + 1)^4}{8}$. Of course, it was crucial that we be in error by no worse than a constant factor, since if the factor is not constant we must use the product rule for differentiation (see, for example, our discussion in Exercise 2.10.2 (L))

In any event we have:

$$\int (2x + 1)^3 dx = \frac{(2x + 1)^4}{8} + c \qquad (1)$$

where c denotes an arbitrary constant. Moreover, it is futile to try to define the constant since $\frac{dc}{dx} = 0$, and, as a result, the addition of an arbitrary constant does not change the derivative. From an intuitive point of view, adding c to f(x) merely raises (or lowers, depending on the sign of c) the curve $y = f(x)$ by $|c|$ units. That is, $y = f(x)$ and $y = f(x) + c$ may be thought of as being "parallel" curves.

An allied question involves the converse of (1). That is, granted that any function, f(x) of the form $f(x) = \frac{(2x + 1)^4}{8} + c$ has the property that $f'(x) = (2x + 1)^3$ (and, by the way, a very

[2.10.1 (L) cont'd]

important aside here is to observe that whenever we integrate, we have as a quick check the fact that if we are correct then the derivative of the answer should yield the original function). Is it possible that there exists another function $g(x)$ not of that form such that $g'(x)$ is also $(2x + 1)^3$? The answer is no! In fact, the appropriate corollary to the mean value theorem tells us that since in this case $g(x)$ and, for example, $\frac{(2x + 1)^4}{8}$ have the same derivative, they must differ by at most a constant. Hence,

$$g(x) = \frac{(2x + 1)^4}{8} + c$$

and this establishes that the right hand side of (1) describes every function and only those functions f for which $f'(x) = (2x + 1)^3$.

To rehash the same results, but more in terms of the language of integrals, let us observe that the recipe:

$$\int u^n du = \frac{u^{n+1}}{n + 1} + c \qquad (n \neq -1)$$

requires that what is being raised to the nth power be EXACTLY the same as what follows the "d". That is:

$$\int (\quad)^n d(\quad) = \frac{(\quad)^{n+1}}{n + 1} + c \qquad (n \neq -1)$$

Thus, in this problem, had we been asked to find

$$\int (2x + 1)^3 d(2x + 1)$$

then the answer would have been

$$\frac{(2x + 1)^4}{4} + c$$

[2.10.1 (L) cont'd]

Moreover, the fact that

$$\int (2x + 1)^3 d(2x + 1) = \frac{(2x + 1)^4}{4} + c \qquad (1)$$

gives us a nice way of using the language of differentials to solve the problem (and while the language seems a bit different, it is equivalent to an earlier way we discussed for solving this exercise). Namely, we observe that $d(2x + 1) = 2dx$, and putting this in (1), we obtain:

$$\int (2x + 1)^3 2dx = \frac{(2x + 1)^4}{4} + c \qquad (2)$$

Since we can take constant factors outside the integral sign, that is, $\int cf(x)dx = c\int f(x)dx$ [to see this let g be any function such that $g' = f$, then we merely observe that $\frac{d(cg(x))}{dx} = \frac{cdg(x)}{dx} = cf(x)$] , (2) becomes:

$$2\int (2x + 1)^3 dx = \frac{(2x + 1)^4}{4} + c \qquad (3)$$

Dividing through by 2, (3) becomes:

$$\int (2x + 1^3 dx = \frac{(2x + 1)^4}{8} + \frac{c}{2} \qquad (4)$$

We finally observe that since c denotes an arbitrary constant, so also does $\frac{c}{2}$; hence, we may write (4) as

$$\int (2x + 1)^3 dx = \frac{(2x + 1)^4}{8} + c \qquad (5)$$

[2.10.1 (L) cont'd]

where we must remember that the c in (4) and the c in (5) are not the same number. Rather, each time c is used to denote an arbitrary constant. If for some reason we want to keep track of these two constants separately, then (5) might better have been written as:

$$\int (2x + 1)^3 dx = \frac{(2x + 1)^4}{8} + c_1 \qquad (c_1 = \frac{c}{2}) \qquad (5')$$

As a check in this exercise, we have that $(2x + 1)^3 = 8x^3 + 12x^2 + 6x + 1$ (by the binomial theorem); hence,

$$\int (2x + 1)^3 dx = \int (8x^3 + 12x^2 + 6x + 1)dx$$

$$= 2x^4 + 4x^3 + 3x^2 + x + c$$

and by writing c as $1/8 + c_1$, we obtain:

$$(2x + 1)^3 dx = \frac{(2x + 1)^4}{8} + c_1$$

which checks with (5').

A most important thing to keep in mind in these exercises is that we can translate any integral problem into a differential counterpart and thus use our previous knowledge to solve new problems.

At any rate, turning our attention to part (b) now, we observe that our last procedure would be hardly palatable for "simplifying" $(2x + 1)^{100}$. This is why we replaced the exponent of part (a) by 100. In other words, it was not too complicated to rewrite $(2x + 1)^3$ as $8x^3 + 12x^2 + 6x + 1$, but the 101 terms that the binomial theorem would yield for $(2x + 1)^{100}$ are far from pleasant to contemplate.

[2.10.1 (L) cont'd]

What we do notice, however, is that the more cumbersome exponent gives us no more trouble in the differential notation than did the smaller exponent. Namely,

$$\int (2x + 1)^{100} d(2x + 1) = \frac{(2x + 1)^{101}}{101} + c$$

whence:

$$\int (2x + 1)^{100} 2dx = \frac{(2x + 1)^{101}}{101} + c$$

or:

$$2\int (2x + 1)^{100} dx = \frac{(2x + 1)^{101}}{101} + c$$

or:

$$\int (2x + 1)^{100} dx = \frac{(2x + 1)^{101}}{202} + c$$

(By the way, this technique is equivalent to the substitution $u = 2x + 1$. In other words, if we have $\int (2x + 1)^{100} dx$ and we let $u = 2x + 1$ then $du = 2dx$ or $dx = \frac{du}{2}$. With this change of variable $(2x + 1)^{100} dx$ becomes

$$\int \frac{u^{100} du}{2} = \frac{1}{2}\int u^{100} du = \frac{1}{2}[\frac{u^{101}}{101} + c]$$

$$= \frac{(2x + 1)^{101}}{202} + c_1 \qquad (\text{where } c_1 = \frac{c}{2}) \quad .$$

In other words, one method of attack is to reduce a given integral $\int f(x)dx$ to one of the form $\int u^n du$ by an appropriate choice of u.

Again, in a manner analogous to what we discussed in part (a), we could have taken a guess that to wind up with $(2x + 1)^{100}$ after differentiating, we should have started with $(2x + 1)^{101}$. But, the derivative of $(2x + 1)^{101}$ with respect to x is

[2.10.1 (L) cont'd]

$$\frac{d(2x + 1)^{101}}{d(2x + 1)} \frac{d(2x + 1)}{dx} = (101(2x + 1)^{100})(2)$$

$$= 202(2x + 1)^{100}$$

Hence, if we start with $\frac{(2x + 1)^{101}}{202}$ we would obtain the desired answer.

2.10.2 (L)

At first glance it might seem that $\int \sqrt{x^2 - 4}\, dx$ would have been an easier problem. That is,

$$\int \sqrt{x^2 - 4}\, dx = \int (x^2 - 4)^{1/2} dx$$

We might then figure that to wind up with $(x^2 - 4)^{1/2}$ we should start with $(x^2 - 4)^{3/2}$. However, use of the chain rule shows us that the derivative of $(x^2 - 4)^{3/2}$ with respect to x is given by

$$\frac{d(x^2 - 4)^{3/2}}{d(x^2 - 4)} \frac{d(x^2 - 4)}{dx} = \frac{3}{2}(x^2 - 4)^{1/2}(2x)$$

$$= 3x(x^2 - 4)^{1/2}$$

Thus, it would appear that our correction factor would be 1/3x but this is incorrect since 1/3x is not a constant and hence cannot be taken outside the integral sign. In still other words we must differentiate $(1/3x)(x2 - 4)^{3/2}$ by the product rule and this

[2.10.2 (L) cont'd]

yields two terms, one of which is the desired $(x^2 - 4)^{1/2}$, but the other term introduces an error.*

