NARRATOR:

PROFESSOR: Hi. Our lecture today is entitled Implicit Differentiation. And aside from other considerations, this lecture bares a very strong relationship to many of the comments that we've made about single-valued functions and one-to-one functions.

Let's get right into the topic by talking about one particular phase that somehow or other we may have taken for granted, but which really is a lot more sophisticated than we really imagine at first glance. Let's look, for example, at an expression such as 'y' equals 'f of x'. A rather harmless expression, we write it down quite frequently, and we say things like what? Given that 'y' equals 'f of x', find 'dy dx'. And you see the point that we've made here is we have assumed that 'y' can be solved for explicitly in terms of 'x'. In other words, in terms of our function machine, we visualize an 'f' machine. 'x' goes in as the input, 'y' comes out as the output, and everything is fine.

Now, the interesting point is simply this. Let's look at a slightly more complicated algebraic equation. For example, 'x to the eighth' plus ''x to the sixth' 'y to the fourth'' plus 'y to the sixth' equals 3. Now, in this particular example, notice that 'x' and 'y' are not given at random. For example, if I say let me pick 'x' to be 2 and 'y' to be 2, and I put 2 in for 'x' and 2 in for 'y', just look at what happens to the left-hand side here. It makes, in particular, 3 a pretty big number.

In other words, when 'x' is 2 and 'y' is 2, this equation-- and we'll talk about the meaning of the equation more in just a few minutes, but this equation does not balance, so to speak. In other words, while we cannot pick 'x' at random for a given value of 'y', once 'x' is given, notice that we wind up with a sixth degree polynomial equation in 'y', which determines at most six values of 'y'. And if we're given a value of 'y' as a fixed number, we wind up with an eighth degree polynomial equation in 'x', which means that what? 'x' is no longer random either.

The point is it is very, very difficult, if not impossible, to try to solve this particular equation for 'y' explicitly in terms of 'x' or for 'x' explicitly in terms of 'y'. And the question is how do we tackle something along these lines? And we make a very tacit assumption. And by the way, you'll notice that when you read this section in the text, there are several remarks saying that many of the proofs are beyond the scope of the textbook at this particular point. And they'll say we'll talk about this later, and there are references to more advanced textbooks. The point is that from a more rigorous point of view, this is true. However, from a geometric intuitive point of view, it's quite easy to see what's really going on here, as we shall do with as the lecture progresses. But for the time being, let's review what that tacit assumption is.

Essentially what we say is let's assume that 'y' is a particular function of 'x'. What particular function of 'x' is it? Well, it's that function of 'x' such that when you replace 'y' by that function of 'x', this becomes an identity. And this is what I want to talk about next. You see, assume that 'y' is that function of 'x' such that when you replace 'y' in terms of 'x', 'x to the eighth' plus ''x to the sixth' 'y the fourth'' plus 'y to the sixth' is identically 3. And notice the use of the three lines here to indicate the identity as opposed to the two lines to indicate the equality.

Now, obviously, there's a more important difference than to say that one is indicated by three lines and the other by two. Let me take a few moments to digress on a very important topic, namely, the difference between what we call an equation, or perhaps more appropriately, a conditional equality, and that which we call an absolute equality or an identity. For example, let's take a look at an expression such as 'x squared' equals 4.

You say, ah, these two things are equals, and if I do the same thing to equals, I get equals. So the fellow says I think I'll differentiate both sides of this equality. If he differentiates 'x squared', he gets '2x', and if he differentiates 4, he gets 0. Now notice that if you solve the equation 'x squared' equals 4, you get what? 'x' equals minus 2, or 2. On the other hand, if you solve the equation '2x' equals 0, you get x equals 0, and there seems to be no correlation between these two.

Now, you see the answer to this thing is this. 'x squared' is not a synonym. 'x squared' is not another way of saying 4. To use the language of the new mathematics, the solution set for 'x squared' equals 4 is what? The set of all 'x' such that 'x squared' equals 4. That's another way of saying what? The set whose only two members are 2 and negative 2. In other words, we cannot say that 'x squared' is a synonym for 4. All we can say is given the condition if 'x' is either 2 or minus 2, then 'x squared' equals 4. Otherwise, this is not the case.

