PROFESSOR: Hi. Today's lesson, well, I settled for the title, "Circular Functions." But I guess it could have been called a lot of different things. It could've been called 'Trigonometry without Triangles'. It could have been called 'Trigonometry Revisited'. And the whole point is that much of what today's lecture hinges on is a hang-up that bothered me, and which I think may bother you and is worthwhile discussing.

I remember, when I was in high school, I asked my trigonometry teacher, why would I have to know trigonometry? And his answer was, surveyors use it. And at that particular time, I didn't know what I was going to be, but I knew what I wasn't going to be. I wasn't going to be a surveyor. And I kind of took the course kind of lightly, and really got clobbered a year or two later when I got into calculus and physics courses.

So what I would like to do today is to introduce the notion of what we call circular functions, and point out what the connection is between these and the trigonometric functions that we learned when we studied the subject that we call trigonometry, and which might better have been called numerical geometry. Let me get to the point right away. Let's imagine that I say circular functions to you. I think it's rather natural that, as soon as I say that, you think of a circle.

And because you think of a circle, let me draw a circle here, and let me assume that the radius of the circle is 1. In other words, I have the circle here, 'x squared' plus 'y squared' equals 1. Now, the thing is this. When I talk about-- And I'm assuming now that you are familiar with the trigonometric functions in the traditional sense. And in fact, the first section of our supplementary notes in the reading material that goes with the present lecture takes care of the fact that, if you don't recall some of these things too well, there's ample opportunity for refreshing your minds and getting some review in here.

But the idea is something like this. When we're talking about calculus, we talk about functions of a real variable. We are assuming that our functions have the property that the domain is a set of suitably chosen real numbers, and the image is a suitably chosen set of real numbers.

We do not think of inputs as being angles and things of this type. And so the question is, how can we define, for example-- let's call it the 'sine machine'. Let me come down here. I'll call it the 'sine machine'.

If the input is the number 't', I want the output, say, to be 'sine t'. But you see, now I'm talking about a number, not an angle. Well, one way of doing this thing visually is the old idea of the number line. Let us think of a number as being a length, the same as we do in coordinate geometry. We knock off lengths along the x-axis and the y-axis. Let me think of 't' as being a length.

As such, I can take 't' and lay it off along my circle in such a way that the length originates at 'S' and terminates, shall we say, at some point 'P' whose coordinates are 'x' and 'y'. Now, notice what I'm saying here. I lay the length off along the circumference. I'll talk more about that a little bit later.

Now, so far, so good. No mention of the word "angle" here or anything like this. Now, wherever t terminates-- and again, conventions here, if 't' is positive, I lay if off along the circle in the so-called positive direction, namely, what? Counter-clockwise. If 't' is negative, I'll lay it off in the clockwise direction, et cetera. The usual trigonometric conventions. Now what I do is is, at the point 'P', I drop a perpendicular. And I define the sine of 't' to be the length, 'PR', and the cosine of 'P' to be the length, 'OR'.

In other words, I could write that like this. I could write down that I'm defining 'sine t' to be the length of 'RP' in that direction, meaning, of course, that this is just a fancy way of saying that the sine of 't' will just be the y-coordinate of the point at which the length 't' terminates on the circle. And in a similar way, 'cosine t' will be the directed length from 'O' to 'R', or more conventionally, the x-coordinate.

Now, notice I can do this with any length. Whatever length I'm given, I just mark this length off. It's a finite length. Eventually, it has to terminate some place on the circle. Wherever it terminates, the x-coordinate of the point of termination is called the cosine of 't', and the y-coordinate is called the sine of 't'. And notice that, in this way, both the sine and the cosine are functions which map real numbers into real numbers.

So that part, I hope, is clear. Notice again, I can mimic the usual traditional trigonometry. I can define the tangent of t to be the number 'sine t', divided by the number 'cosine t', et cetera. And I'll leave those details to the reading material. I can ascertain rather interesting results the same way as I could in regular traditional trigonometry. In fact, I can get some certain results very nicely.

I remember, for example-- Well, I won't even go into these. But how did you talk about the sine of 0 when one talked about traditional trigonometry? How did you embed a 0-degree angle into a triangle, and things of this type. Notice that in terms of my tradition here-- and we'll summarize these results in a minute-- but notice, for example, that the sine of 0 comes out to be 0 very nicely, because when 't' is 0, the length 0 terminates at 'S'. 'S' is on the x-axis. That makes, what? 'y' equal to 0.