Now what we could have correctly said is that

$$\int (x^2 - 4)^{1/2} d(x^2 - 4) = \frac{(x^2 - 4)^{3/2}}{3/2} + c$$

That is,

$$\int \sqrt{x^2 - 4}\, d(x^2 - 4) = \frac{2}{3}(x^2 - 4)^{3/2} + c \qquad (1)$$

Now, since $d(x^2 - 4) = 2xdx$, (1) becomes:

$$\int \sqrt{x^2 - 4}\, 2xdx = \frac{2}{3}(x^2 - 4)^{3/2} + c \qquad (2)$$

or:

$$2\int \sqrt{x^2 - 4}\, xdx = \frac{2}{3}(x^2 - 4)^{3/2} + c \qquad (3)$$

---

*Notice that if $\int f(x)dx = g(x) + c$ then $g'(x) = f(x)$. Hence if $\int \sqrt{x^2 - 4}\, dx = \frac{(x^2 - 4)^{3/2}}{3x} + c$ then $\frac{d}{dx}[\frac{1}{3x}(x^2 - 4)^{3/2}] = \sqrt{x^2 - 4}$. However

$$\begin{aligned}\frac{d}{dx}[\frac{1}{3x}(x^2 - 4)^{3/2}] &= \frac{1}{3x}\frac{d}{dx}(x^2 - 4)^{3/2} + \frac{d}{dx}(\frac{1}{3x})\cdot(x^2 - 4)^{3/2} \\ &= \frac{1}{3x}[\frac{3}{2}(x^2 - 4)^{1/2}(2x)] + [-\frac{1}{3x^2}](x^2 - 4)^{3/2} \\ &= \sqrt{x^2 - 4} - \frac{(x^2 - 4)^{3/2}}{3x^2} \\ &\neq \sqrt{x^2 - 4}\end{aligned}$$

[2.10.2 (L) cont'd]

$$\therefore \quad \int x\sqrt{x^2 - 4}\,dx = \frac{1}{3}(x^2 - 4)^{3/2} + c \qquad (4)$$

As a check:

$$\frac{d}{dx}[\frac{1}{3}(x^2 - 4)^{3/2}] = \frac{1}{3}[\frac{3}{2}(x^2 - 4)^{1/2}(2x)]$$

$$= x\sqrt{x^2 - 4}$$

Using "u" we could have let $u = x^2 - 4$. Then $du = 2xdx$ or $\frac{du}{2} = xdx$

$$\therefore \quad \int \sqrt{x^2 - 4}\,xdx = \int \sqrt{u}\,\frac{du}{2} = \frac{1}{2}\int u^{1/2}du$$

$$= \frac{1}{2}[\frac{2}{3}u^{3/2} + c]$$

$$= \frac{1}{3}u^{3/2} + c$$

$$= \frac{1}{3}(x^2 - 4)^{3/2} + c$$

Thus, surprising as it may seem at first glance it is easier to handle $\int x\sqrt{x^2 - 4}\,dx$ (since it can easily be transformed into an integral of the form $\int u^n du$) than to handle $\int \sqrt{x^2 - 4}\,dx$.

Keep in mind that given $\int \sqrt{x^2 - 4}\,dx$ we can still let $u = x^2 - 4$. Then $du = 2xdx$ or $dx = \frac{du}{2x}$. Moreover, $u = x^2 - 4$ implies that $x^2 = u + 4$ or $x = \sqrt{u + 4}$. Thus $dx = \frac{du}{2\sqrt{u + 4}}$. In any event, then:

$$\int \sqrt{x^2 - 4}\,dx = \int \frac{\sqrt{u}\,du}{2\sqrt{u + 4}}$$

[2.10.2 (L) cont'd]

We have thus transformed our original integral into one which is expressed in terms of u. There is nothing wrong with this approach. The trouble lies in the fact that the "new" integral $\int \frac{\sqrt{u}\,du}{2\sqrt{u+4}}$ is no easier to handle than the "old" one.

In other words, we can always make the substitution, but unless the new integral reduces to the special form $\int u^n du$, we are no better off than before. (In the language of functions, we are saying that $\int f^n(x)f'(x)dx = \frac{f^{n+1}(x)}{n+1} + c \quad (n \neq 1)$. Thus the "u" substitution helps us only if it converts the remainder of the integral to within a constant multiple of $\frac{du}{dx}$.)

As for part (b), a paraphrase of part (a) is that if $\frac{dy}{dx} = x\sqrt{x^2 - 4}$ then

$$y = \frac{1}{3}(x^2 - 4)^{3/2} + c \qquad (5)$$

What (5) tells us is that our curve must belong to the "family"

$$y = \frac{1}{3}(x^2 - 4)^{3/2} + c$$

The fact that our given curve passes through the point (2,3) merely means that equation (5) must be satisfied when $x = 2$ and $y = 3$.

Thus:

$$3 = \frac{1}{3}(4 - 4)^{3/2} + c \qquad \text{or } c = 3$$

Hence the equation of our curve is:

$$y = \frac{1}{3}(x^2 - 4)^{3/2} + 3$$

[2.10.2 (L) cont'd]

In essence, the choice of c merely raises or lowers the curve. The point is that since c is an arbitrary constant we have a "degree of freedom" in the sense that we can prescribe, within reason (for example, in this exercise x must be at least of magnitude 2), a point through which the curve must pass.

2.10.3

$$\frac{dy}{dx} = x^2 \sqrt{x^3 + 1}$$

$$dy = x^2 \sqrt{x^3 + 1}\, dx$$

$$y = \int x^2 \sqrt{x^3 + 1}\, dx + c$$

Let $u = x^3 + 1 \therefore du = 3x^2dx \therefore x^2dx = \frac{1}{3}du$

$$\therefore \int x^2 \sqrt{x^3 + 1}\, dx = \int (x^3 + 1)^{1/2}(x^2dx) = \int u^{1/2} \frac{du}{3} = \frac{1}{3}\int u^{1/2}du$$

$$= \frac{1}{3}[\tfrac{2}{3}u^{3/2} + c] = \frac{2}{9}u^{3/2} + c$$

$$\therefore y = \frac{2}{9}(x^3 + 1)^{3/2} + c \qquad (1)$$

Finally, since (1,4) is on the curve, equation (1) must be satisfied when x = 1 and y = 4.

$$\therefore 4 = \frac{2}{9}(1 + 1)^{3/2} + c$$

or $c = 4 - 2\,\frac{2^{3/2}}{9} = 4 - \frac{2^{5/2}}{9} = 4 - 4\frac{\sqrt{2}}{9} = 4\left(1 - \frac{\sqrt{2}}{9}\right)$

$\therefore$ Equation of curve is:

$$y = \frac{2}{9}(x^3 + 1)^{3/2} + 4\left(1 - \frac{\sqrt{2}}{9}\right)$$

2.10.4(L)

This problem is basically the same as the others we have done so far, except for the fact that x and y are not quite as separated as we would like them. That is, in the other problems we had things like $dy = f(x)dx$ so that we could obtain at once $y = \int f(x)dx + c$. Here if we try this we obtain $dy = \frac{x^2}{y}dx$ and we still have a mixture of x's and y's on the right hand side of our differential equation. The major point here is to see just how advantageous the differential notation is. We need only cross-multiply to obtain

$$ydy = x^2dx \qquad (1)$$

We then invoke the corollary to the mean value theorem which tells us that if two functions have identical derivatives (differentials) then they differ only by an additive constant.

Making this observation in (1) and noticing that $d(y^2/2) = ydy$ and $d(x^3/3) = x^2dx$, we obtain

$$y^2/2 = x^3/3 + c \qquad (2)$$

or:

$$3y^2 = 2x^3 + c \qquad (3)$$

(where we get (3) from (2) by multiplying both sides of (2) by 6 and recalling that since c is an arbitrary constant so also is 6c).

We then complete the exercise as before simply by letting $x = 2$ and $y = 1$ in (3) and solving for c. Thus, $3 = 2(2)^3 + c$ or $c = -13$. So our curve is given by

$$3y^2 = 2x^3 - 13$$

2.10.5

Separating the variables we obtain:

$$\frac{y\,dy}{\sqrt{y^2+4}} = dx$$

or

$$\int \frac{y\,dy}{\sqrt{y^2+4}} = x + c$$

Letting $u = y^2 + 4$, $du = 2y\,dy$ $\quad y\,dy = \frac{1}{2}du$

$$\therefore \int \frac{y\,dy}{\sqrt{y^2+4}} = \int \frac{\frac{1}{2}\,du}{\sqrt{u}} = \frac{1}{2}\int u^{-1/2}du = u^{1/2} + c_1 = \sqrt{y^2+4} + c_1$$

$$\therefore \sqrt{y^2+4}^{\,*} = x + c \qquad (1)$$

Since (1) must be satisfied when $x = y = 0$, we obtain $\sqrt{0^2+4} = 0 + c$ or $c = 2$. (Notice that the curve passing through the origin does not mean $c = 0$.)

$$\therefore \underline{\sqrt{y^2+4} = x + 2} \qquad (2)$$

If we wish we can square both sides of (2) to obtain

$$y^2 + 4 = (x+2)^2$$

or

$$y^2 = x^2 + 4x$$

---

*Technically we should write $\sqrt{y^2+4} + c_1 = x + c$, but the difference of two arbitrary constants is still an arbitrary constant. That is:

$\sqrt{y^2+4} + c_1 = x + c \rightarrow \sqrt{y^2+4} = x + c - c_1 \rightarrow \sqrt{y^2+4} = x + c_2$
where $c_2 = c - c_1$.