For example, if you subtract 4 from 'x squared', you do not get identically 0. In other words, 'x squared' minus 4 is not another way of saying 0. For example, if I put 'x' equals 3 in here, this says what? 3 squared equals 4, which is certainly a meaningful statement, but nonetheless a false statement.

You see, somehow or other, an equation is something which can be true for a certain values of your variable but false for others. On the other hand, an identity, as the name may imply, is something that's true for all values of the variable. For example, let's go back to something we learned in elementary school algebra: 'x squared minus 1' equals 'x + 1' times 'x - 1'.

Notice I wrote this with the three lines here. It's my way of saying that for any number 'x' whatsoever, 'x squared - 1' always names the same number as 'x + 1' times 'x - 1'. In other words, the solution set of this particular equation includes all real numbers. Or another way of saying it is that if you were to subtract 'x + 1' times 'x - 1' from the 'x squared minus 1', the result would be 0 independently of what the value of 'x' was. In more colloquial terms, what we're saying is that 'x squared minus 1' and 'x + 1' times 'x - 1' are two different ways of saying the same thing.

Now, the beauty of an identity versus a conditional equality is this. That if two expressions are just two different names for the same thing, then whatever is true for one expression will be true for the other. You see, in other words, if the concept is the same but only the names are different, then certainly anything that depends on the concept will not depend on the particular name that's involved.

Well, as a case in point, let's suppose now I take 'x squared minus 1' and I differentiate it. The result is '2x'. On the other hand, if I take 'x + 1' times 'x - 1' and differentiate it, I get what? By use of the product rule, it's the first factor times the derivative of the second plus the derivative of the first factor times the second. That's what? 'x + 1' plus 'x - 1', and that comes out to be '2x'.

In other words, since these two expressions were just two different names for the same thing, the derivative of one of the expressions must equal the derivative of the other because it's still the same function that you're differentiating. And this is what we mean when we say let's assume that 'y' is that function of 'x' that makes the resulting equation an identity.

Let's look at a particularly simple illustration, and we'll do this quite often in this course. Namely, whenever we want to illustrate a new topic, we will always, when possible, pick an illustrative example that could have been solved by a previous method. As a case in point, let's look at the identity 'x' times 'y' is 1. See, in other words, 'y' is that particular function of 'x', such that whenever you replace 'y' by that function, this becomes an identity.

Now, if that seems hard, notice that we could turn this into an explicit relationship just by dividing both sides of this equation, our identity by 'x'. Of course, that assumes 'x' is not equal to 0. But if we do that, 'y' becomes '1/x'. Now notice that '1/x' has the property that as long as 'x' is not 0, if you multiply that by 'x', you get identically 1. 'x' times '1/x' is 1. Two different ways of saying 1.

Now, how does the method of implicit differentiation proceed? We say OK, since this is an identity, if I differentiate both sides with respect to 'x', the derivative of the left-hand side should equal the derivative of the right-hand side. Now, how do we differentiate 'x' times 'y'? Observe that we're assuming that 'y' is a function of 'x'. Therefore, 'x' times 'y' is a product of two functions of 'x'.

To differentiate a product, we use the product rule. Namely, we will take the first factor times the derivative of the second with respect to 'x'. That's 'dy dx', because 'y' is the second factor, plus the derivative of the first factor with respect to 'x'. Well, the derivative of 'x' with respect to 'x' is 1 times the second factor, which is 'y', that's now the derivative of the left-hand side. That must equal identically the derivative of the right-hand side. The right-hand side is 1. The derivative of a constant is 0. So we wind up with the relationship that 'x' times 'dy dx' plus 'y' must be identically 0. And now solving for 'dy dx' in terms of 'x' and 'y', we find that 'dy dx' is equal to minus 'y/x'.