Notice also that, if the radius of my circle is 1, the circumference is 2 pi. So for example, what I usually think of a 90-degree angle would be the length pi/2. And without making any fuss over this, again, leaving most of the details to the reading and to the simplicity of just plugging these things in, we arrive at these rather familiar results. We also get, very quickly, in addition to these results, things like the fundamental result that we always like with trigonometric functions. That's 'sine squared t' plus 'cosine squared t' is 1.

And how do we know that? Remember that 'cosine t' was just another name for the x-coordinate at which the point terminated on the circle. In other words, notice that 'cosine squared t' is 'x squared'. 'Sine squared t' is 'y squared'. The x-coordinate and the y-coordinate are related by the fact that, what? The sum of the squares to be on the circle is equal to 1.

We could even graph 'sine t' without any problem at all. Namely, we observe that when 't' is 0, 'sine t' is 0. Notice that as we go along the circle, the sine increases up until we get to pi/2, at which it peaks at 1, then decreases at pi, back down to 0. And if that's giving you trouble to follow, let's simply come back to our diagram to make sure that we understand this.

In other words, all we're saying is, as 't' gets longer, its y-coordinate increases from 0 to a maximum of 1, when the particle was over here. Then, as 't' goes from to pi/2 to pi, the length of the y-coordinate decreases until it again becomes 0. And again, without making much more ado over this, we get the usual curve that we associate with the sine function even when we thought of it as a traditional trigonometric problem.

But the major point that I want you to see right now-- and we won't worry about why I want to do this-- I can define the trigonometric functions in such a way that their domains are real numbers rather than angles. And in fact, this is the main reason why people invented the notion of radian measure. Let me see if I can't make that a little bit clearer, once and for all.

You see, the question is this. Let's suppose I'm talking-- Oh, let me give you some letters over here. We'll put a 'Q' over here. Let's talk about angle, 'QOS'. That's a right angle. It's 1/4 of a rotation of the circle. Now, the question that I have in mind is, if something is 1/4 of a rotation, why do you need two different ways of saying that? Why do we have to say it's 90 degrees or pi/2 radians, and bring in a new measure when we already have another way of measuring circles, angles of circles?

The idea is something like this. Let's again mimic the idea of taking the length 't' and laying it off along the circle like this. Now, here's the idea. Remember, the radius of this circle is 1. So notice that 'PR', in other words, this y-coordinate, is what we call, by definition, the sine of 't'. In other words, just above we said that 'sine t' was the length 'R' to 'P' in that direction.

Now, the point is, I said disregard traditional trigonometry, but we can't really disregard it. It exists. For the person who's had traditional trigonometry, how would he tend to look at this length divided by this length? He would think of that as being what? It's side opposite over hypotenuse. That also suggests sine. And the sine of what? Well, the sine of what angle this is.

Now, the thing is this. Somehow or other, to avoid ambiguity, if we could have called whatever measure this angle was measured in terms of, if we could have called that unit 't', then notice that the sine of the angle 't' would have been numerically the same as the sine of the number 't'. And again, if this seems like a hard point to understand, we explore this in great detail in our notes.

But the idea is this. You see, somehow or other, if 'sine t' is going to have two different meanings, we would like to make sure that we pick the kind of a unit where it makes no difference whether you're thinking of 't' as being a number or thinking of 't' as being a length. For example, suppose I now invent the word "radian" to mean the following. An angle is said to have 't' radians. If, when made the central angle of a unit circle, a circle whose radius is 1, it subtends an arc whose length is 't' units of length.

See, in other words, I would define the measure called radians so that an angle of 't' radians intercepts the length 't' over here. In that way, 'sine t' is unambiguous whether you're talking about an angle or a length. For example, when I say the sine of pi/1 radians, what do I mean? I mean the angle which is the sine of the angle which intercepts a length, an arc, pi/2 units long.

Well, see, pi/2 is this length. I'm now talking about this angle here. And the sine, therefore, of pi/2 radians, in terms of classical trigonometry, is 1. But that's also what the sine of a number pi/2 was. This explains the convention that one says when one uses radians, you can leave the label off.

