Unit 3: Space Curve and the Bi-Normal Vector (Optional

The exercises in this unit extend the concept of tangential and normal vectors in the plane into a 3-dimensional analog. The results discussed here are not needed in any of our later work, and it is for this reason that we have made this unit optional. On the other hand, the material presented here does have practical application, and much of it serves as an introduction to Differential Geometry. In addition, by trying to solve the exercises in this unit, you will get additional review and reinforcement of the major principles of the previous unit.

1. Exercises:

2.3.1(L)

a. Mimic our previous definitions to show what it means for the space curve $\vec{R}(t)$ to be a continuous function of t. In particular, if the equation is given in the form, $\vec{R}(t) = x(t)\vec{1} + y(t)\vec{j} + z(t)\vec{k}$, show that $\vec{R}$ is a continuous function of t as soon as x(t), y(t), and z(t) are continuous functions of t.

b. (1) Again, mimicking the 2-dimensional case, find formulas for velocity, speed, and acceleration of a particle that moves along the curve C according to the equation, $\vec{R} = x(t)\vec{1} + y(t)\vec{j} + z(t)\vec{k}$.

(2) In particular, find $\vec{v}$, $\vec{a}$, and $|\vec{v}|$ if C is given by $\vec{R} = \cos 3t\ \vec{1} + \sin 3t\ \vec{j} + t\ \vec{k}$.

(3) In the example above, show that $\vec{v} \cdot \vec{a} = 0$ and explain why this had to happen.

c. With the curve C as defined in b.(2) above, find a unit tangent vector to the curve.

d. Letting $\vec{T}$ denote the unit tangent vector $\frac{d\vec{R}}{ds}$, show that for a space curve, $\frac{d\vec{T}}{ds} = \kappa\ \vec{N}$, where $\vec{N}$ is a unit vector which has the same sense and direction as $\frac{d\vec{T}}{ds}$. In particular, determine κ and $\vec{N}$ for the curve given in b.(2) above.

e. In the same way that we need three unit vectors in Cartesian coordinates for 3-space ($\vec{1}$, $\vec{j}$, and $\vec{k}$), we now need a third unit vector to go along with $\vec{T}$ and $\vec{N}$. To this end we define a <u>binormal</u> <u>vector</u>, $\vec{B}$, by $\vec{B} = \vec{T} \times \vec{N}$. Show that $\frac{d\vec{B}}{ds}$ can be written in the form, $\tau\vec{N}$, and interpret this geometrically.

2.3.2

a. Show that the formula $\kappa = \dfrac{|\vec{v} \times \vec{a}|}{|v|^3}$ applies to 3-space also.

b. Given the curve $\vec{R} = t\vec{i} + 2t^2\vec{j} + t^3\vec{k}$, find the equation of the plane which is perpendicular to $\vec{B}$ at the point corresponding to $t = 1$.

c. Compute the curvature of the above curve at $t = 1$.

d. Express $\vec{B}$ and $\vec{N}$ for this same curve at $t = 1$.

2.3.3

Our space geometry now includes the formulas: $\dfrac{d\vec{R}}{ds} = \vec{T}$, $\dfrac{d\vec{T}}{ds} = \kappa\vec{N}$, and $\dfrac{d\vec{B}}{ds} = \tau\vec{N}$. Complete this list by computing $\dfrac{d\vec{N}}{ds}$.

2.3.4

A curve is given in the form $\vec{R} = \vec{R}(s)$ where s denotes arclength.

a. Compute $\dfrac{d\vec{R}}{ds}$, $\dfrac{d^2\vec{R}}{ds^2}$ and $\dfrac{d^3\vec{R}}{ds^3}$ in terms of $\vec{T}$, $\vec{N}$, and $\vec{B}$.

b. From a. compute

$$\frac{d\vec{R}}{ds} \cdot \left(\frac{d^2\vec{R}}{ds^2} \times \frac{d^3\vec{R}}{ds^3}\right)$$

to find the formula for τ.

c. Use b. to find formulas for κ and τ if $\vec{R}$ is expressed in the form $\vec{R} = x(s)\vec{i} + y(s)\vec{j} + z(s)\vec{k}$.

2.3.5

a. Assume $\vec{R}$ is given as a function of time, t, rather than of arclength s. Mimic the discussion on Exercise 2.2.4 to show that

$$\vec{v} \cdot \left(\vec{a} \times \frac{d\vec{a}}{dt}\right) = -\kappa^2\tau\left(\frac{ds}{dt}\right)^4$$

(continued on next page)

2.3.5 continued

$$\tau = \frac{-|\vec{v}|^2\left[\vec{v}\cdot(\vec{a}\times\frac{d\vec{a}}{dt})\right]}{|\vec{v}\times\vec{a}|^2}.$$

b. Compute $|\tau|$ for the curve $\vec{R} = t\vec{i} + 2t^2\vec{j} + t^3\vec{k}$ at $t = 1$.