PROFESSOR: Hi. Our lecture today is on the one hand, deceptively simple, and on the other hand, deceptively hard. That from a certain point of view, to talk about the equations of lines and planes is, first of all, a topic that many people say, "I'm not interested in studying geometry. When are we going to get back to calculus?" And this is very analogous to our problem of part one in calculus, if you recall, where we started with analytic geometry, and the question came up, why do we need geometry when what we really wanted to study was calculus?

And the idea of graphs played a very important role in calculus. We found out, for example, also, that the relatively harmless straight line was a rather crucial curve. Namely, given any curve, no matter how random in the plane, provided only that it was smooth, we were able to approximate that curve by a sequence of tangent lines to various points. And in a similar way, one finds that planes will do for functions of two real variables what the equations of lines did for functions of a single real variable. As I say, we'll talk about that in more detail as we go along.

The other nice part about this lesson is the fact that we can now take some of our vector properties that we've learned from the past lessons and hit what I call the home run ball, once and for all now, finish off what we've started. And that is, in the study of three-dimensional geometry, the crucial characteristics are those building blocks called lines and planes. And by finding convenient recipes for expressing lines and planes, we'll be most of the way home as far as using this material is concerned. Again, we'll hit mainly the highlights today. The remainder of the material will be covered, I hope, adequately between the text and the exercises.

At any rate then, the lesson today is "Equations of Lines and Planes." And to refresh what I just said before, the little ratio-- planes are to surfaces what lines are to curves-- that we can approximate curves by tangent lines, we can approximate smooth surfaces by tangent planes.

Now what we would like to do is go back to Cartesian coordinates and find the equation of a plane. The first question that we asked before we even use any coordinate system is just how much information do you need before you can determine a plane? Well one way we know of is to know three points in the plane. Another way is analogous to the line case, where to determine a line, you needed a point and a slope. One way of determining a plane is to know what? A point in the plane and the direction of the plane. And one way of getting the direction of the plane is to fix a normal, a perpendicular to the plane.

In other words, the approach that we're going to use is, let's suppose that we know a point that we want to be in our plane. And we'll call that point p0. We'll denote it generically by (x0, y0, z0). And let's let capital N denote the vector Ai plus Bj plus Ck, which is perpendicular to our plane. OK? So we're given a vector perpendicular to the plane, we're given a point in the plane, and now in Cartesian coordinates, we would like to know the equation of the plane.

And as we do so often, we simply come back to our little diagram here that will utilize vector geometry. What we say is, OK, here's a little diagram here. Here's P0 in the plane. Here's N, normal to the plane. Now we pick any point P whatsoever, any point in the whole world whatsoever, as long as it's in space, three-dimensional space. And what we say is-- what does it mean for the point P to be in the plane? Well, in vector language, for the point P to be in the plane, I think it's rather obvious that P0-P had better be perpendicular to N. On the other hand, in dot product language, what does it mean to say that N is perpendicular to P0-P? That says that the dot product of N and P0-P must be 0.

Now just to economize space, let me utilize the notation that we've talked about in the notes and in the exercises where in Cartesian coordinates I'll abbreviate a vector in i, j, and k components, just by writing down its components. In other words, let me abbreviate N by A comma B comma C, which stands for Ai plus Bj plus Ck, et cetera.

Notice the beauty, now, of our Cartesian coordinate system. N is the vector whose components are A, B, and C. What about P0-P? Well, that's the vector whose components are x minus x0, y minus y0, and z minus z0, that beauty of Cartesian coordinates again. Consequently, I want the dot product of these two vectors to be 0. But in Cartesian coordinates, we know that to dot two vectors, we simply multiply corresponding components and add, and that results in what? A times (x minus x0) plus B times (y minus y0), plus C times (z minus z0) equals 0.

By the way, each of these steps is reversible, meaning that under these given conditions, the point (x,y,z) is in the given plane if and only if this equation here is satisfied. Another way of saying that is what? If A, B, C, x0, y0, and z0 are given constants, then A times (x minus x0) plus B times (y minus y0) plus C times (z minus z0) equals 0 is the equation of the plane which passes through the point (x0, y0, z0), and which has the vector Ai plus Bj plus Ck as its normal vector.

And perhaps the easiest way to illustrate this is again, by means of an example. Let me simply write down a linear expression. I'll write down 2 times (x minus 1) plus 3 times (y plus 2) plus 4 times (z minus 5). My claim is that this is the special case where A, B, and C are played by 2, 3, and 4, respectively. X0, y0, and z0 are played by 1, negative 2, and negative 5-- the same thing as in our previous study of ordinary two-dimensional geometry. Remember that the standard form of our equation uses a minus sign. Consequently, to use the equation, where we see y plus 2, we should rewrite that as y minus (-2).