All we're saying is that, if we had used degrees, there would have been an ambiguity. Certainly, the sine of 3 degrees is not the same as the sine of 3. You see, 3 degrees is a rather small angle. But 3 is a rather great length when you're talking about the arc of the unit circle here. Remember, 1/2 circle is pi units long, so 3 would be just about this long.

In other words, notice that 3 radians and 3 degrees are entirely different things. But the beauty is what? That if we agreed to use radian measure, then we have no ambiguity when we talk about the sine. The sine of the number 't' will equal the sine of the angle 't' radians. The cosine of a number 't' will equal the cosine of the angle 't' radians.

In a certain sense, it was analogous to when we talked about the derivative 'dy/dx', then wanted to define differentials 'dy' and 'dx' separately, so that 'dy' divided by 'dx' would be the same as 'dy/dx', that we wanted to avoid any ambiguity where the same symbol could be interpreted in two different ways to give two different answers.

By the way, again, there is nothing sacred about our choice of why we pick circular functions. We could have picked hyperbolic functions. Namely, why couldn't we have started, say, with one branch of the hyperbola, 'x squared' minus 'y squared' equals 1. Given a length 't', why couldn't we have measured 't' off along the hyperbola? Say this way if 't' is positive, the other way if 't' is negative.

And then what we could have done is drop the perpendicular again. And we could have defined what? The y-coordinate to be the hyperbolic. Well, we couldn't call it cosine anymore because it would be confused with the circular functions. We could have invented a name, as we later will, called the 'hyperbolic cosine'. I won't go into any more detail on this. See, this is an abbreviation for hyperbolic cosine, meaning this-- I'm sorry, I got this backwards. Call the x-coordinate the hyperbolic cosine, the y-coordinate the hyperbolic sine.

You don't have to know anything about advanced mathematics to see this. All I'm saying is, I could just as easily have taken any geometric figure, marked off lengths along it, taken the x-coordinates and the y-coordinates, and seen what relationships they obey. You see, as such, there's nothing sacred about working on a circle.

Not only that, but even after you agree to work on the circle, there are many other ways that one could have done this. For example, someone might have said, look it, when you take this length called 't', why did you elect to mark it off along the circle? Why couldn't you have taken a radius equal to 't', taken 'S' as a center, and swung an arc that met the circle, and call this length 't'? You see, instead of measuring along the circle, measure along the straight line. Again, you could have done this if you wanted to. Why you would've wanted to do this? Well, you have the same right to do this as I had to do mine.

Of course, you have to be a little bit careful. For example, in this particular configuration, notice that, if this is how you're going to define your trigonometric function, your input, your domain, has to be somewhere between 0 and 2. In other words, you cannot have a length longer than 2, because notice that the diameter of the circle is only 2. And therefore, if 't' were greater than 2, when you swung an arc from the point 'S', it wouldn't meet the circle at all.

Well, that's no great handicap. It's no great disaster. You still have the right to make up whatever functions you want. I will try to make it clearer why we chose these circular functions from a physical point of view as we go along. What I thought I'd like to do now is, having motivated, that we can invent the trigonometric functions in terms of numbers definitions along this circle. And coupled with the fact that, in radian measure, you can have a very nice identification between what's happening pictorially and what's happening analytically, to show, for example, that in terms of our subject called calculus, that we're pretty much home free once we learn these basic ideas.

You see, the important point is that, in a manner of speaking, we have finished differential calculus. We know what all the recipes are We know what properties things have. So all of the rules that we learned will apply to any particular type of function that we're talking about. For example, let's suppose we define 'f of x' to be 'sine x'. And we want to find the derivative of 'sine x'. Notice that 'f prime of x' evaluated at any number 'x1' has already been defined for us.

It's the limit as 'delta x' approaches 0, 'f of 'x1 plus delta x'', minus 'f of x1' over 'delta x'. This is true for any function 'f'. In particular, if 'f of x' is 'sine x', all we get is what? That the derivative is the limit as 'delta x' approaches 0, sine of 'x1 plus delta x' minus sine of 'x1' over 'delta x'. Now you see, on this particular score, nobody can fault us. This is still the basic definition.