By the way, notice that by picking a problem that could be solved explicitly for 'y' in terms of 'x', we have a very simple check on this particular problem. Namely, we know that 'y', if 'x' is not 0, that y is another name for '1/x'. Therefore, minus 'y/x' is another name for ''minus 1' over 'x squared''. But we already know that if 'y' equals 'x to the minus 1' by another method, we know that 'dy dx' is 'minus x to the minus 2', which is also ''minus 1' over 'x squared''. And so we see that the new method does give us the same answer as the old method.

By the way, let me make a rather important aside over here. And that is when you look at something like this, you might say something like I wonder what happens if I try to compute 'dy dx', say, when 'x' equals 2 and 'y' equals 3? Now, you see if you mechanically plug into something like this, you get what? Minus 3/2, which is minus three-halves.

The point that I want to bring out, and we'll come to this at the conclusion of today's lecture also, is the concept of related rates and related variables. Notice that whereas from this equation it looks as if we can let 'y' equal 3 and 'x' equal 2, notice that if we go back to our basic definition, 'x' and 'y' are related so that they are not independent. Notice that as soon as we say let 'x' equal 2, we have what? That '2y' is 1, and 'y' must be 1/2. In other words, when you use implicit differentiation, never forget that whenever you're going to compare 'x' and 'y', you must go back to the equation or the identity which implicitly relates 'x' to 'y'.

Well, so far this may look rather easy and straightforward. But the fact remains that there are certain subtleties here which we have not hit yet. So what I'd like to do now is pick a second example, slightly more complicated than this one, which can still be solved explicitly but which leads to a wrinkle which we may not have observed before. With this in mind, what I would like to do is the following.

Let's consider the relation 'x squared' plus 'y squared' equals 25 and ask the question how do you find 'dy dx' in this particular case? Again what we do is we assume, and this is the big word here. There are lots of things that you can assume, but whether they exist or not is another question. That's the part that the textbook means is more advanced and is hard to justify. But let's take a look here. We'll assume that 'y' is a particular function of 'x' with the property that when 'y' is that function of 'x', 'x squared' plus 'y squared' is identically 25, OK? And what that means is that whatever 'x' and 'y' are, they're related in such a way that 'x squared' plus 'y squared' is a synonym for 25.

If we now proceed by implicit differentiation here, you see the left-hand side is a function which is the sum of two functions of 'x'. The derivative of a sum is the sum of the derivatives. The derivative of 'x squared' with respect to 'x' is '2x'. The derivative of 'y squared' with respect to 'x' is not '2y'. The derivative of 'y squared' with respect to 'y' is '2y'. By the chain rule, the derivative of 'y squared' with respect to 'x' is what? The derivative of 'y squared' with respect to 'y' times 'dy dx'. In other words, this is simply what? ''2y' 'dy dx''.

So the derivative of the left-hand side is '2x' plus ''2y' 'dy dx''. The derivative of the right-hand side, the right-hand side being a constant, is 0. And if we now solve for 'dy dx' in terms of 'x' and 'y', we find that 'dy dx' is 'minus x/y', OK? This is all there is to this thing mechanically. By the way, of course, it happens as you probably remember, that 'x squared' plus 'y squared' equals 25 is a circle centered at the origin with radius equal to 5.

OK, so far so good. We'll come back to this diagram in a little while. But the thing now is could we have solved the same problem by solving for 'y' explicitly in terms of 'x'? The answer in this case, of course, is yes. Namely, if 'x squared' plus 'y squared' is 25, that says that 'y squared' is 25 minus 'x squared', and therefore, the desired function of 'x'.

'y' is what function of 'x'? It's plus or minus the square root of '25 minus 'x squared''. In other words, that's what? It's the positive '25 minus 'x squared'' to the 1/2 power. That's one of the solutions. And the other solution is the negative '25 minus 'x squared'' to the 1/2 power. I simply use the exponent notation because it's more familiar to us in terms of differentiation to differentiate the exponent rather than the radical sign.

But the idea is this: Notice now that we begin to see a multivalued function creeping in over here. In other words, we now find that we want to solve for 'y' explicitly in terms of 'x', that we do get the problem that y might be a multivalued function of 'x'. Well, we'll look at that in a moment. For the time being, let's simply do a double check over here. Let's actually differentiate this thing explicitly and see what the derivative turns out to be.

