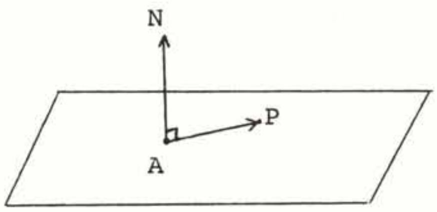
Unit 6: Equations of Lines and Planes

1.6.1(L)

From Exercise 1.5.2 we know that a vector $\vec{N}$ normal to the desired plane is (-14,10,9). We also know that (1,2,3) is in the plane. Hence, P(x,y,z) is in the plane if and only if $\vec{N} \cdot \vec{AP} = 0$. Pictorially,



Since $\vec{AP} = (x - 1, y - 2, z - 3)$ and $\vec{N} = (-14,10,9)$, $N \cdot AP = 0$ implies that

$$-14(x - 1) + 10(y - 2) + 9(z - 3) = 0$$

or $\underline{-14x + 10y + 9z = 33}$. (1)

As a "quick" check of (1), we know that it must be satisfied when x = 1, y = 2, z = 3 [since (1,2,3) belongs to the plane], when x = 3, y = 3, z = 5 [since (3,3,5) belongs to the plane], and when x = 4, y = 8, z = 1 [since (4,8,1) belongs to the plane].

We obtain

(1) $-14(1) + 10(2) + 9(3) = -14 + 20 + 27 = 33$

(2) $-14(3) + 10(3) + 9(5) = -42 + 30 + 45 = 33$

(3) $-14(4) + 10(8) + 9(1) = -56 + 80 + 9 = 33$

and we see that equation (1) is satisfied in each case.

You may have noticed that we were solving problems of this type in Unit 4. One of the major differences between then and now is that earlier we talked about a vector being perpendicular to a

1.6.1(L) continued

plane even though we may not have been able to compute such a vector. Now, by use of the cross product, we may actually determine a vector which is perpendicular to a given plane.

Quite in general, to find the equation of a plane determined by three non-collinear points A, B, and C, we form the vectors $\vec{AB}$ and $\vec{AC}$, whereupon $\vec{AB} \times \vec{AC}$ yields a normal vector $\vec{N}$ to the plane. We then pick any point P(x,y,z) in space and form, say, the vector $\vec{AP}$. The equation of the plane in vector form is then simply

$$\vec{N} \cdot \vec{AP} = 0. \quad (2)$$

The "standard equation" follows from (2) merely by expressing (2) in Cartesian coordinates.

In summary, in Cartesian coordinates

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

represents the equation of the plane which has the vector $a\vec{i} + b\vec{j} + c\vec{k}$ as the normal vector and which passes through the point (x_0,y_0,z_0), for in this case $\vec{N} \cdot \vec{AP} = (a,b,c)\cdot(x - x_0, y - y_0, z - z_0) = a(x - x_0) + b(y - y_0) + c(z - z_0)$.

1.6.2

We are probably conditioned to think of $y = 2x$ as denoting the equation of a line. In terms of what we call the universe of discourse, $y = 2x$ denotes a line if we think of it as an abbreviation for

$$\{(x,y): y = 2x\}. \quad (1)$$

On the other hand, if we think of it as an abbreviation for

$$\{(x,y,z): y = 2x\} \quad (2)$$

then we should have a plane rather than a line. That is, the basic difference between (1) and (2) is that in (1) our elements

1.6.2 continued

are ordered pairs (two-dimensional) while in (2) they are ordered triplets (three-dimensional).

With a little geometric intuition we may recognize that equation (2) is the plane which passes through the line $y = 2x$ and is perpendicular to the xy-plane. That is, equation (2) seems to say that a point belongs to our collection as soon as $y = 2x$ regardless of the choice of z.