And what we're saying is that this equation is then what? It passes through the point (1, -2, 5). And has as its normal the vector 2i plus 3j plus 4k. Actually, this is not a very difficult concept once you try a few examples and see what's happening over here.

By the way, before I go on, I just want to make a note that I'm going to return to later on. Notice, by the way, there is nothing sacred about the right-hand side of this equation being 0. Notice that somehow or other, the really important factor was 2x plus 3y plus 4z. In other words, notice that the other terms led to a constant which could have been transposed onto the right-hand side of the equation, and that somehow or other, I can change the constant. But if these multipliers in front-- the 2, 3, and 4-- stay the same, see? Notice what I'm driving at here. No matter how I-- let me go back up here for a second. No matter how I change x0, y0, z0, whatever plane this is, it still has Ai plus Bj plus Ck as a normal vector.

What I can change is what point the plane passes through. In other words, somehow or other if I leave A, B, and C fixed, but I vary x0, y0, z0, I generate a family of parallel planes. And that can be restated somewhat differently. Well I guess if it wasn't somewhat differently, that wouldn't be called "restated," would it? Let's just note this way. For fixed A, B, and C, the equation Ax plus By plus Cz equals D. See, forget about the 0 on the right-hand side. Let D be an arbitrary constant. What I'm saying is that this equation is a family of parallel planes. And why is it a family of parallel planes? Because every plane in this family has, as a normal vector, Ai plus Bj plus Ck. Ai plus Bj plus Ck. OK?

The second point I would like to emphasize about the equation of our plane is that it's called a linear equation, meaning-- and I don't know why I suddenly switched to small letters here, but that certainly doesn't make any difference. With A, B, C, and D as constants, observe that Ax plus By plus Cz equals D is what we call a linear algebraic equation in the variables x, y, and z. Namely, each variable appears multiplied only by a constant. And we add these things up.

And notice that in a way, a plane should be linear, meaning there's no curvature to a plane once it's fixed in space. And this sort of generalizes the idea of the line. Remember, the general definition of a line was of the form what? Ax plus By equals a constant. That was a two-dimensional linear equation. The plane is a three-dimensional linear equation.

And one of the subjects that we'll return to later in the course, but I just mention it in passing now-- is that even though you can't draw in more than three-dimensional space, if you have 15 variables, you can certainly have a linear equation in 15 unknowns. And the interesting point in calculus of several variables is that even when you run out of pictures-- when you can't draw the situation-- the linear equation plays a very, very special role in the development of calculus of several variables, analogous to what a line does for a curve in the case of one variable and what a plane does for a surface in the case of two variables.

What I wanted to emphasize though, also-- and we'll come back to this in very short order-- is that a plane has two degrees of freedom. Meaning what? That in a plane, observe that given the linear equation, you have what? One linear equation and three unknowns. It requires that you pick two of the variables before the rest of the equation is determined. In other words, if given $x + 2y + 3z = 6$, if I say, let x be 15, notice that I have what? $15 + 2y + 3z = 6$, which gives me one equation and the two unknowns y and z. That to actually uniquely fix anything, I must specify what two of the three variables are. In other words, in this equation, I can choose any two of the three variables at random, and solve for the third.

By the way, if anybody is having difficulty understanding the difference between the 6 being on the right-hand side as we have it now and the 0 as we used it originally, notice that we can very quickly find a point which passes--is playing passes through. For example, among other things, just set y and z equal to 0, in which case x is 6. So certainly one point in this plane is 6, 0, 0. What is a vector perpendicular to this plane? It's the vector 1i plus 2j plus 3k. So using the standard form, we could write what? $(x - 6) + 2 * (y - 0) + 3 * (z - 0) = 0$.

Notice, by the way, that algebraically, these two equations are equivalent. But that in this form, this tells me what? This specifies a point that the plane passes through, namely what? This is the plane that passes through 6, 0, 0), and has the vector i plus 2j plus 3k as a normal. By the way, you could do this in different ways. Some person might say, why couldn't you have transposed the 6 over here, then taken 3z minus 6? You see? Why don't you let x and y be 0 and solve this equation, and get that z equals 2?

In other words, isn't (0, 0, 2) also a point in the plane? And the answer is, yes it is.

And you could have written the equation now as what? x minus 0 plus twice y minus 0 plus 3 times z minus 2 equals 0. That would be what? The equation of the plane that passed through (0, 0, 2), and had as its normal i plus 2j plus 3k. Of course what happens is that (6, 0, 0) and (0, 0, 2) belong to the same plane.