If we differentiate this, we bring the 1/2 down. We replace it to an exponent one less. And by the chain rule, we must multiply by the derivative of what's inside here, 25 minus x squared with respect to 'x'. That's 'minus 2x'. Collecting terms and simplifying, we get what? 'Minus x' over ''25 minus 'x squared'' to the 1/2'. And recalling that ''25 minus 'x squared'' to the 1/2' is equal to the 'y' value in this case, we get the derivative of 'y1' with respect to 'x' is 'minus x' over 'y1', just calling the function this. That at any rate shows what? The negative x-coordinate over the y-coordinate, and that certainly checks with the result that we had before.

In a similar way, if we now differentiate 'y2' with respect to 'x', bringing down the exponent, multiplying by the derivative of what's inside with respect to 'x', we wind up with the same expression as we did before, only now we remember that 'y2' is defined to be negative ''25 minus 'x squared'' to the 1/2', so this is really negative y2.

By the way, notice again that we get the same answer here as we did here. I like to keep the minus sign with the appropriate term. In other words, notice it was the fact that 'y2' was multivalued that caused us to have two different possibilities here in the first place.

Now, let's go back to our little graph over here and take a look to see just what happened here. Remember, all through this course, we've said what? Be aware of what happens when 'y' equals 0. Notice in terms of our picture and our related rates here, when 'y' is 0, 'x' is not some arbitrary number. When 'y' is 0, since 'x squared' plus 'y squared' equals 25 when 'y' is 0, 'x' must either be 5 or minus 5. So what's happening is the bad point when 'y' is 0, or the bad points, are right here. And notice the rather interesting thing here, and this is what comes up more analytically later.

The only time you're in trouble when you assume that 'y' is a function of 'x'-- remember, function mean single-valued-- is that if you happen to be in the neighborhood of 5 comma 0 or minus 5 comma 0, notice that for any neighborhood surrounding this point, there is no way-- if this point here is included, there is no way of breaking up this function to be single valued. In other words, if you must have a small portion of the curve that includes 5 comma 0 as an interior point, no matter how you do it, the resulting curve is going to be multivalued, and then we're in the same predicament again.

In other words, from a theoretical point of view, geometrically speaking, it's very easy to assume that 'y' is a function of 'x' that makes our equation an identity. But from a real point of view, what may frequently happen is that the only type of function that would work is what we call a multivalued function. That's exactly what's going on over here. And that's why a textbook, which is trying to be rigorous, has to be very, very careful in explaining what happens in neighborhoods of points like this. But again, we'll talk about that in more detail in a little while.

I would like to make one aside over here. You may recall that we learned in one of our previous lectures that by inverse functions, we can differentiate things like 'x' to the 1/2, 'x' to the 1/3. This generalizes very nicely and very simply in terms of implicit differentiation. Namely, suppose you have 'y' equals 'x to the 'p/q'' power where 'p' and 'q' are integers. In other words, your exponent is now a fraction. Raise both sides to the q-th power, and you get 'y' to the 'q' equals 'x to the p'.

Now, we know how to differentiate powers of 'x' or 'y' if the power happens to be an integer. So by implicit differentiation, assuming that this is now an identity, which it is if 'y' equals 'x to the 'p/q'', if we differentiate both sides with respect to 'x', the derivative of the left-hand side is 'qy to the 'q - 1'' times the derivative of 'y' with respect to 'x'. That is, we're differentiating with respect to 'x'. And the derivative of the right-hand side is 'px to the 'p - 1''. And since this is an identity, these two expressions must be equal.

If we solve now for 'dy dx', we obtain this expression, replacing 'y' by 'x to the 'p/q'', and multiplying out here, and remembering that when you have an exponent in the denominator, it can come up into the numerator by changing the sign. A little bit of arithmetic shows that 'dy dx' is ''p/q' x to the ''p - 1' minus 'p' plus 'p/q'' power. The 'p' and the 'minus p' cancel out, and we find what? That 'dy dx' is 'p/q' times 'x to the ''p/q' - 1''.