Our main aim in this section is to show how this result actually follows from the standard equation of a plane. Specifically, we rewrite $y = 2x$ as

$$2x - y = 0$$

and to bring in the third dimension, we rewrite this as

$$2x - y + 0z = 0.$$

Finally, to put this into the standard form, we rewrite it as

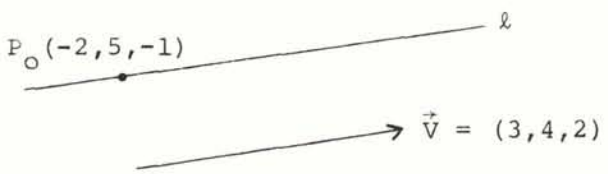
$$2(x - 0) + (-1)(y - 0) + 0(z - 0) = 0$$

and this is the plane which passes through (0,0,0) and has the vector $2\vec{i} - \vec{j}$ as a normal.

As a check notice that the slope of $2\vec{i} - \vec{j}$ is -1/2 and this is the negative reciprocal of 2 which is the slope of the line $y = 2x$.

1.6.3(L)

a. The line passes through (-2,5,-1) and is parallel to the vector $3\vec{i} + 4\vec{j} + 2\vec{k}$ (see note at the end of this exercise for a further discussion). Pictorially, we have:



1.6.3(L) continued

Now P(x,y,z) is on ℓ if and only if $\overrightarrow{P_oP}$ is parallel to $\vec{V}$. In vector language this says that $\overrightarrow{P_oP}$ and $\vec{V}$ must be scalar multiples of one another. That is, P is on ℓ if and only if there exists a number (scalar) t such that

$$\overrightarrow{P_oP} = t\vec{V}. \tag{1}$$

Writing (1) in Cartesian coordinates we have

$$(x - (-2), y - 5, z - (-1)) = t(3,4,2)$$

$$\text{or } (x + 2, y - 5, z + 1) = (3t, 4t, 2t). \tag{2}$$

Hence, by comparing components in (2), we have

$$\left.\begin{array}{l} x + 2 = 3t \\ y - 5 = 4t \\ z + 1 = 2t \end{array}\right\} \text{ or } \left\{\begin{array}{l} \frac{x+2}{3} = t \\ \frac{y-5}{4} = t \\ \frac{z+1}{2} = t \end{array}\right. \tag{3}$$

Thus (x,y,z) belongs to our line if and only if

$$\frac{x+2}{3} = \frac{y-5}{4} = \frac{z+1}{2} \qquad (= t). \tag{4}$$

Quite in general

$$\frac{x - x_o}{a} = \frac{y - y_o}{b} = \frac{z - z_o}{c} \qquad (= t)$$

represents the line which passes through the point (x_o,y_o,z_o) and is parallel to the vector $a\vec{i} + b\vec{j} + c\vec{k}$.

b. The point in question is determined from (4) with z = 0. In this event

$$\frac{x+2}{3} = \frac{y-5}{4} = \frac{0+1}{2} = \frac{1}{2}$$

$$\frac{x+2}{3} \rightarrow x = -2 + \frac{3}{2} = \frac{-1}{2}$$

1.6.3(L) continued

$\frac{y-5}{4} \to y = 5 + 2 = 7.$

Therefore, the required point is $(-\frac{1}{2},7,0)$.

THE SLOPE OF A LINE IN THREE-DIMENSIONAL CARTESIAN COORDINATES

The statement that a line is determined by two distinct points or by one point and the slope is independent of any particular coordinate system. It is also independent of whether we are dealing in two-dimensional space or three-dimensional space. In the solution of the exercise just concluded, we talked about the slope of a line in three-dimensional space indirectly by specifying that the line be parallel to a given vector.

This idea is very much akin to a traditional topic known as directional cosines. Given a line in space, we considered the line parallel to the given line that passed through the origin (the advantage to vector notation is that if we view the line as vectorized we may assume that it passes through the origin without having to think of it as a different vector). In any event, the slope of the line was well-defined as soon as we could measure the angle it made with each of the three coordinate axes. Traditionally, the convention is to let α denote the angle the line makes with the positive x-axis, β the angle it makes with the positive y-axis, and γ the angle it makes with the positive z-axis.

The point is that if we use vector notation it is particularly easy to find the cosines of these three angles (even without vectors there are simple formulas for cosines of these angles in terms of x, y, and z). Namely, we have already seen that if $\vec{A}$ and $\vec{B}$ denote unit vectors, then $\vec{A} \cdot \vec{B}$ is the cosine of the angle between A and B. That is,

$$\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}| \cos \angle(\vec{A}, \vec{B})$$

and by definition of unit vectors, both $|\vec{A}|$ and $|\vec{B}|$ equal 1 in this case, and the result follows.