I mean, that's another thing to keep in mind here, that the plane that we're talking about passes through more than one point. So x0, y0, z0 can be played by an infinity of choices. At any rate, let's let that go now as the equation of our plane, and let's talk now about the equation of a line. That's a plane. Let's talk about the equation of a line.

How do we determine a line? In two-dimensional space, we said we needed to know a point on the line and the slope. And another way of saying that is, we need to know a point on the line. And we would like to know a line parallel to the given line. In vector language, what we say is, OK, let's suppose we're given the line l And we know that the vector V whose components are A, B, and C, that that vector is parallel to l and that the point P0 whose coordinates are (x0, y0, z0), that that point is on the line l.

Then the question is, how do we find the equation of the line l? And again, vector methods come to our aid very nicely. What we say is, let's pick any other point P in space. All right? What does it mean if the point P is on the line l? If the point P is on the line l, since we want to use vector methods, let's simply observe that since l is parallel to V, the vector P0-P, being parallel to V, must be a scalar multiple of V. That's what parallel means for vectors, scalar multiple. So P0-P is equal to some constant times V.

And let me pause here for a moment to point out that this constant is really a variable. That sounds awful. How can a constant be a variable? What I mean of course, is that P was any point in this line. Notice that t determines the length of P0-P, and how long P0-P is, is going to depend on where I choose P. In other words, for different choices of P, I get a different scalar multiple. And by the way, if I choose P on the wrong side of P0, as I've deliberately done over here, notice that P0-P has the opposite sense of V. So that t can even be negative. In other words, not only is t a variable, but if it's negative, it means that P0-P has the opposite sense of V. If it's positive, they have the same sense.

But I'm not going to belabor that point. What I'm now going to do is in Cartesian coordinates, see what this equation tells me. And right away, it tells me what? That P0-P is that vector whose components are x minus x0, y minus y0, and z minus z0. What vector is t times V? Well, V, we saw, had as components, A, B, and C. And in Cartesian coordinates, multiplying a vector by a scalar simply multiplies each component by that scalar. So in other words, t times V is the vector whose components are tA, tB, and tC.

We also know in Cartesian coordinates that the only way that two vectors can be equal is component by component. And that tells us what? That x minus x0 must equal (t times A. y minus y0 must equal t times B. And z minus z0 must equal t times C. That's these three equations here.

What do all of these three equations have in common numerically? They all have the factor t. And consequently, I can solve each of these three equations for t. Namely, what? Divide both sides of this equation by A, both sides of this equation by B, both sides of this equation by C, being very careful that neither A, B, nor C are 0. By the way, if they are 0, straightforward ramifications take place that we'll leave for the textbook to explain. Don't worry about that part right now. We don't want to get bogged down in that.

But at any rate, if we now go from here to see what that says, we now wind up with the standard equation of the straight line. Namely, if you have (x minus x0) over A equals (y minus y0) over B equals (z minus z0) over C equals some constant t, that particular form is called the standard equation for a straight line. What straight line is it? It's the line which passes through the point (x0, y0, z0) and is parallel to the vector Ai plus Bj plus Ck.

By means of an example, (x minus 1) over 4 equals (y minus 5) over 3 equals (z minus 6) over 7 is the equation. It's one equation, really. It's the equation of a line which has what property? It passes through the point (1, 5, 6). And it's parallel to the vector 4i plus 3j plus 7k. And by the way, I have to be very, very on my guard here. There's something very deceptive here. The equation of a plane and a line are very, very much different. But they look enough alike so it may confuse you.

You know, it reminds me of my daughter, who I get a lot of stories from, was eating a sandwich one day. And I asked her what kind of a sandwich she was eating. And she said it was like a peanut butter and jelly sandwich. And I never heard of a sandwich that was like a peanut butter and jelly sandwich. So I looked at it to see what it was, and it was ham and cheese. And I say, why did you say it was like peanut butter and jelly? And she says well, it was two things in it. All right?

Look at it. The equation of a line and the plane have three things in it-- x, y, and z. But to juxtaposition these, let me write down the two things that may look confusing. Let's suppose I write this down. You see what I'm doing now? What I'm doing now is I'm changing the equal signs here to plus signs and bringing up the denominators here. See, this is a line. This is a plane. What plane is this? This is the plane which passes through the point 1 comma 5, comma 6, and has the line of the vector 4i plus 3j plus 7k as its normal.