[2.10.5 cont'd]

This can also be written as:

$$x^2 + 4x - y^2 = 0$$

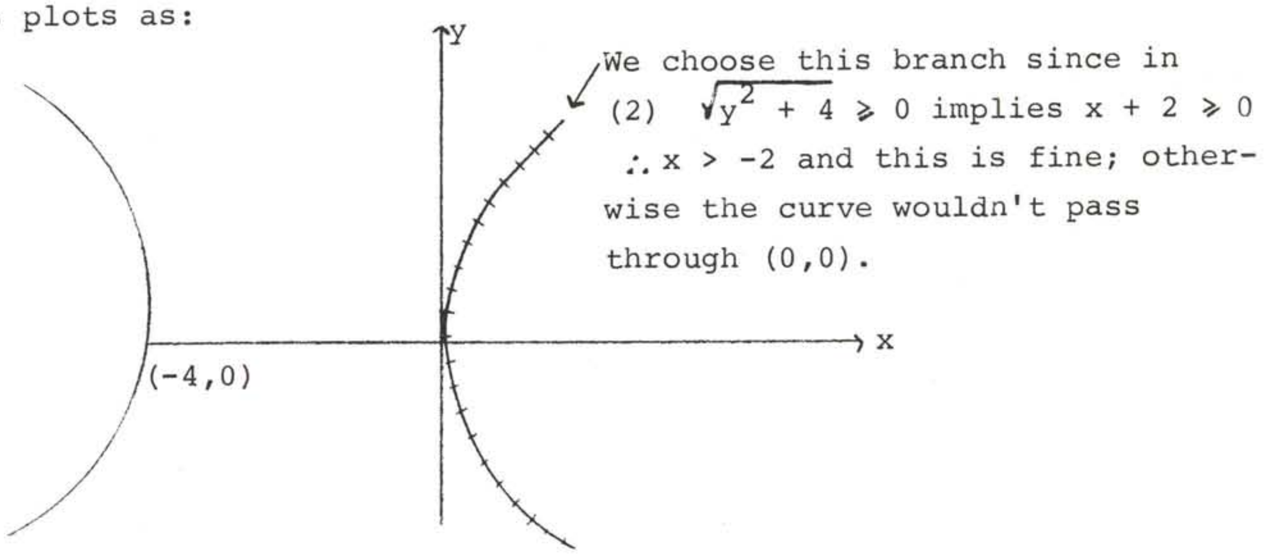
or:

$$x^2 + 4x + 4 - y^2 = 4$$

or:

$$(x + 2)^2 - y^2 = 4$$

This plots as:



We choose this branch since in (2) $\sqrt{y^2 + 4} \geqslant 0$ implies $x + 2 \geqslant 0$ $\therefore x > -2$ and this is fine; otherwise the curve wouldn't pass through (0,0).

2.10.6

This is the same type of problem as before, but in terms of a different physical example. Namely, in terms of x (distance) and t (time), $\frac{dx}{dt}$ is speed.

Thus,

$$v = t^2 \rightarrow \frac{dx}{dt} = t^2 \rightarrow x = \frac{1}{3}t^3 + c \qquad (1)$$

Finally, since when $t = 0$ x must equal 1, (1) becomes:

[2.10.6 cont'd]

$$1 = \frac{1}{3}(0)^3 + c$$

or $c = 1$

Putting this result into (1), we obtain

$$x = \frac{1}{3}t^3 + 1$$

as the desired solution.

(In still other words, this is also the solution to the problem: determine the curve c if $\frac{dy}{dx} = x^2$ and (0,1) is on c. The curve is then $y = \frac{1}{3}x^3 + 1$.)

2.10.7

Again, $a = \frac{dv}{dt}$; hence, $a = -t^2$ implies $v = -\frac{1}{3}t^3 + c_1$ where c_1 is any constant. But $v = \frac{dx}{dt}$, hence:

$$\frac{dx}{dt} = -\frac{1}{3}t^3 + c_1 \quad , \text{ or:}$$

$$x = -\frac{1}{12}t^4 + c_1t + c_2 \qquad \left\{\begin{array}{l}\text{where } c_1 \text{ and } c_2 \text{ are} \\ \text{arbitrary constants}\end{array}\right.$$

Unless additional information is given, we can make no refinements on what c_1 and c_2 must equal. In essence, we have two "degrees of freedom" which means that we can more or less prescribe two conditions which must be obeyed by the motion of the particle.

We shall say more about this in the next exercise.

2.10.8 (L)

In the previous exercise, we had shown

$$v = -\frac{1}{3}t^3 + c_1 \quad (1)$$

and

$$x = -\frac{1}{12}t^4 + c_1t + c_2 \quad (2)$$

We are now given the additional information that when $t = 0$, $v = 9$ and $x = 0$. This makes (1) become:

$$9 = -\frac{1}{3}(0)^3 + c_1$$

or $\underline{c_1 = 9}$, while (2) then becomes

$$0 = -\frac{1}{12}(0)^4 + c_1(0) + c_2$$

or $c_2 = 0$*.

Hence, under the given conditions, the answer to part (2) is $x = -\frac{1}{12}t^4 + 9t$.

As for part b., we observe that x is maximum when $\frac{dx}{dt} = 0$. Since $\frac{dx}{dt} = v$, we already know that $\frac{dx}{dt} = 0 \longleftrightarrow -\frac{1}{3}t^3 + 9 = 0$, but this occurs if and only if $t = 3$.

---

*In the previous exercise, we talked about two "degrees of freedom." Notice here that the choice of c_2 was not affected by the value of c_1 (since $c_1(0) = 0$ for any value of c_1). That is, at $t = 0$, we could prescribe, say, x and v independently of one another.

[2.10.8 (L) cont'd]

Thus, the particle attains its maximum positive displacement when t = 3, at which time

$$x = -\frac{1}{12}(3)^4 + 9(3) = 27 - \frac{81}{12} = 27 - \frac{27}{4} = \frac{3}{4}(27)$$

$$= \frac{81}{4} \text{ feet}$$

The important thing to observe is that in this unit we learned how to find x in terms of t once the acceleration was given as a function of t. Yet once this is done we do part b. just as would have solved any max-min problem.

Finally, fn part c., we need only observe that x = 0 implies

$$-\frac{1}{12}t^4 + 9t = 0 \quad .$$

Hence, t = 0, or

$$-\frac{1}{12}t^3 + 9 = 0$$

$\therefore\ t^3 = 108$ or $t = \sqrt[3]{108} = \sqrt[3]{27 \times 4} = 3\sqrt[3]{4}$

Thus, the particle starts at the origin (t = 0) and returns there when $t = 3\sqrt[3]{4}$ (or approximately 4 seconds).

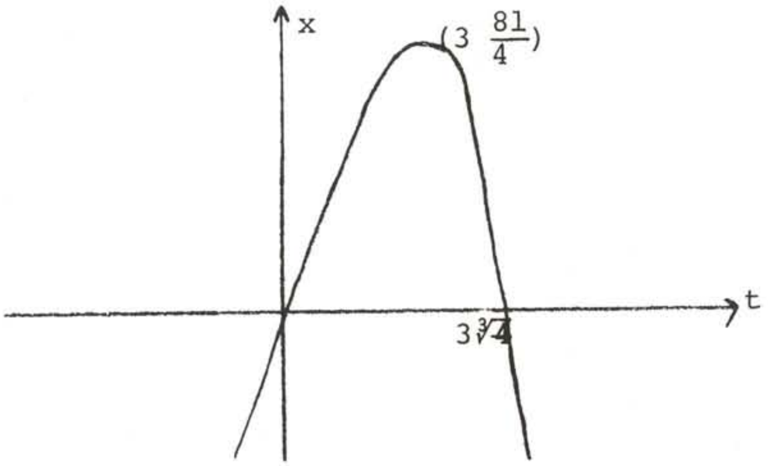
Again, the technique hinges only on setting x(t) = 0. All we learned that was new in this section was how to determine x(t).

d. $x = -\frac{1}{12}t^4 + 9\ t = 0 \longleftrightarrow t = 0$ or $\sqrt[3]{108} = \sqrt[3]{27 \times 4} = 3\sqrt[3]{4}$

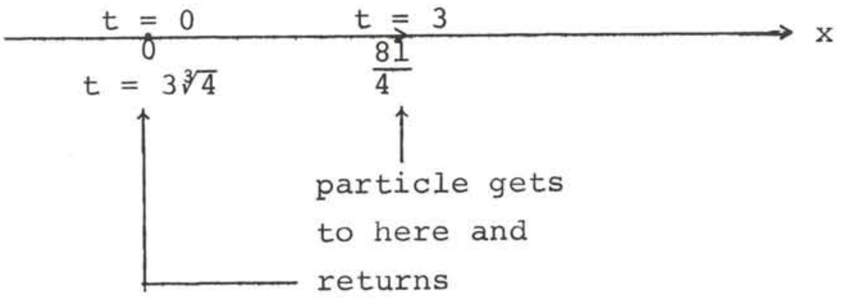
$x' = -\frac{1}{3}t^3 + 9 = 0 \longleftrightarrow t = 3$

$x'' = -\ t^2$

[2.10.8 (L) cont'd]



However, do not confuse the graph of x versus t with the path followed by the particle. Recall that we are told that the particle moves along the x-axis.