In other words, then, if $\vec{u}$ denotes a unit vector parallel to a given line then $\vec{u} \cdot \vec{1}$ represents the cosine of the angle between

1.6.3(L) continued

$\vec{u}$ and $\vec{i}$; and since $\vec{u}$ has the direction of the given line and since $\vec{i}$ has the direction of the positive x-axis, if follows that $\vec{u} \cdot \vec{i}$ is also the cosine of the angle between the line and the positive x-axis.

That is,

$$\cos\alpha = \vec{u} \cdot \vec{i}.$$

Similarly,

$$\cos\beta = \vec{u} \cdot \vec{j}$$

and

$$\cos\gamma = \vec{u} \cdot \vec{k}.$$

1.6.4

Since the line

$$\frac{x - 1}{6} = \frac{y + 2}{3} = \frac{z - 4}{2}$$

is parallel to the vector $6\vec{i} + 3\vec{j} + 2\vec{k}$, and since $|6\vec{i} + 3\vec{j} + 2\vec{k}| = \sqrt{6^2 + 3^2 + 2^2} = 7$, we have that $\vec{u} = \frac{6}{7}\vec{i} + \frac{3}{7}\vec{j} + \frac{2}{7}\vec{k}$ is a unit vector along the given line. In accord with our note at the end of Exercise 1.6.3, we have:

$$\cos\alpha = \vec{u} \cdot \vec{i} = \frac{6}{7}$$

$$\cos\beta = \vec{u} \cdot \vec{j} = \frac{3}{7}$$

$$\cos\gamma = \vec{u} \cdot \vec{k} = \frac{2}{7}$$

Therefore, $\alpha = \cos^{-1}\frac{6}{7} \approx 31°$

$\beta = \cos^{-1}\frac{3}{7} \approx 64° \; 40'$

$\gamma = \cos^{-1}\frac{2}{7} \approx 73° \; 20'$

1.6.5

The line in question is given by the equation

$$\frac{x+1}{2} = \frac{y-3}{3} = \frac{z+2}{5} \qquad (=t). \tag{1}$$

This equation may be written parametrically by

$$\left.\begin{aligned} \frac{x+1}{2} &= t \\ \frac{y-3}{3} &= t \\ \frac{z+2}{5} &= t \end{aligned}\right\}$$

$$\text{or } \left.\begin{aligned} x &= 2t - 1 \\ y &= 3t + 3 \\ z &= 5t - 2. \end{aligned}\right\} \tag{2}$$

In Exercise 1.6.1(L) we saw that the plane was given by the equation

$$-14x + 10y + 9z = 33. \tag{3}$$

Since the point we seek belongs to both the line and the plane its coordinates must satisfy both equations (2) and (3). Putting the values of x, y, and z from (2) into (3) we obtain:

$$-14(2t - 1) + 10(3t + 3) + 9(5t - 2) = 33$$

and this yields

$$47t + 26 = 33$$

$$\text{or } t = \frac{7}{47}. \tag{4}$$

Putting the value of t in (4) into (2), we see that

$$x = \frac{14}{47} - 1 = \frac{-33}{47}$$

1.6.5 continued

$y = \frac{21}{47} + 3 = \frac{162}{47}$

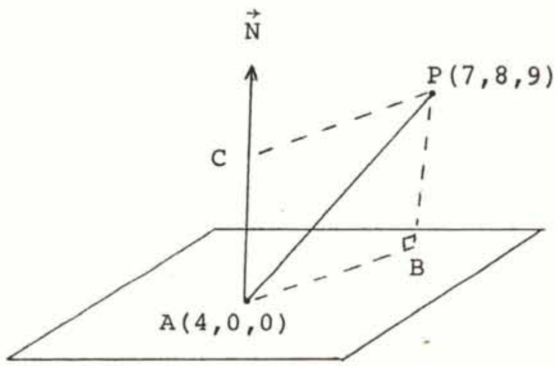
$z = \frac{35}{47} - 2 = \frac{-59}{47}$

so that $(\frac{-33}{47}, \frac{162}{47}, \frac{-59}{47})$ is the point at which the line meets the plane.