Remembering that 'p/q' was our fractional exponent, we again see what? That to differentiate 'x' to a fractional exponent-- 'p/q'-- you bring the exponent down and replace the 2 by one less. So that gives us an alternative method for finding the derivative of a fractional exponent by namely using implicit differentiation.

Now, the reason I put that aside in is we could certainly get along without it. But I felt that before we go any further, that I would like at least to add it on the more elementary levels to be sure that we understand explicitly what implicit differentiation means. With this in mind, let's now go back to the problem that we started off with in this lecture, namely, the curve whose equation is 'x to the eighth' plus ''x to the sixth' 'y to the fourth'' plus 'y the sixth' equals 3. And we say what? Let's find the equation of the line tangent to this curve at the point 1 comma 1.

By the way, I did something that teachers are allowed to do here. I rigged the problem to come out rather simply. You see, what I did was by picking 1 and 1 here, it turns out what? That if I made the right-hand side just equal to the number of terms here, I would get this thing to check out. But notice, by the way, if I had just picked the point at random to try to even check where that point is on this curve, or if it were on this curve, to find out what its coordinates are, this is a rather difficult problem.

In other words, if I just pick a random value of 'x', as I said before, I get a sixth degree polynomial equation to solve, and if I pick a random value of 'y', an eighth degree polynomial equation to solve. You know, it might be nice to pretend that the 8 was a 2 or something like this and solve it as a quadratic equation. But quite frankly, solving polynomial equations, which are higher than the degree two is a very difficult task. In fact, if the degree is greater than four, it may even be an impossible task.

I'm not going to go into that now because it's not that crucial. What is crucial is to observe that this particular problem makes sense. It's a meaningful problem. It would be difficult, if not impossible, to solve for 'y' explicitly in terms of 'x' here, OK? Yet to solve this problem, we can now use implicit differentiation except for the fact that we can no longer check back by another method to see if the answer is right. That's why I wanted to pick a few introductory examples that we could check by other means. Now that we have a feeling for this, maybe we will trust the technique in a case where we have no recourse other than to use the technique.

So what we do now is this: We say OK, let's assume that 'y' is that function of 'x', that differentiable function of 'x' that makes this particular equation an identity. And assuming now that this is an identity, let me differentiate both sides with respect to 'x'. See, this is a sum. One of the terms in the sum happens to be a product.

But by now, hopefully how we go about something like this will be old hat. Namely, to differentiate the left-hand side with respect to 'x', we get what? '8x to the seventh' plus what? The derivative of 'x to the sixth', which is '6x to the fifth', times the second factor, which is 'y the fourth', plus the first factor, which is 'x to the sixth', times the derivative of 'y to the fourth' with respect to 'x'-- that's ''4y cubed' 'dy dx''-- plus the derivative of 'y to the sixth' with respect to 'x', which is ''6y to the fifth' 'dy dx''. That must be identically 0. And if we now solve this for 'dy dx', just by transposing, we wind up again with a rather messy expression, but which does show what 'dy dx' looks like in terms of 'x' and 'y'.

We were interested in knowing what the slope was not only for any old value of 'x' and 'y' but rather for what? When 'x' is 1 and 'y' is 1. And that works out very nicely computationally. It's just minus 14/10. In other words, at the point 1 comma 1, the slope of the curve is minus 7/5. The curve passes through the point (1, 1). Hence, it's equation is 'y - 1' over 'x - 1' equals minus 7/5, or more explicitly, '7x + 5y' equals 12.

And the point that I wanted to make here is notice that nothing changed in principle from our first few lectures. Notice that to find the equation of the line, we still use the recipe that we have to know a point on the line and the slope. The only thing that's changed with today's lesson is that we can now find the slope of a curve at a particular point that we could not find prior to today's lesson. All that has changed is that we have one more technique for finding the derivative of a particular type of function.

I would like to analyze this problem in more detail. In particular, I would like to see where the numerator of this expression can be 0 and where the denominator can be 0. Because you see in terms of slopes, where the numerator is 0, it means the slope will be 0. That means we have a horizontal tangent line. Where the denominator is 0, that means the slope is infinite, OK? And where the slope is infinite, that means you have a vertical line, and that means you have a vertical tangent, OK.