2.10.9 (L)

We know that $a = -k^2\sqrt{v}$ at, say, t = 0 (that is, we elect to measure time from the instant the particle starts to decelerate). We write $a = -k^2\sqrt{v}$ to indicate that we have deceleration. Had we written $a = -h\sqrt{v}$, k could have been negative, hence -k would be positive. The point is that if k is real, k^2 can never be negative.

Thus: $\frac{dv}{dt} = -k^2v^{1/2}$ (0)

[2.10.9 (L) cont'd]

Separating variables, we obtain:

$$v^{-1/2}dv = -k^2dt \qquad (1)$$

or

$$2v^{1/2} = -k^2t + c \qquad (2)$$

(Again, notice that in going from (1) to (2) we proceed as in each of the preceding exercises. The only new thing in this exercise is that we had a new physical model which led to (1).)

Now, if we read the exercise and translate the prose into appropriate mathematics, we have that when $t = 0$, $v = 36$. From (2), this means:

$$2(36)^{1/2} = -k^2 0 + c \qquad \text{or} \qquad c = 12$$

$$\therefore\ 2v^{1/2} = 12 - k^2t \qquad (3)$$

Then the fact that the particle comes to rest in 8 seconds* means that when $t = 8$, $v = 0$. This causes (3) to become:

$$2(0)^{1/2} = 12 - 8k^2$$

or $k^2 = \frac{12}{8} = \frac{3}{2}$

---

*With this in mind, equation (0) should really read

$$\frac{dv}{dt} = -k^2v^{1/2}\ , \quad \underline{\underline{0 \leqslant t \leqslant 8}}$$

In other words, $v = 0$ if $t > 8$, and $v = 36$ if $t < 0$

[2.10.9 (L) cont'd]

Hence (3) becomes:

$$2v^{1/2} = 12 - \frac{3}{2}t \qquad (4)$$

This in turn may be written as:

$$v = \left(6 - \frac{3}{4}t\right)^2 \quad , \quad v \geqslant 0 \qquad (4)$$

(recalling as usual that $v^{1/2} = 6 - \frac{3}{4}t$ is only the positive branch of $v = \left(6 - \frac{3}{4}t\right)^2$)

As a check, (4) yields that $v = 36$ when $t = 0$, and that $v = 0$ when $t = 8$. In any event, (4) allows us to express v for any t such that $0 \leqslant t < 8$. In particular, when $t = 4$, (4) yields:

$$v = \left(6 - \frac{3}{4}(4)\right)^2 = 9 \text{ ft/sec}$$

(Once (4) is developed, we see that speed is proportional to the square of the time. Thus, when the time is half way to the stopping time, the speed is only $\frac{1}{4}$ of the original speed.)

Finally, we may rewrite (4) as

$$\frac{dx}{dt} = \left(6 - \frac{3}{4}t\right)^2$$

whereupon

$$x = -\frac{4}{9}\left(6 - \frac{3}{4}t\right)^3 + c \qquad (5)$$

(Recall that in obtaining (5) form (4) we proceed just as in the previous exercises. Namely, $\frac{d}{dt}\left[\left(6 - \frac{3t}{4}\right)^3\right] = 3\left(6 - \frac{3t}{4}\right)^2\left(-\frac{3}{4}\right) = -\frac{9}{4}\left(6 - \frac{3t}{4}\right)^2$. Hence $\frac{d}{dt}\left[-\frac{4}{9}\left(6 - \frac{3t}{4}\right)^3\right] = \left(6 - \frac{3t}{4}\right)^2$.)

[2.10.9 (L) cont'd]

We recall that our problem starts at $t = 0$ and at this time $x = 0$. (In this problem x is not the displacement of the particle but more precisely it is the displacement from the time of the deceleration.)

Thus (5) must be obeyed when $t = x = 0$ and this says:

$$0 = -\frac{4}{9}(6)^3 + c \quad \text{or} \quad c = \frac{4(6)^3}{9} = 96$$

∴ t seconds from the start of the deceleration, the particle has moved $x = -\frac{4}{9}(6 - \frac{3}{4}t)^3 + 96$.

When $t = 4$, we obtain $x = -\frac{4}{9}(6 - 3)^3 + 96 = -\frac{4}{9}(27) + 96 = 84$ feet and it stops when $t = 8$, at which time $x = 96$ feet.

Thus the particle travels 96 feet before coming to rest in 8 seconds, and during the first 4 seconds it travels 84 feet.

UNIT 11: The Definite Indefinite Integral

2.11.1

a. $\int_0^3 (t^2-1)dt = G(3)-G(0)$ where $G'(t) = t^2-1$

$\therefore$ we may pick $G(t) = \frac{1}{3}t^3-t$

$\therefore G(3) = \frac{1}{3}(3)^3-3 = 6$

$G(0) = 0$

$\therefore \int_0^3 (t^2-1)dt = 6-0 = 6$

b. $\int_0^2 (x+1)^2dx = G(2)-G(0)$ where $G'(x) = (x+1)^2$

$\therefore$ we may pick $G(x) = \frac{1}{3}(x+1)^3$

$\therefore G(2) = \frac{1}{3}(2+1)^3 = \frac{1}{3}(3)^3 = 9$

$G(0) = \frac{1}{3}(0+1)^3 = \frac{1}{3}$ (Note: G(0) need not equal 0. In terms of curve plotting G(0) is the y-intercept of $y = G(x)$, and $G(0) = 0$ merely means that the curve passes through the origin.)

[Solution 2.11.1 cont'd]

$$\therefore \int_0^2 (x+1)^2 dx = 9 - \frac{1}{3} = \frac{26}{3}$$

c. $$\int_1^2 X \sqrt{x^2+3} = G(2)-G(1) \text{ where } G'(x) = x\sqrt{x^2+3}$$

$$= x(x^2+3)^{1/2}$$

$$\therefore \text{we may let } G(x) = \frac{1}{3}(x^2+3)^{3/2} \quad \left(\text{check: } \left[\frac{1}{3}(x^2+3)^{3/2}\right]'\right.$$

$$\left. = \frac{1}{3}\left(\frac{3}{2}\right)(x^2+3)^{1/2}(2x) = x(x^2+3)^{1/2}\right)$$

$$G(2) = \frac{1}{3}(4+3)^{3/2} = \frac{1}{3}\, 7^{3/2} = \frac{1}{3}\, 7\sqrt{7} = \frac{7}{3}\sqrt{7}$$

$$G(1) = \frac{1}{3}(1+3)^{3/2} = \frac{1}{3}(4)^{3/2} = \frac{8}{3}$$

$$\therefore \int_1^2 x\sqrt{x^2+3} = \frac{7\sqrt{7}}{3} - \frac{8}{3} = \frac{1}{3}(7\sqrt{7}-8)$$

d. $$\int_{-1}^1 (4x^3+3x^2)dx = G(1)-G(-1): \quad G'(x) = 4x^3+3x^2$$

$$\text{Let } G(x) = x^4+x^3$$

$$G(1) = 1^4+1^3 = 2$$

$$G(-1) = (-1)^4+(-1)^3 = 1-1 = 0$$

[Solution 2.11.1 cont'd]

$$\therefore \int_{-1}^{1} (4x^3+3x^2)\,dx = 2-0 = 2$$

2.11.2(L)

a. $\Delta x\Big|_{t=0}^{t=2} = G(2)-G(0)$ where $G'(t) = t-t^2$

Hence we may take $G(t) = \frac{1}{2}t^2 - \frac{1}{3}t^3$

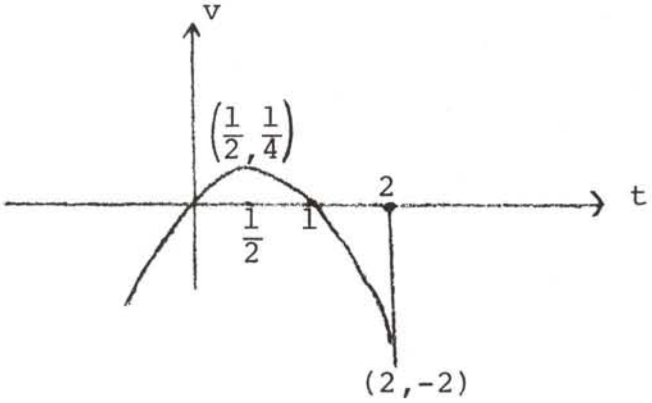
$$\therefore G(2) = \frac{1}{2}(2)^2 - \frac{1}{3}(2)^3 = 2 - \frac{8}{3} = -\frac{2}{3}$$

$$G(0) = 0$$

$$\Delta x\Big|_{t=0}^{t=2} = -\frac{2}{3}$$

Hence the particle, at t=2, is $\frac{2}{3}$ feet to the left of its position at t = 0.