All that happens computationally is that, if we're not familiar with our new functions called the trigonometric functions, we might not know how to express sine of 'x1 plus delta x' in a more convenient form. What do we mean by a more convenient form? Well, notice again, as is always the case when we take a derivative, as delta x approaches 0, our numerator becomes 'sine x1' minus 'sine x1', which is 0/0. And we're back to our familiar taboo form of 0/0.

Somehow or other, we're going to have to make a refinement on our numerator that will allow us to get rid of a 0/0 form. Well, to make a long story short, if we happen to know the addition formula for the sine-- in other words, 'sine 'x1 plus delta x'' is ''sine x1' 'cosine delta x'', plus ''sine delta x' 'cosine x1''-- then we subtract off 'sine x1' and divide by 'delta x', and then we factor and collect terms.

We see what? Without any knowledge of calculus at all, but just what? By our definition of derivative, just by our definition, coupled with properties of the trigonometric functions, we wind up with the fact that 'f prime of x1' is this particular limit.

Now certainly, our limit theorems don't change. The limit of a sum is still going to be the sum of the limits. The limit of a product will still be the product of the limits. So all in all, what we have to sort of do is figure out what these limits will be. Certainly, as 'delta x' approaches 0, this will stay 'cosine x1'. Certainly this will stay 'sine x1', because 'x1' is a fixed number that doesn't depend on 'delta x'. But notice, rather interestingly, that both of my expressions in parentheses happen to take on that 0/0 form if we're not careful. Namely, if you replace 'delta x' by 0, sine 0 is 0, 0/0 is 0, and we run into trouble here if we replace 'delta x' by 0, which of course we can't do.

This is the same definition of limit as we had before. 'Delta x' gets arbitrarily close to 0, but never is allowed to get there. Well, you see, if nothing else, this motivates why we would like to learn this particular type of limit. In other words, what we would like to know is, how do you-- the 'delta x' symbol here isn't that important. 'Delta x' just stands for any number.

Notice that what we would like to know is, if you take the sine of something over that same something, and take the limit as that same something goes to 0, we would like to know what that becomes. In a similar way, we would like to know how to handle this quotient here, because notice that when 'delta x' is 0, cosine 0 is 1. This is 1 minus 1 over 0. It's another 0/0 form.

So the problem that we're confronted with is that, what we would like to do is to figure out how to handle the limit of 'sine t' over 't' as 't' approaches 0. Now, what's 't' here? 't' is a number. Remember that. This is the big pitch I've been making. We're thinking of 't' as a number. If, on the other hand, you feel more comfortable thinking in terms of traditional trigonometry-- and let's face it, the more background you've had in traditional trigonometry, the more comfortable you're going to feel using it. Let's simply agree to do this, that if it bothers you to think of this as a length divided by a length, et cetera, and that this is a length or a number, let's agree that we will go back to angles but use radian measure.

Why? Because if the angle is measured in radians, the sine of the angle 't' radians is the same as the number, the sine, of the number 't'. Well again, here's how this problem is tackled. What we do is we mark off the angle of 't' radians. Remember that we have the unit circle. And what we very cleverly do is we catch our wedge, our circular wedge, between two right triangles.

Again, without making a big issue over this, notice that this length is 'sine t', this length is 'cosine t', so the area of the small triangle is 'sine t' times 'cosine t' over 2. See, 'sine t' times 'cosine t' over 2. Now, on the other hand, since that's caught in our wedge, what is the area of our wedge? Well, since the area of the entire circle is pi-- see, pi 'R squared' and 'R' is 1-- since the area of the entire circle is pi-- and we're taking what? 't' of the 2pi. So there are two pi radians in a circle.

So the sector of the circle that we have, it's 't/2pi' of the entire circle. And by the way, this is done more rigorously and carried out in detail in the notes. Let me point out that, if we insisted on working with degrees, instead of 't/2pi', we just would have had 't/360'. Because, you see, if we're dealing with degrees, the entire angle and measure of the circle is 360 degrees, and we would have had 't/360'.