So let's take a look and see what that means. Keep this particular equation in mind, because now you see on the next board, all I want to do is work with what this thing means. In other words, 'dy dx' will be 0 when my numerator is 0. My numerator can be written in this particular form. And by the way, here's again an interesting point. When will this expression be 0? And the answer is when either of these two factors is 0. Well, the first factor is 0 when 'x' is 0. And the second factor is 0, since these are even powers, only when 'x' and 'y' are both 0.

Notice, however, that 'x' and 'y' cannot both be 0. Recall that the equation was what? Whatever it was, notice that (0, 0) is not a point which satisfies the equation. Remember, (0, 0) does not satisfy 'x to the eighth' plus ''x to the sixth' 'y squared'' plus 'y to the sixth' equals 3. It's equal to 0, you see. But at any rate, notice that the slope is 0 only when 'x' is 0. And because of the particular equation, when 'x' is 0-- well, we'll go back to that in a minute. Let's just check to see what's happening here.

The denominator will be 0. In other words, when 'dx dy' is 0-- '1 over 'dy dx'', you see-- only when this factor here is 0, and that occurs again only when 'y' is 0. Now, the point to keep in mind is this. Remember, I put this down here so we could refer to it, and I forgot that I put it here, and that's why I didn't see it until just now. All I'm saying here is that notice that when 'x' is 0, 'y' must be plus or minus the sixth root of 3. And when 'y' is 0, these two terms drop out. 'x' must be plus or minus the eighth root of 3.

Coming over to a graph here then, what we see is that the curve crosses the x-axis at this point with a vertical tangent. It crosses the y-axis at this point with a horizontal tangent. Notice, by the way, that this curve is also symmetric with respect to both the x- and the y-axes, because if I replace 'x'-- well, it's not important, and I don't want to obscure the lecture by taking time out for this now. But the point is a quick check shows that this curve is symmetric with respect to both the x- and the y-axes, that if I could plot this curve just in the first quadrant, the mirror image with respect to the y-axis would then show me the second quadrant. If I then took the mirror image of this upper half with respect to the x-axis, that would give me the lower portion of this curve. The curve tends to look something like this.

And by the way, all we've done, if we look back over here if you can see this OK, the '7x + 5y' equals 12 simply turned out to be what? The equation of the line which was tangent to this curve at this particular point. What I'd like to show you in terms of one-to-oneness and single-valuedness is this. I told you I was interested in what was happening at this curve at the point 1 comma 1. Suppose I had said instead find the equation of the line tangent to this curve at the point whose x-coordinate is 1?

Well, you see, there are two points on this curve whose x-coordinate is 1. You see, this is a double-valued curve. A given value of 'x' between these two extremes yields the two values of 'y'. I would have had no way of knowing which of the two y-values I meant, you see? And correspondingly, if somebody had said find the slope of the curve, of the equation of a line tangent to the curve, at the point whose y-coordinate is 1, notice that this is not 1 to 1. In other words, if the y-coordinate is 1, notice that I cannot distinguish between the point 1 comma 1 and the point what? Minus 1 comma 1.

You see, there's again our problem with inverse functions and things of this particular type. If we're told the neighborhood of the point that we're interested in, we're fine. If all we're told are one of the coordinates and have to find the other, there is a certain amount of ambiguity. And keep in mind, by the way, that I deliberately rigged this problem to get something I could graph at least. In many cases, it's much more difficult to even visualize what the graph looks like, much more complicated. You see, computationally, this can be come quite a mess.

The important point is that what we were doing implicitly here assumed on the explicit fact I could take this curve and break it down at the points where I have vertical tangents and look at this as two separate curves: 'c1', namely, the original curve, but restricted to 'y' being non-negative, and 'c2', the original curve, but restricted to 'y' being negative. In other words, I can look at 'c1' as being this piece and 'c2' as being this piece. And again, the point is what? That whenever you're in the neighborhood of these points, you're in trouble. Because notice that no matter how small a neighborhood I pick, if I can't tell one of these branches from the other, no matter how I do this, I'm going to be caught on a multivalued part of the curve over here.