b. If we graph v versus t, we find



[Solution 2.11.2(L) cont'd]

Hence, $v \geqslant 0$ for $0 \leqslant t \leqslant 1$

$v \leqslant 0$ for $1 \leqslant t \leqslant 2$

During $0 \leqslant t \leqslant 1$, the displacement is given by

$$\Delta x\Big|_{t=0}^{t=1} = \frac{1}{2}t^2 - \frac{1}{3}t^3\Big|_0^1 = \frac{1}{6}\text{ feet (from left to right)}$$

Since the velocity never changes sign during this time interval, Δx is also the total distance travelled during $0 \leqslant t \leqslant 1$. When $1 \leqslant t \leqslant 2$,

$$\Delta x\Big|_{t=1}^{t=2} = \frac{1}{2}t^2 - \frac{1}{3}t^3\Big|_1^2 = \left(\frac{-2}{3}\right) - \left(\frac{1}{6}\right) = \frac{-5}{6}\text{ ft. (from right to left)}$$

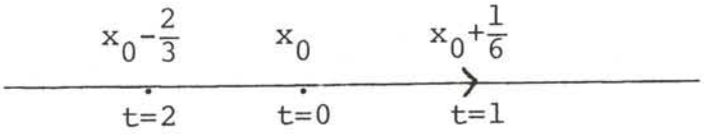
Again since v doesn't change sign on this time interval, the total distance is also $\left|\frac{-5}{6}\right|$ ft, where the absolute value sign indicates that we take distance to be positive (in fact an alternative way of working with negative speed as in this example is to write that when $1 \leqslant t \leqslant 2$, the total distance is given by

$$\int_1^2 |t - t^2|\,dt = \left|\frac{t^2}{2} - \frac{t^3}{3}\Big|_1^2\right| = \frac{5}{6}\ ).$$

[Solution 2.11.2(L) cont'd]

In any event the total distance travelled is $\frac{1}{6}$ ft (during the 1st second) + $\frac{5}{6}$ ft (during the 2nd second) = 1 foot.

Pictorially;



That is, the particle starts at x_0 moves steadily to the right until it reaches the point $x_0+\frac{1}{6}$ at t=1. It then reverses direction and moves steadily to the left, reaching the position $x_0-\frac{2}{3}$ when t=2.

2.11.3

$v = t^2 - 3t + 2 \qquad 0 \leqslant t \leqslant 4$

a. $\therefore$ displacement $= \Delta x\Big|_{t=0}^{t=4} = G(4)-G(0)$ where $G'(t)=t^2-3t+2$

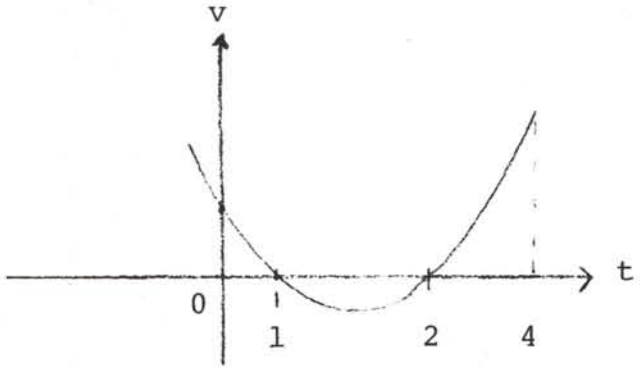
$$= \frac{1}{3}t^3 - \frac{3}{2}t^2 + 2t\Big|_0^4$$

$$= \frac{64}{3} - 24 + 8 = \frac{64}{3} - 16$$

$$= \frac{16}{3} = 5\frac{1}{3} \text{ feet}$$

[Solution 2.11.3 cont'd]

b.



$v \geqslant 0$ when $0 \leqslant t \leqslant 1$

$v \leqslant 0$ when $1 \leqslant t \leqslant 2$

$v \geqslant 0$ when $2 \leqslant t \leqslant 4$

$$\therefore \text{ total distance } = \left|\int_0^1 (t^2-3t+2)\,dt\right| + \left|\int_1^2 (t^2-3t+2)\,dt\right|$$

$$+ \left|\int_2^4 (t^2-3t+2)\,dt\right| = \left|\left.\tfrac{1}{3}t^3-\tfrac{3}{2}t^2+2t\right|_0^1\right| + \left|\left.\tfrac{1}{3}t^3-\tfrac{3}{2}t^2+2t\right|_1^2\right|$$

$$+ \left|\left.\tfrac{1}{3}t^3-\tfrac{3}{2}t^2+2t\right|_2^4\right| = \left|\tfrac{1}{3}-\tfrac{3}{2}+2\right| + \left|\left(\tfrac{8}{3}-6+4\right)-\left(\tfrac{1}{3}-\tfrac{3}{2}+2\right)\right|$$

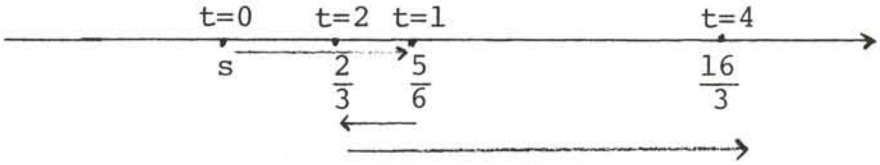
[Solution 2.11.3 cont'd]

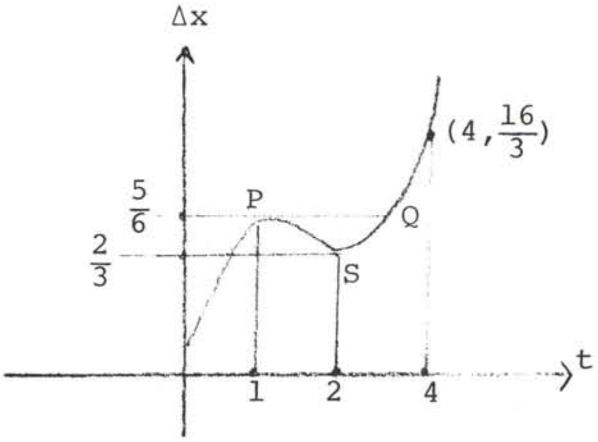
$$+ \left|\frac{64}{3}-24+8)-(\frac{8}{3}-6+4)\right| = \left|\frac{5}{6}\right|+\left|\frac{2}{3}-\frac{5}{6}\right|+\left|\frac{16}{3}-\frac{2}{3}\right|$$

$$= \frac{5}{6} + \frac{1}{6} + \frac{14}{3}$$

$$= \frac{17}{3} \text{ feet}$$

More pictorially,





In going from P to Q the net $\Delta x = 0$ but the vertical distances from P to S and S to Q are each $\frac{1}{6}$

$\frac{1}{6}+\frac{1}{6} = \frac{1}{3}$ = difference between total distance travelled and the displacement

2.11.4(L)

$$\int_0^1 (x+1)\,dx = \tfrac{1}{2}x^2 + x\Big|_0^1 = \left(\tfrac{1}{2}+1\right) - 0 = \tfrac{3}{2}$$

$$\therefore 2\int_0^1 (x+1)\,dx = 2\left(\tfrac{3}{2}\right) = 3$$

$$\int_{-1}^1 (x+1)\,dx = \tfrac{1}{2}x^2 + x\Big|_{-1}^1 = \left(\tfrac{1}{2}+1\right) - \left[\tfrac{1}{2}(-1)^2+(-1)\right]$$

$$= \tfrac{3}{2} - \left(+\tfrac{1}{2} - 1\right) = \tfrac{3}{2} - \left(-\tfrac{1}{2}\right)$$

$$= 2$$

Hence

$$\int_{-1}^1 (x+1)\,dx \neq 2\int_0^1 (x+1)\,dx$$

In particular, then, it need not be true that

$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx \quad .$$

Of course, it might happen that $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$.

