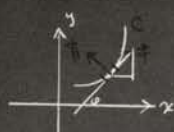


Unit 2: Tangential and Normal Vectors

1. Lecture 2.020

Tangential and Normal Vectors



$\vec{r} = \cos \theta \vec{i} + \sin \theta \vec{j}$

$\frac{d\vec{r}}{d\theta} = -\sin \theta \vec{i} + \cos \theta \vec{j}$

$= \cos(\theta + 90^\circ) \vec{i} + \sin(\theta + 90^\circ) \vec{j}$

$= \vec{N}$

$\vec{v} = \frac{d\vec{r}}{dt} = \frac{ds}{dt} \vec{T}$


$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2s}{dt^2} \vec{T} + \frac{ds}{dt} \frac{d\vec{T}}{dt}$

$= \frac{d^2s}{dt^2} \vec{T} + \frac{ds}{dt} \left[\frac{d\vec{T}}{ds} \frac{ds}{dt} \right]$

$\vec{a} = \frac{d^2s}{dt^2} \vec{T} + k \left(\frac{ds}{dt} \right)^2 \vec{N}$

$\left| \frac{d\vec{T}}{ds} \right| = k = \text{curvature}$

$\frac{1}{k} = \rho = \text{radius of curvature}$



a.

More generally,

$|\vec{T}| = 1 \rightarrow \frac{d\vec{T}}{ds} \perp \vec{T}$

$\therefore \text{Let } \vec{N} = \frac{d\vec{T}}{ds} / \left| \frac{d\vec{T}}{ds} \right|$

$\therefore \frac{d\vec{T}}{ds} = \left| \frac{d\vec{T}}{ds} \right| \vec{N}$

$\left| \frac{d\vec{T}}{ds} \right| = \left| \frac{d\vec{T}}{ds} \right|$

$\therefore \frac{d\vec{T}}{ds} = k \vec{N}$

3-dimensional Curves

$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

is vector form of the curve

$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$

Again,


$\vec{T} = \frac{d\vec{r}}{ds}$

$\vec{B} = \vec{T} \times \vec{N}$

$(\vec{k} = \vec{i} \times \vec{j})$

$\left| \frac{d\vec{B}}{ds} \right|$ measures the "twist" (torsion) of the curve.

If $\frac{d\vec{B}}{ds} = \vec{0}$, the curve lies in a plane.



b.

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2. Read Thomas, Sections 14.3, 14.4, 14.5 (Note: if you are not too familiar with these vectors, you may find it helpful to view the lecture a second time, after you have read the appropriate sections of Thomas.)

3. Exercises:

2.2.1(L)

A particle moves in the plane according to the equation of motion:

$$\vec{R} = t \vec{i} + \frac{1}{3}(t^2 + 2)^{3/2} \vec{j}.$$

- a. Find the tangent vector to the curve at any point $(x(t), y(t))$.
b. How far does the particle travel between $t = 0$ and $t = 6$?

2.2.2

A particle moves according to the equation of motion:

$$\vec{R} = \frac{t^2}{2} \vec{i} + \frac{1}{3}(2t + 1)^{3/2} \vec{j}.$$

- a. Find the position, velocity, speed, and acceleration of the particle at $t = 4$.
b. How far does the particle travel between $t = 0$ and $t = 4$?
c. Find a unit tangent vector to the curve $t = 4$.

2.2.3(L)

- a. Show that if $\vec{v} = \frac{ds}{dt} \vec{T}$, then the acceleration, \vec{a} , is given by

$$\vec{a} = \frac{d^2s}{dt^2} \vec{T} + \kappa \left(\frac{ds}{dt}\right)^2 \vec{N} \text{ where } \vec{T} \cdot \vec{N} = 0 \text{ and } \kappa = \left| \frac{d\vec{T}}{ds} \right|.$$

- b. Use the expressions for \vec{a} and \vec{v} in part a. to find an expression for $\vec{v} \times \vec{a}$. From this deduce an expression for κ in terms of \vec{v} and \vec{a} .
c. Use b. to find the curvature of the path followed by the particle which moves according to the equation in Exercise 2.2.2 at $t = 4$.

2.2.4

Use part a. to Exercise 2.2.3 to show that the acceleration of a particle is always normal to its path of motion if and only if its speed is constant.

2.2.5(L)

Show that if the equation of a curve is given in the Cartesian form $y = f(x)$, the curvature, κ , as defined in Exercise 2.2.3 a. can be computed from the "recipe"

$$\kappa = \left| \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}} \right| .$$

2.2.6

- Use the result of Exercise 2.2.5 to find the curvature of $y = e^{2x}$ at the point $(0,1)$.
- If a particle moves according to the equation $\vec{R} = t \vec{i} + e^{2t} \vec{j}$, its path of motion in Cartesian coordinates is $y = e^{2x}$ (this should be easily checked by the reader). Use this fact together with the result of Exercise 2.2.3 b. to solve part a. by a different method.

2.2.7

- Find the curvature of $y = ax + b$, where a and b are given constants.
- Find the curvature of $y = \sqrt{a^2 - x^2}$ where a is a positive constant.

2.2.8(L)

A particle moves according to the equation

$$\vec{R} = t \vec{i} + (t^2 + 1) \vec{j} .$$

- Find its normal and tangential components of acceleration at any time t .
- Find the curvature of the path of motion both from the results of part a. and from Exercise 2.2.3 b.

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2.2.9

Compute the tangential and normal components of acceleration of a particle which moves according to the equation, $\vec{R} = t^3 \vec{i} + \sin t \vec{j}$.

2.2.4

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<http://ocw.mit.edu>

Resource: Calculus Revisited: Multivariable Calculus
Prof. Herbert Gross

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