Well, at any rate, I think this begins to show us what implicit differentiation means, why we have to be careful of points at which vertical tangent occur, but I would like before closing this lecture to generalize the concept of related rates and make this a little bit more applicable from a physical point of view. And that is when we say in something like this that let's assume that 'y' is a differentiable function of 'x', there is no reason to have to assume that the variable you want to relate things to is 'x' itself.

For example, let me do something which uses the same kind of an equation that we had before, but only from a different point of view. See, now instead of asking for the slope of this circle or what have you, let's work the question this way. Let's suppose we have a particle. The particle is moving along the curve 'x squared' plus 'y squared' equals 25 where for physical reasons we'll say 'x' and 'y' are in feet, and that we know at the point 3 comma 4, 'dx dt'-- the horizontal component of the speed of the particle-- is 8 feet per second.

And the question is we'd like to find 'dy dt', the vertical component. In other words, the particle is moving along the circle. We know that at the point 3 comma 4, 'x' is increasing at the rate of 6 feet per second while the increasing x-direction is this way. So somehow or other, we know that the particle is moving along the curve in this direction. You see, if it were moving in this direction, it's x-coordinate would be decreasing, not increasing. And the question that comes up is how do we find 'dy dt' in this case?

Well, see, all we do in this case is we say look-it, instead of assuming that 'y' is a differentiable function of 'x', why don't we assume that both 'x' and 'y' are differentiable functions of 't'? In other words, let's assume that 'x' and 'y' are differentiable functions of 't' that make this an identity in terms of 't'.

Now again, as long as this is an identity and we're assuming that 'x' and 'y' are differentiable functions of 't', there is absolutely no reason why we can't differentiate both sides of this equation with respect to 't' instead of with respect to 'x'. If we do that, we get what? '2x' 'dx dt' plus '2y' 'dy dt' equals 0. You see the same principle as before even though we're now differentiating implicitly evidently what we could call parametric equations. We're assuming that 'x' and 'y' are differentiable functions of 't'.

You see, from this identity now, we can conclude that 'dy dt' is 'minus x/y' times 'dx dt'. And now, you see, to polish off our particular problem, namely, we're interested in what? When 'x' was 3, 'y' was 4, and 'dx dt' is 8. In other words, I find out from this that 'dy dt' is minus 6 feet per second. You see, the idea was that 'x' and 'y' as positions of the point were related by the fact that 'x squared' plus 'y squared' had to equal 25. In other words, this is what we physically call a constraint.

And by the way, this is what makes calculus such an important, powerful, analytical tool. Notice that in doing this particular problem, I never had to know explicitly what functions 'x' and 'y' were of 't'. All I had to know was what? That 'x' and 'y' were differentiable functions of 't'. I don't care how the particle was moving away from the point 3 comma 4 as long as 'x' and 'y' were differentiable functions of 't', this is how the rates 'dy dt' and 'dx dt' had to be related.

Well, let me summarize what was really important conceptually about today's lecture. The most important thing conceptually was what? With all of our talk about explicitly writing an output and an input, in many important mathematical relationships, the variables that we're concerned with are implicitly related. In other words, we are not told what 'y' looks like in terms of 'x', but rather how 'x' and 'y' are interrelated, and from here we have to find a derivative. And the way we do this is that we make the assumption that an appropriate identity exists, that 'y' is an appropriate function of 'x' that makes the relationship an identity.

The validity of when you can do this, for example, geometrically, when do you wind up not having a point that includes a vertical tangent, a point where you can't separate the curve doubling back? That turns out to be a rather difficult point from an analytical point of view, a point that we will return to in a more advanced context when we deal with functions of several variables.

But for the time being, all I want you to be left with in this lecture is the feeling that we can, given an implicit relationship under the proper conditions, assume that the appropriate explicit relationship exists and that we can differentiate both sides as an identity. Well, enough said for today, and until next time, goodbye.

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