[Solution 2.11.4(L) cont'd)

For example, consider

$$\int_0^1 x^2dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}$$

$$\therefore 2\int_0^1 x^2dx = \frac{2}{3}$$

On the other hand

$$\int_{-1}^{1} x^2dx = \frac{1}{3}x^3\Big|_{-1}^{1} = \left[\frac{1}{3}(1)^3\right] - \left[\frac{1}{3}(-1)^3\right]$$

$$= \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}$$

$$\therefore \int_{-1}^{1} x^2dx = 2\int_0^1 x^2dx \quad \left(= \frac{2}{3}\right)$$

While we will not discuss this idea in too much detail we would like to make a few remarks about even and odd functions. Observe that, in any event,

[Solution 2.11.4(L) cont'd]

$$\int_{-a}^{a} f(x)dx = G(a)-G(-a)^{*} \quad \text{where } G'(x) = f(x)$$

$$\text{Let } h(x) = G(x)-G(-x). \quad \left(\text{Hence, } \int_{-a}^{a} f(x)dx = h(a)\right) \qquad (1)$$

Now,

$$h'(x) = G'(x) - [G(-x)]'$$

By the chain rule

$$\frac{dG(-x)}{dx} = \frac{dG(-x)}{d(-x)} \quad \frac{d(-x)}{dx} = - \frac{dG(-x)}{d(-x)}$$

But $G'=f$ means $\frac{dG(u)}{du} = f(u)$

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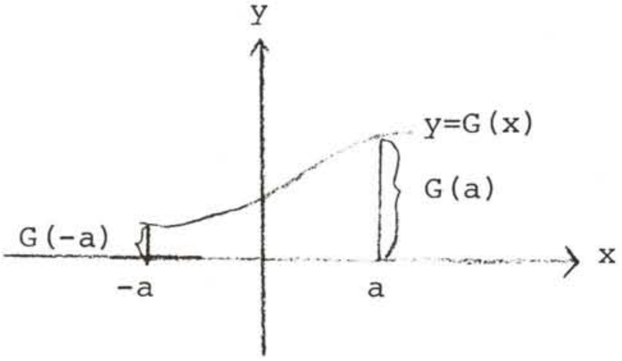
[*] Notice that there need be no special relationship between G(a) and G(-a). That is:



[Solution 2.11.4(L) cont'd]

$$\therefore G'(x) = f(x)$$

$$\frac{dG(-x)}{d(-x)} = f(-x)$$

Hence, in any case, $h'(x) = f(x) + f(-x)$ (2)

Equation (2) gives us the key clue. Namely if $f(-x) = -f(x)$ (i.e., f is an odd function) then (2) says:

$$\begin{aligned} h'(x) &= f(x) + f(-x) \\ &= f(x)-f(x) \\ &= 0 \end{aligned}$$

$$\therefore h(x) = \text{constant}$$

But $h(0) = G(0)-G(-0) = G(0)-G(0) = 0$

$\therefore$ since $h(0) = 0$ and $h(x)$ is a constant, $\underline{h(x)= 0}$

$$\therefore G(a)-G(-a) = 0$$

In other words if f is an odd function then

$$\int_{-a}^{a} f(x)dx = 0$$

[Solution 2.11.4(L) cont'd)

Example

$$f(x) = x \to f \text{ is odd; that is } f(x) = -f(-x)$$

$$\int_{-a}^{a} x dx = \frac{1}{2}x^2 \Big|_{-a}^{a} = \frac{1}{2}a^2 - \left(\frac{1}{2}(-a)^2\right) = \frac{1}{2}a^2 - \frac{1}{2}a^2 = 0$$

The other "extreme" case as far as (2) is concerned occurs when f is even (i.e., f(x) = f(-x)). Then (2) becomes:

$$\begin{aligned} h'(x) &= f(x) + f(-x) \\ &= f(x) + f(x) \\ &= 2f(x) \end{aligned} \qquad (3)$$

But $G'(x) = f(x) \to$

$$[2G(x)]' = 2f(x) \qquad (4)$$

Comparing (3) and (4) we see that h(x) and 2G(x) have the same derivative

Hence,

$$h(x) = 2G(x) + c$$

[Solution 2.11.4(L) cont'd]

But

$$h(0) = G(0)-G(-0) = 0 \rightarrow \ .$$

$$0 = h(0) = 2G(0) + c \rightarrow$$

$$c = -2G(0) \rightarrow$$

$$h(x) = 2G(x)-2G(0) \rightarrow$$

$$h(a) = 2G(a)-2G(0) \rightarrow$$

$$\int_{-a}^{a} f(x)dx = h(a) = 2G(a)-2G(0) \qquad (5)$$

On the other hand,

$$\int_{0}^{a} f(x)dx = G(a)-G(0)$$

$$\therefore 2\int_{0}^{a} f(x)dx = 2G(a)-2G(0) \qquad (6)$$

Comparing (5) and (6) we see that if f is even,

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

[Solution 2.11.4(L) cont'd]

In summary, there is no fixed way of expressing $\int_{-a}^{a} f(x)dx$ in terms of $\int_{0}^{a} f(x)dx$.

However,

(1) if f(x) is an even function $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

(2) if f(x) is an odd function $\int_{-a}^{a} f(x)dx = 0$

(To see this result more physically, think in terms of $v = f(t)$ so that $\Delta x\Big|_{t=-a}^{t=a} G(a)-G(-a)$ where $G' = f$. If the speed is the same at t as at -t for all values of t, then it seems reasonable that the particle moves the same distance between -t and 0 as it does between 0 and t. Moreover if the speeds are the same at t and -t but in opposite directions, the displacement between t = -a and t = a should be zero since the particle moves equal distances but in opposite directions.)

SOLUTIONS: Calculus of a Single Variable - Block II: Differentiation

QUIZ

1. If $x^3 + 3xy + y^3 = 5$, then $3x^2 + 3x\frac{dy}{dx} + 3y + 3y^2\frac{dy}{dx} = 0$. Hence, $\frac{dy}{dx} = \frac{-(3x^2 + 3y)}{(3x + 3y^2)}$ and $\frac{dy}{dx}$ at (1,1) is -1.

Thus, L_1 passes through (1,1) with slope -1, so the equation of L_1 is:

$$\frac{y - 1}{x - 1} = -1 \qquad \text{or} \qquad y = -x + 2 \qquad (1)$$

If $y = \frac{x + 1}{x - 1}$ then $\frac{dy}{dx} = \frac{(x - 1)(1) - (x + 1)(1)}{(x - 1)^2} = \frac{-2}{(x - 1)^2}$ Hence, at (2,3), $\frac{dy}{dx} = -2$. Thus L_2 passes through (2,3) with slope -2. This means that the equation of L_2 is:

$$\frac{y - 3}{x - 2} = -2 \qquad \text{or} \qquad y = -2x + 7 \qquad (2)$$

The desired point is the solution of

$$\left.\begin{aligned} y &= -x + 2 \\ y &= -2x + 7 \end{aligned}\right\} \qquad (3)$$

This occurs if $-x + 2 = -2x + 7$, or $x = 5$, but when $x = 5$, (3) implies that $y = -3$.

Therefore, the point at which L_1 and L_2 intersect is (5,-3).

2. $y = t^5 + 2t + 1 \rightarrow \dfrac{dy}{dt} = 5t^4 + 2$

$t = x^6 + x^5 + 3 \rightarrow \dfrac{dt}{dx} = 6x^5 + 5x^4$

Hence, by the chain rule,

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = (5t^4 + 2)(6x^5 + 5x^4)$$

Then, $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}\left[(5t^4 + 2)(6x^5 + 5x^4)\right]$

$$= (5t^4 + 2)\frac{d}{dx}(6x^5 + 5x^4) + (6x^5 + 5x^4)\frac{d(5t^4 + 2)}{dx}$$

$$= (5t^4 + 2)\frac{d}{dx}(6x^5 + 5x^4) + (6x^5 + 5x^4)\frac{d(5t^4 + 2)}{dt}\frac{dt}{dx}$$

$$= (5t^4 + 2)(30x^4 + 20x^3) + (6x^5 + 5x^4)(20t^3)(6x^5 + 5x^4) \qquad (1)$$

Now, when x = 1, t = 5 (since $t = x^6 + x^5 + 3$). Putting this into (1) we obtain:

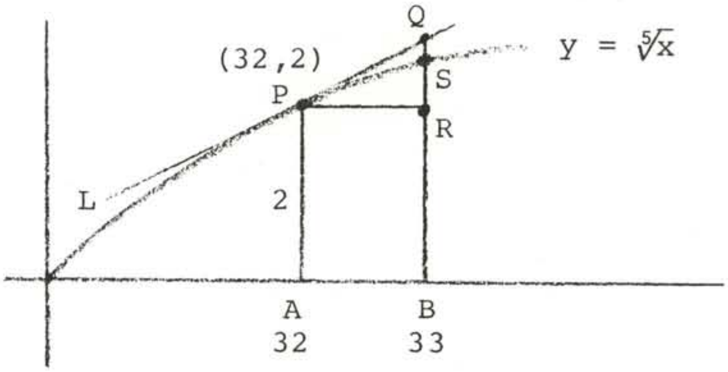
$$\left(\frac{d^2y}{dx^2}\right)_{x=1} = [5(5)^4 + 2](50) + (11)[20(5)^3](11)$$

$$= (3,127)(50) + 121(2500)$$

$$= 156,350 + 302,500$$

$$= 458,850$$

3. We let $y = f(x) = \sqrt[5]{x} = x^{\frac{1}{5}}$. Then when $x = 32$, $y = 2$



(Figure 1)

$$f'(x) = \frac{1}{5}x^{-\frac{4}{5}} \qquad \therefore f'(x)\Big|_{x=32} = \frac{1}{5}(32)^{-\frac{4}{5}} = \frac{1}{5}(2)^{-4} = \frac{1}{80}$$