How can I best explain this to you to keep this straight in your mind? Well, I think the easiest way-- and again, notice what I'm saying, see the x, y's, and z's here, the x, y's and z's here. Which is which? The easiest way is to keep track of degrees of freedom. Remember in the plane, we said look at it. You can pick two of the variables at random and solve for the third.

I claim in this system-- in this system here, there is only one degree of freedom. The line has one degree of freedom. Namely, let's repeat this example, so we don't have to keep looking back to the board here. Let's take (x minus 1) over 4 equals (y minus 5) over 3 equals (z minus 6) over 7. And since we don't like to work with fractions, I'll pick a number that works out nicely.

I say, OK, let's see what happens when x is 9. Now here's the whole point. As soon as I say that x equals 9, as soon as I let x equal 9, this is fixed. Right? In fact, what does it become fixed as soon as I do this? As soon as x equals 9, (x minus 1) over 4 is 2. Now notice that (y minus 5) over 3 has to equal 2. Well I have no more choice then. If (y minus 5) over 3 has to equal 2, and also (z minus 6) over 7 has to equal 2-- you see what I'm saying here? This fixes the fact that y must be 11, and that z must be 20. In other words, the choice of x equals 9 forces me to make y equal 11 and z equal 20. One degree of freedom.

And by the way, if you want to see this thing from a geometrical point of view, what we're saying is, visualize this line cutting through space, all right? Notice that directly on that line, only one point will have its x-coordinate equal to 9. And what we're saying is, the point on that line whose x-coordinate is 9 is the point 9 comma 11 comma 20. OK? One degree of freedom again. That's very, very crucial for you to see.

By the way, I guess one thing that bothers a lot of students is the fact that they read this as two separate equations. They say, you know, why isn't this (x minus 1) over 4 equals (y minus 5) over 3? Why can't I treat that as one equation? Or why couldn't I take (y minus 5) over 3 equals (z minus 6) over 7? Or why couldn't I take (x minus 1) over 4 and say that equals (z minus 6) over 7? And the answer is, that by itself isn't enough. But rather than give you a negative answer, let me give you a positive one.

Let me close today's lesson with this particular illustration. Suppose we had solved (x minus 1) over 4 equals (y minus 5) over 3. What we would have obtained is the equation 4y minus 3x equals 17. Now this is very dangerous. When you look at the equation 4y minus 3x equals 17, I'll bet you dollars to doughnuts you tend to think of this as a line rather than as a plane. But the interesting thing is, notice that the way we got this equation was ignoring the z-coordinate of our points. And what we're really saying is, let's forget about the z-coordinate. In other words, 4y minus 3x equals 17 may be viewed as a line, but in this particular case, it's a plane.

In fact, what plane is it? It's the plane that goes through the line 4y minus 3x equals 17, which lies on the xy-plane. It's the plane that goes through that line perpendicular to the xy-plane. By the way, again, if you go back to part one of this course where we stress sets, the language of sets comes to our rescue very nicely. The difference between whether 4y minus 3x equals 17 is a plane or whether it's a line hinges on whether we're talking about the set of pairs x comma y such that 4y minus 3x equals 17, or whether we're talking about the set of triplets x, y, z such that 4y minus 3x equals 17.

In this particular example, we're talking about points in space. In other words, our universe of discourse are the points x comma y comma z, not the points x comma y. Well anyway, rather than to belabor this point, what I'm saying is, in the same way that this equation represents a plane, in a similar way, had we equated y minus 5 over 3 equals z minus 6 over 7, we would've obtained the plane 7y minus 3z equals 17.

So if you don't like to look at our set of three equations, if you'd like to look at these three equations in pairs, you see, if you want to look at these three equations in pairs, another way of saying it is this, that the triple equality-- the x minus 1 over 4 equals y minus 5 over 3 equals z minus 6 over 7-- that that may be viewed as the intersection of the two planes-- namely the plane determined by this equation and the plane determined by this equation.

Of course, someone can also say, isn't there a plane determined by this one and this one? And the answer is yes, there is. Notice that whereas you have three equalities here, only two of them are independent. Namely, as soon as the first equals the second and the second equals the third, the first must equal the third. OK?

But the whole idea is, can you now see the difference? The easiest way I know of to distinguish the difference between the equation of a line and a plane. The plane has two degrees of freedom. The line has but one degree of freedom. And that triple equality says as soon as you've picked one of the unknowns, you've determined all of the others. Whereas that string of plus signs says that once you've determined one, you still have some freedom left.

Now what we're going to do is next time start a new phase of vectors. For the time being, what we have now done is finished, at least for the moment, our preliminary investigation of three-dimensional space as seen through the eyes of Cartesian coordinates.

At any rate, until next time, goodbye.

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