1.6.6(L)

a. Since the equation of the plane is $2x + 3y + 6z = 8$, we see that $\vec{N} = (2,3,6)$ is normal to the plane.* In particular $\vec{u}_N = (\frac{2}{7}, \frac{3}{7}, \frac{6}{7})$ is a unit vector normal to the plane.

We can now locate a point in the plane by arbitrarily picking values for two of the variables in $2x + 3y + 6z = 8$ and then solving the equation for the third. For example, if we let $y = z = 0$, then $x = 4$. Hence, A(4,0,0) is in the plane, Pictorially,



We seek the length of PB which in turn is the length of $\vec{AC}$. Vectorially, this is the magnitude of the projection of $\vec{AP}$ onto $\vec{N}$. [We say "magnitude" since if the measure of ∢PAC exceeds 90° (that is, if P is "below" the plane) the projection will fall in the opposite sense of $\vec{N}$.]

*In general to convert $ax + by + cz = d$ into the standard form we may (if $a \neq 0$) rewrite the equation as

$(ax - d) + by + cz = 0$

or $a(x - \frac{d}{a}) + b(y - 0) + c(z - 0) = 0$.

Hence, our plane passes through $(\frac{d}{a}, 0, 0)$ and has $a\vec{i} + b\vec{j} + c\vec{k}$ as a normal vector.

1.6.6(L) continued

Now we have already seen that to project a vector $\vec{A}$ onto a vector $\vec{B}$ the magnitude of the projection is given by $|\vec{A} \cdot \vec{u}_B|$ where $\vec{u}_B$ is a unit vector in the direction of $\vec{B}$. Hence, the distance we seek is given by

$$|\vec{AP} \cdot \vec{u}_n|$$

which is equal to

$$\left|(7 - 4,\ 8 - 0, 9 - 0)\cdot\left(\frac{2}{7},\frac{3}{7},\frac{6}{7}\right)\right| =$$

$$\left|(3,8,9)\cdot\left(\frac{2}{7},\frac{3}{7},\frac{6}{7}\right)\right| =$$

$$\left|\frac{6}{7} + \frac{24}{7} + \frac{54}{7}\right| =$$

$$\left|\frac{84}{7}\right| =$$

$$12.$$

b. The line through P perpendicular to the plane is, of course, the line through P parallel to $\vec{N}$ (where $\vec{N}$ is as in a.). Hence the equation of this line is:

$$\frac{x - 7}{2} = \frac{y - 8}{3} = \frac{z - 9}{6} \qquad (= t). \qquad (1)$$

In parametric form this becomes:

$$\left.\begin{aligned} x &= 2t + 7 \\ y &= 3t + 8 \\ z &= 6t + 9. \end{aligned}\right\} \qquad (2)$$

If B(x,y,z) is the point at which the line through P and parallel to $\vec{N}$ intersects the plane $2x + 3y + 6z = 8$, then B must satisfy both the equation of the plane and equation (2). That is, we must have:

$$2(2t + 7) + 3(3t + 8) + 6(6t + 9) = 8.$$

1.6.6(L) continued

Hence,

$49t + 92 = 8$

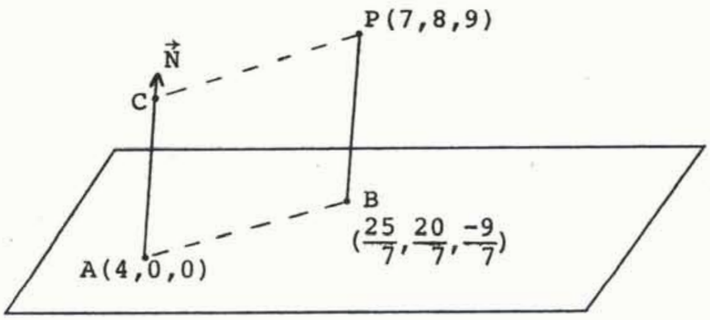
or $t = -\frac{84}{49} = -\frac{12}{7}$. (3)

Putting (3) into (2) yields

$$\left.\begin{aligned} x &= -\frac{24}{7} + 7 = \frac{25}{7} \\ y &= -\frac{36}{7} + 8 = \frac{20}{7} \\ z &= -\frac{72}{7} + 9 = -\frac{9}{7}. \end{aligned}\right\} \quad (4)$$

From (4) we find that B is the point $\left(\frac{25}{7}, \frac{20}{7}, \frac{-9}{7}\right)$.