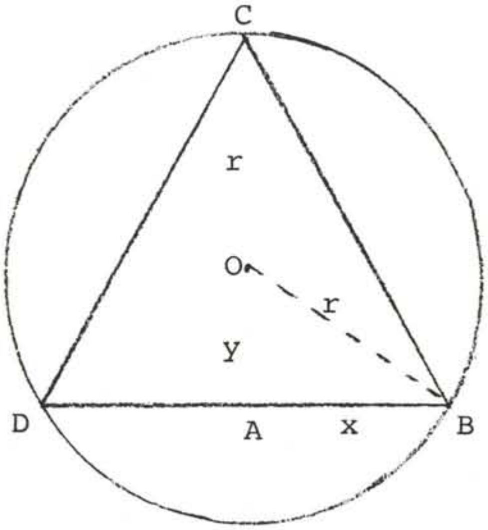
(In Figure 1, this means that the slope of L is $\frac{1}{80}$.)

Thus $\Delta y_{\tan}$ ($\overline{QR}$ in Figure 1) $= \left(\frac{dy}{dx}\right)_{x=32} \Delta x = \frac{1}{80}(1) = \frac{1}{80}$.

$$\therefore \overline{QB} = 2\frac{1}{80} = 2.0125 \approx \overline{BS} = \sqrt[5]{33}$$

Thus our approximation for $\sqrt[5]{33}$ is 2.0125 and this approximation is greater than $\sqrt[5]{33}$ since L lies above the curve in Figure 1. (The curve spills water since $\frac{d^2y}{dx^2} = \frac{-4}{25} x^{-\frac{9}{5}} < 0$ for $x > 0$.)

4. (a)



$$A = \frac{1}{2}bh = x(r + y)$$

$$\therefore \frac{dA}{dx} = x\frac{d(r + y)}{dx} + \frac{dx}{dx}(r + y) = x\frac{dy}{dx} + r + y \qquad (1)$$

The constraint is:

$$x^2 + y^2 = r^2$$

$$\therefore 2x + 2y\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{x}{y} \qquad (2)$$

Substituting the value for $\frac{dy}{dx}$ from (2) into (1), we obtain:

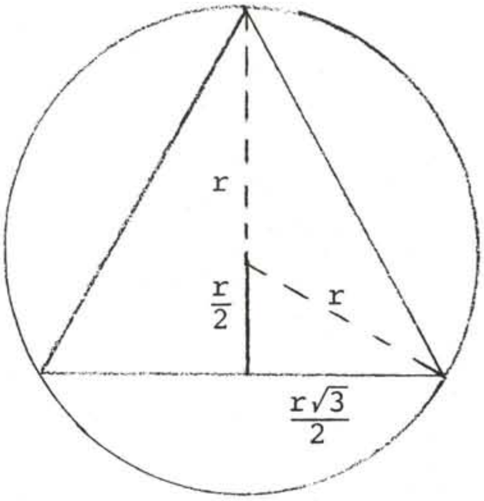
$$\begin{aligned}\frac{dA}{dx} &= x(\frac{-x}{y}) + r + y \\ &= \frac{-x^2 + ry + y^2}{y} = \frac{-(r^2 - y^2) + ry + y^2}{y} \\ &= \frac{2y^2 + ry - r^2}{y} \\ &= 0 \leftrightarrow 2y^2 + ry - r^2 = 0 \leftrightarrow (2y - r)(y + r) = 0\end{aligned}$$

[4. cont'd]

Since $y > 0$ in order for the problem to make sense, we have that $\frac{dA}{dx} = 0 \leftrightarrow y = \frac{r}{2}$ and this is precisely the condition that ΔOAB is a 30° - 60° - 90° triangle, and this in turn means that ΔBCD is an equilateral triangle.

Thus, if the area is to be maximum, the isosceles triangle must, in fact, be equilateral.

(b)



Volume of cone is:

$$\frac{1}{3}\pi R^2 h =$$

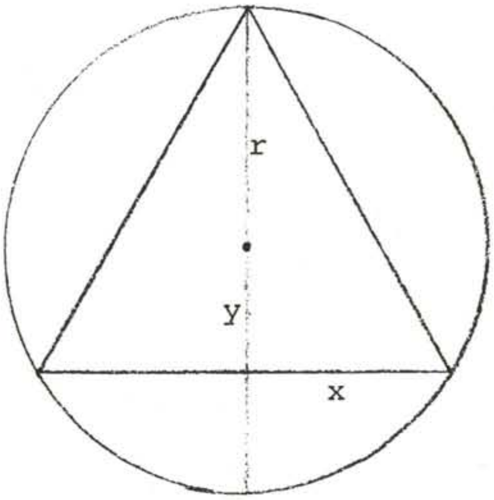
$$= \frac{1}{3}\pi \left(\frac{r\sqrt{3}}{2}\right)^2 \left(r + \frac{r}{2}\right)$$

$$= \frac{1}{3}\pi \left(\frac{3r^2}{4}\right) \left(\frac{3}{2}r\right)$$

$$= \frac{3}{8}\pi r^3$$

[4. cont'd]

(c)



$$V = \frac{1}{3}\pi x^2(y + r) \quad (1)$$

The constraint is:

$$x^2 + y^2 = r^2 \quad \text{or} \quad x^2 = r^2 - y^2 \quad (2)$$

Replacing x^2 in (1) by its value in (2) we obtain:

$$V = \frac{1}{3}\pi(r^2 - y^2)(y + r) = \frac{1}{3}\pi[r^2y + r^3 - y^3 - y^2r]$$

$$\therefore \frac{dV}{dy} = \frac{1}{3}\pi[r^2 - 3y^2 - 2ry] = \frac{1}{3}\pi(r - 3y)(r + y)$$

$$\therefore \frac{dV}{dy} = 0 \leftrightarrow r - 3y = 0 \text{ or } y = \frac{r}{3}$$

[4. cont'd]

If $y = \frac{r}{3}$, we have $x^2 + (\frac{r}{3})^2 = r^2$ or $x^2 = \frac{8r^2}{9}$. Putting these values of x and y into (1) we obtain:

$$V = \frac{1}{3}\pi(\frac{8r^2}{9})\ (\frac{r}{3} + r) = \frac{1}{3}\pi(\frac{8r^2}{9})\ (\frac{4r}{3}) = \frac{32}{81}\pi r^3$$

In summary, the inscribed cone of maximum volume is $\frac{32\pi r^3}{81}$ and its dimensions are $R = \sqrt{\frac{8}{9}}r = \frac{2\sqrt{2}r}{3}$ and $h = (r + \frac{r}{3}) = \frac{4r}{3}$.

(In passing, we note that $\frac{32}{81} > \frac{3}{8}$; hence a comparison with the answer to (b) shows that the greatest area need not generate the greatest volume.)

5\. (a)

$$y = x^4 + x^3 = x^3(x + 1) \qquad (1)$$

$$y' = 4x^3 + 3x^2 = x^2(4x + 3) \qquad (2)$$

$$y'' = 12x^2 + 6x = 6x(2x + 1) \qquad (3)$$

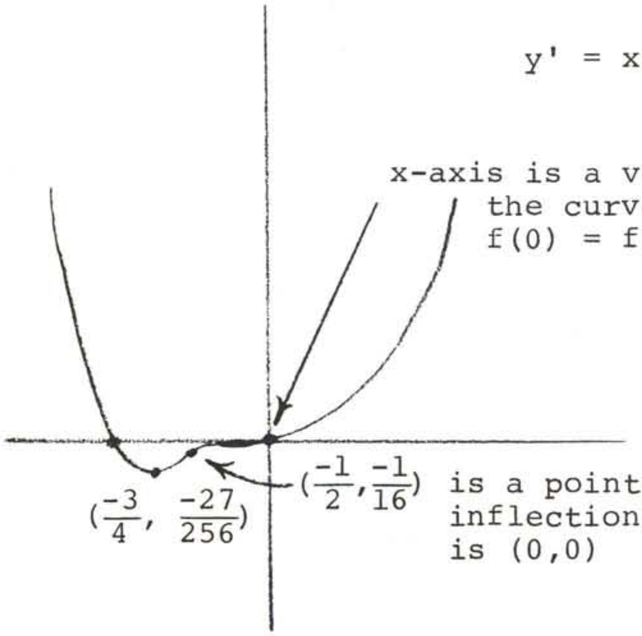
From (1) we see that the curve crosses the x-axis when $x = 0$ or $x = -1$.