But here we've used the fact that we're dealing with radians. And finally, the bigger triangle, which includes the wedge, has, as its base, 1, so that's the radius. And since the tangent is side opposite over side adjacent, this length is 'tangent t'. And so what we have is what? That ''sine t' 'cosine t/2' must be less than this, which in turn must be less than this, multiplying through by 2 and dividing through by 'sine t'. And by the way, this hinges on the fact that 't' is positive. Again, in our notes, we treat the case where 't' is negative to arrive at the same result.

Remembering that 'tan t' is 'sine t' over 'cosine t', we wind up with this result. And now, observing that as 't' approaches 0, this approaches 1. This also approaches 1. And 't' over 'sine t' is caught between these two. We get that the limit of 't' over 'sine t' as 't' approaches 0 is 1. Now of course, since this limit is 1, the limit of the reciprocal of this will be the reciprocal of this. But what's very nice about the number 1 is that it's equal to its own reciprocal.

In other words, what we've now shown is that the limit of 'sine t' over 't' as 't' approaches 0 is 1. That, as I said before, is done in the text. We do it in our notes. But the thing that I hope this motivates is why we want to do this in the first place. Notice that this was a limit that we had to compute if we wanted to compute the derivative of the sine.

Now, the next thing was, how do we handle '1 - cosine t' over 't' as 't' approaches 0? Again, leaving the details to you to sketch in as you see fit, let me point out simply what the mathematics involved here is. You see, what we can handle is 'sine t' over 't'. That means that what we would like to do is, whenever we're given an alien form, we would somehow or other like to figure some way of factoring a sine t over t out of this thing.

When you look at '1 - cosine t', the identity, 'sine squared' equals '1 - 'cosine squared t'', should suggest itself. Now, how do you get from '1 - cosine t' to '1 - 'cosine squared t''? You have to multiply by '1 + cosine t'. And if you multiply by '1 + cosine t' upstairs, you must multiply by '1 + cosine t' downstairs.

By the way, the only time you can't multiply by something is when the thing is 0. You can't put that into the denominator. Notice that 'cosine t' is not 0 in a neighborhood of 't' equals 0. See, 'cosine t' behaves like 1 when 't' is near 0, so this is a permissible step in this particular problem.

The point is, we now factor '1 - 'cosine squared t'' as 'sine t' times 'sine t'. See, that's 'sine squared t'. We break up our 't' times '1 + cosine t' this way. Now we know that the limit of a product is the product of the limits. This we already know goes to 1. And as 't' approaches 0, from our previous limit work on the like, notice here, the limit of a quotient is the quotient of the limits, the numerator goes to 0, the denominator goes to 2, because as t approaches 0, cosine 0 is 1.

At any rate, that's 0/2, which is 0. And so this limit is 0. Now, at the risk of giving you a slight headache as I take the board down here, let me just review what it was that we did. You see, notice that, without any knowledge of these limits at all, we were able to show that whatever the derivative of 'sine x' was, it was this particular thing here. Now what we've done is we've shown that this is 1, and we've shown that this is 0. And using our limit theorems, what we now see is what? That if 'f of x' is 'sine x', 'f prime of x' is 'cosine x'.

Let me just write that down over here, that if 'y' equals 'sine x', 'dy/dx' is 'cosine x'. And again, notice how much of the calculus involved here was nothing new. It goes back to the so-called baby chapter that nobody likes, where we go back to epsilons, deltas, you see derivatives by 'delta x', et cetera. See, those recipes always remain the same.

What happens is, as you invent new functions, you need a different degree of computational sophistication to find the desired limits. By the way, once you get over these hurdles, everything again starts to go smoothly as before. For example, our chain rule. Suppose we have now that 'y' equals 'sine u', where 'u' is some differentiable function of 'x'. And we now want to find the 'dy/dx'.

Well, you see, the point is that we know that the derivative of 'sine u' with respect to 'u' would be 'cosine u'. What we want is the derivative of 'sine u' with respect to 'x'. And we motivate the chain rule the same way as we did before. It happens to be that we're dealing with the specific value called sine, but it could've been any old function. How would you differentiate 'f of u' with respect to 'x' if you know how to differentiate 'f of u' with respect to 'u'?

And the answer is, you would just differentiate with respect to 'u', and multiply that by a derivative of 'u' with respect to 'x'. In other words, we get the result what? That since 'dy/du' is 'cosine u', we get that the derivative of 'sine u' with respect to 'x' is 'cosine u' times 'du/dx'. And by the way, one rather nice application of this is that it gives us a very quick way of getting the derivative of 'cosine x'.