Again pictorially,



By the way, since $\overline{BP} = \overline{AC}$, the result of b. should supply us with an alternative solution for a. Namely,

$$\overline{BP} = |\vec{BP}| = \sqrt{\left(7 - \frac{25}{7}\right)^2 + \left(8 - \frac{20}{7}\right)^2 + \left(9 - \left[\frac{-9}{7}\right]\right)^2}$$

$$= \sqrt{\left(\frac{24}{7}\right)^2 + \left(\frac{36}{7}\right)^2 + \left(\frac{72}{7}\right)^2}$$

$$= \frac{1}{7}\sqrt{(12)^2(2^2 + 3^2 + 6^2)}$$

$$= \frac{1}{7}\sqrt{(12)^2\ 49}$$

$$= 12$$

1.6.6(L) continued

which checks with our previous result.

c. Looking at

$$2x + 3y + 6z = 22 \tag{5}$$

and

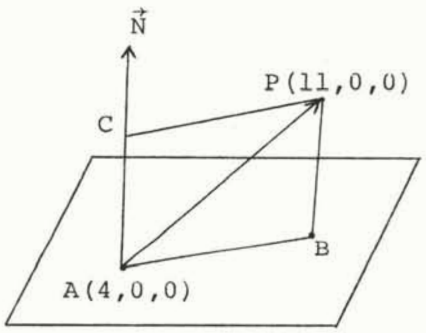
$$2x + 3y + 6z = 8 \tag{6}$$

one might be tempted to subtract (6) from (5) and obtain 14 as an answer.

Rather than argue (at least for the moment) against this reasoning let us actually derive the correct answer.

Since the planes are parallel (i.e., both are perpendicular to $2\vec{i} + 3\vec{j} + 6\vec{k}$) it is sufficient to find the (perdendicular) distance from a point in one plane to the other plane.

For example P(11,0,0) belongs to $2x + 3y + 6z = 22$. Hence imitating our approach in part a. we have:



Notice our orientation. Certainly (11,0,0) and (4,0,0) should lie on the x-axis. That is AP is a segment of the x-axis. The point is that PB does not represent the direction of the z-axis in our diagram.

$$\overline{AC} = |\vec{AP} \cdot \vec{u}_N| = |(7,0,0)\cdot(\tfrac{2}{7},\tfrac{3}{7},\tfrac{6}{7})|$$

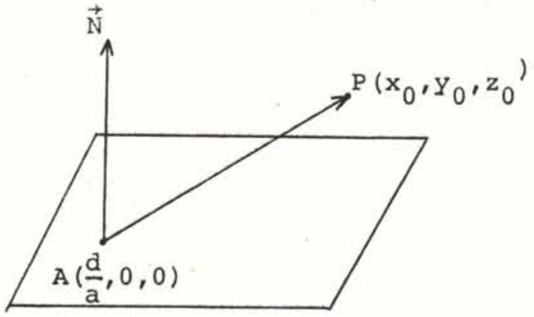
$$= 7(\tfrac{2}{7}) + 0(\tfrac{3}{7}) + 0(\tfrac{6}{7})$$

$$= 2.$$

In other words the distance between the planes is 2 units; not 14 units.

To see what happened here, let's look at the general situation:

1.6.6(L) continued



Equation of plane is $ax + by + cz = d$. Therefore, we may let $A(\frac{d}{a},0,0)$ denote a point in the plane (if $a = 0$ we work with $(0,\frac{d}{b},0)$, etc.).