From (2) we see that its slope is 0 when $x = 0$ or $x = \frac{-3}{4}$ $\left(\text{and since } y = x^4 + x^3\text{, the points are } (0,0) \text{ and } (\frac{-3}{4}, \frac{-27}{256})\right)$.

From (3) the candidates for points of inflection are (0,0) and $(\frac{-1}{2}, \frac{-1}{16})$.

[5. cont'd]

$$y' = x^2(4x + 3)\begin{cases} >0 \text{ if } x > \frac{-3}{4} \\ <0 \text{ if } x < \frac{-3}{4} \end{cases} \text{since } x^2 \geqslant 0$$



x-axis is a very good approximation for the curve near the origin since $f(0) = f'(0) = f''(0) = 0$.

$(\frac{-1}{2}, \frac{-1}{16})$ is a point of inflection as is (0,0)

$y'' = 6x(2x+1) < 0$ for $\frac{-1}{2} < x < 0$. So curve spills water only on the interval $(\frac{-1}{2}, 0)$

----	--	++++	x
---	++++	+++	2x + 1
	$\frac{-1}{2}$	0	

(b) $\frac{dy}{dt} = (4x^3 + 3x^2)\ \frac{dx}{dt}$

$= 5(4x^3 + 3x^2)$

$\therefore$ At (1,2) $\frac{dy}{dt} = 5(4 + 3) = 35$ ft/sec.

6. $x^5 + 3x^2 = 2$

is equivalent to

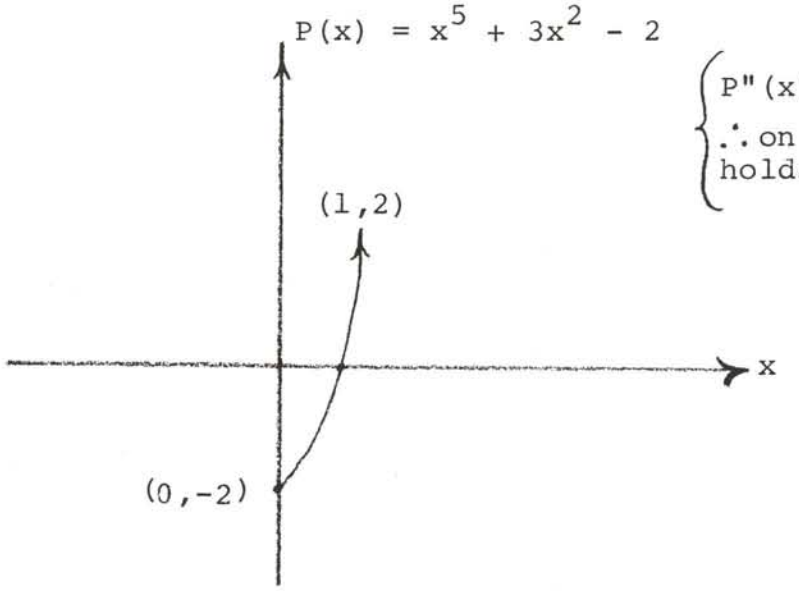
$$x^5 + 3x^2 - 2 = 0$$

Let $P(x) = x^5 + 3x^2 - 2$. Now $P(0) = -2$, $P(1) = 1 + 3 - 2 = +2$.

Since $P(x)$ is continuous and it changes sign between $x = 0$ and $x = 1$, we can be sure that $P(x) = 0$ has at least one root between 0 and 1.

We next observe that $P'(x) = 5x^4 + 6x$ and this is never negative when $x > 0$. Hence for $x > 0$, $P(x)$ is 1-1 (since the curve $y = P(x)$ is always rising).

Pictorially,



$$\left\{\begin{array}{l} P''(x) = 20x^3 + 6 > 0 \text{ for } x > 0 \\ \therefore \text{ on } [0,1] \text{ the curve always} \\ \text{holds water} \end{array}\right.$$

(We could, for example, use Newton's Method to locate this root more exactly if we so desired.)

7. (a) $$\frac{dy}{dx} = y^3$$

$$\therefore y^{-3}dy = dx$$

$$\frac{y^{-2}}{-2} = x + c$$

or $$y^{-2} = -2x + c$$

or $$\frac{1}{y^2} = -2x + c \qquad (1)$$

When $x = 3$ and $y = 4$, (1) yields

$$\frac{1}{16} = -6 + c$$

$$\therefore c = 6 + \frac{1}{16} = \frac{97}{16}$$

$$\therefore \frac{1}{y^2} = -2x + \frac{97}{16} = \frac{97 - 32x}{16}$$

$$\therefore y^2 = \frac{16}{97 - 32x} \qquad (2)$$

(To conform with the notion of single-valued functions, we have that (2) implies that

$$y = \frac{4}{\sqrt{97 - 32x}} \quad \text{or} \quad y = \frac{4}{-\sqrt{97 - 32x}}$$

In this event, since we know that $y = 4$ when $x = 3$, we must pick $y = \frac{4}{\sqrt{97 - 32x}}$ if we require a single-valued branch.)

[7. cont'd]

(b) $$\frac{dy}{dx} = x\sqrt[3]{x^2 + 18}$$

$$\therefore y = \int x(x^2 + 18)^{\frac{1}{3}}dx$$

Letting $u = x^2 + 18$, $du = 2xdx$ or $xdx = \frac{1}{2}du$

$$\therefore \int x(x^2 + 18)^{\frac{1}{3}}dx = \int \frac{1}{2}u^{\frac{1}{3}}du = \frac{1}{2}\int u^{\frac{1}{3}}du = \frac{1}{2}\left[\frac{3}{4}u^{\frac{4}{3}} + c\right]$$

$$= \frac{3}{8}(x^2 + 18)^{\frac{4}{3}} + c$$

$$\therefore y = \frac{3}{8}(x^2 + 18)^{\frac{4}{3}} + c \qquad (3)$$

Letting x = 3 and y = 4 in (3), we obtain

$$4 = \frac{3}{8}(9 + 18)^{\frac{4}{3}} + c, \quad \text{or:}$$

$$4 = \frac{3}{8}(27)^{\frac{4}{3}} + c = \frac{3}{8}(81) + c, \quad \text{or:}$$

$$c = 4 - \frac{3(81)}{8} = \frac{32 - 243}{8} = \frac{-211}{8}$$

[7. cont'd]

$$\therefore y = \frac{3}{8}(x^2 + 18)^{\frac{4}{3}} - \frac{211}{8} \quad \text{or:}$$

$$y = \frac{1}{8}\left[3(x^2 + 18)^{\frac{4}{3}} - 211\right]$$

8. (a) $$a = \frac{dv}{dt} \qquad (1)$$

By the chain rule, (1) may be rewritten as:

$$a = \frac{dv}{dx}\frac{dx}{dt} \qquad (2)$$

Recalling that $v = \frac{dx}{dt}$, (2) becomes:

$$a = v\frac{dv}{dx} \qquad (3)$$

In (3), if we let $v = f(x)$, we obtain the desired result:

$$a = f(x)\ f'(x)$$

The major use of part (a) is in the case where acceleration is expressed as a function of x rather than t. That is, $\frac{dv}{dt} = g(t)$ is fine, but $\frac{dv}{dt} = g(x)$ gives us one too many variables.

In particular, in part (b),

$$a = x^2$$

[8. cont'd]

That is, $v\frac{dv}{dx} = x^2$. (Had we written $\frac{dv}{dt} = x^2$, we would have to be able to express x in terms of t before we could proceed. In this example we have no explicit representation of x in terms of t.)

At any rate:

$$v\frac{dv}{dx} = x^2$$

implies

$$vdv = x^2dx$$

or:

$$\frac{1}{2}v^2 = \frac{1}{3}x^3 + c_1$$

or:

$$3v^2 = 2x^3 + c \tag{4}$$

The fact that we are told v = 0 when x = 0 allows us to conclude that c = 0 in (4).

That is:

$$v^2 = \frac{2}{3}x^3$$

or

$$v = \pm\sqrt{\frac{2x^3}{3}} \tag{5}$$

[8. cont'd]

The fact that we are also told that the motion is in the direction of the positive x-axis allows us to choose the positive root in (5).

$$\therefore v = \sqrt{\frac{2x^3}{3}} = \sqrt{\frac{6x^3}{9}} = \frac{x}{3}\sqrt{6x} \qquad (6)$$

Letting x = 6 in (6), we obtain:

$$v = \frac{6}{3}\sqrt{36} = 12 \text{ ft/sec}$$

Resource: Calculus Revisited
Herbert Gross