After all, our basic identity is that 'cosine x' is sine pi/2 minus 'x'. Again, a number or an angle, either way. As long as the measurement is in radians, it makes no difference whether you think of this as being an angle or being a number. The answer will be the same. The idea is this. To take the derivative of 'cosine x' with respect to 'x', all I have to differentiate is sine pi/2 minus 'x' with respect to 'x'.

But I know how to do that. Namely, the derivative of sine pi/2 minus 'x' is cosine pi/2 minus 'x,' and by the chain rule, times the derivative of this with respect to 'x'. Well, pi/2 is a constant. The derivative of 'minus x' is minus 1. And then, remembering that the cosine of pi/2 minus 'x' is 'sine x', I now have the result that the derivative of the cosine is minus the sine.

And again, I can do all sorts of things this way. If I want the derivative of a tangent, I could write tangent as sine over cosine. Use the quotient rule. You see, as soon as I make one breakthrough, all of the previous body of calculus comes to my rescue, so to speak. By the way, what I'd like to do now is point out why, from a physical point of view, we like circular functions to be independent of angles and the like.

With the results that we've derived so far, it's rather easy to derive one more result. Namely, let's assume that a particle is moving along the x-axis-- I'm going to start with the answer, sort of, and work backwards-- according to the rule, 'x' equals 'sine kt', where 't' is time and 'k' is a constant. Then its speed, 'dx/dt', is what? It's the derivative of 'sine kt', which is 'cosine kt', times the derivative of what's inside with respect to 't'. In other words, it's 'k cosine kt'.

The second derivative of 'x' with respect to 't', namely, the acceleration is what? How do you differentiate the cosine? The derivative of the cosine is minus the sine. By the chain rule, I must multiply by the derivative of 'kt' with respect to 't', which gives me another factor of 't' over here. Remembering that 'x' equals 'sine kt', I arrive at this particular so-called differential equation.

And what does this say? It says that 'd2x/ dt squared', the acceleration, is proportional to the displacement, the distance traveled, but in the opposite direction. You see, 'k squared' can't be negative, so 'minus 'k squared'' can't be positive. This says what? The acceleration is proportional to the displacement, but in the opposite direction. Does that problem require any knowledge of angles to solve?

Notice that this is a perfectly good physical problem. It's known as simple harmonic motion. And all I'm trying to have you see is that, by inventing the circular functions in the proper way, not only can we do their calculus, but even more importantly, if we reverse these steps, for example, we can show that, to solve the physical problem of simple harmonic motion, we have to know the so-called circular trigonometric functions.

And this is a far cry, you see, from using trigonometry in the sense that the surveyor uses trigonometry. You see, this ties up with my initial hang-up that I was telling you about at the beginning of the program. By the way, in closing, I should also make reference to something that we pointed out in our last lecture, namely, inverse differentiation.

Keep in mind, also, that as you read the calculus of the trigonometric functions, that the fact that we know that the derivative of sine u with respect to 'u' was 'cosine u' gives us, with a switch in emphasis, the result that the integral 'cosine u', 'du' is 'sine u' plus a constant. And in a similar way, since the derivative of cosine is minus the sine, the integral of 'sine u' with respect to 'u' is 'minus cosine u' plus a constant.

Be careful. Notice how the sines can screw you up. Namely, they're in the opposite sense when you're integrating as when you were differentiating. But again, these are the details which I expect you can have come out in the wash rather nicely. We can continue on this way, from knowing how to differentiate 'sine x' to the nth power. Namely, it's 'n - 1' 'x', times the derivative of 'sine x', which is 'cosine x'. We don't want this in here. That's a differential form. Without going into any detail here, notice that a modification of this shows us that, if we differentiate this, we wind up with this.

We could now take the time, if this were the proper place, to develop all sorts of derivative formulas and integral formulas. As you study your study guide, you will notice that the lesson after this is concerned with the calculus of the circular functions. My feeling is is that, with this as background, a very good review of the previous part of the course will be to see how much of this you can apply on your own to these new functions called the circular functions.

Next time, we will talk, as you may be able to guess, about the inverse circular functions and why they're important. But until next time, goodbye.

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