Now a vector $\vec{N}$ normal to our plane is (a,b,c). Hence a unit normal is given by $\frac{a\vec{i} + b\vec{j} + c\vec{k}}{\sqrt{a^2 + b^2 + c^2}} = \vec{u}_N$. Thus, the distance from P to the plane is

$$|\vec{AP} \cdot \vec{u}_N| = \left|(x_0 - \frac{d}{a}, y_0, z_0)\cdot \frac{(a,b,c)}{\sqrt{a^2 + b^2 + c^2}}\right|$$

$$= \frac{|ax_0 - d + by_0 + cz_0|}{\sqrt{a^2 + b^2 + c^2}} . \qquad (7)$$

If we now assume that P lies in the plane $ax + by + cz = e$, then since $P(x_0,y_0,z_0)$ is in this plane we have

$$ax_0 + by_0 + cz_0 = e. \qquad (8)$$

Putting (8) into (7) yields that the distance between the planes $ax + by + cz = e$ and $ax + by + cz = d$ is

$$\frac{|e - d|}{\sqrt{a^2 + b^2 + c^2}}. \qquad (9)$$

1.6.6(L) continued

From (9) we see that the correct answer is $|e - d|$ if and only if

$$\sqrt{a^2 + b^2 + c^2} = 1. \qquad (10)$$

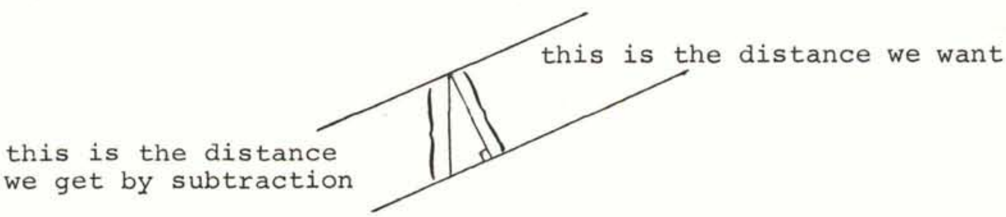
In our example a = 2, b = 3, and c = 6. Hence $\sqrt{a^2 + b^2 + c^2} = 7$. Notice that had we written equations (5) and (5) in the equivalent form

$$\frac{2}{7}x + \frac{3}{7}y + \frac{6}{7}z = \frac{22}{7} \qquad (5')$$

$$\frac{2}{7}x + \frac{3}{7}y + \frac{6}{7}z = \frac{8}{7} \qquad (6')$$

then $a^2 + b^2 + c^2 = 1$, and the correct answer would then be $\left|\frac{22}{7} - \frac{8}{7}\right| = 2$ which checks with the previously obtained result.

In terms of a simple diagram



1.6.7

We have $\vec{AB} = (3 - 1, 4 - 1, 5 - 4) = (2,3,1)$ is parallel to our line and since (1,1,4) is on the line [things will work out the same if we choose (3,4,5) rather than (1,1,4)] our line L has the equation

$$\frac{x - 1}{2} = \frac{y - 1}{3} = \frac{z - 4}{1} \quad (= t) \qquad (1)$$

or in parametric form

$$\left.\begin{aligned} x &= 2t + 1 \\ y &= 3t + 1 \\ z &= t + 4 \end{aligned}\right\} \qquad (2)$$

1.6.7 continued

A vector normal to our plane M is given by

$$\vec{CD} \times \vec{DE} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 4 & 3 & 5 \end{vmatrix}$$

$$= \vec{i}(10 - 9) - \vec{j}(5 - 12) + \vec{k}(3 - 8)$$

$$= \vec{i} + 7\vec{j} - 5\vec{k}.$$

Hence the equation of M is given by

$$1(x - 3) + 7(y - 4) - 5(z + 1) = 0$$

or $x + 7y - 5z = 36$. (3)

The point (x,y,z) we seek, since it belongs to both the line and the plane, must satisfy both (2) and (3). Hence:

$$(2t + 1) + 7(3t + 1) - 5(t + 4) = 36$$

or

$$18t - 12 = 36$$

therefore, $t = \frac{8}{3}$. (4)

Putting (4) into (2) yields

$$x = \frac{16}{3} + 1 = \frac{19}{3}$$

$$y = \frac{24}{3} + 1 = 9$$

$$z = \frac{8}{3} + 4 = \frac{20}{3}.$$

Thus, the point of intersection is $(\frac{19}{3}, 9, \frac{20}{3})$.

Check:

$$\frac{\frac{19}{3} - 1}{2} = \frac{9 - 1}{3} = \frac{\frac{20}{3} - 4}{1} = \frac{8}{3}$$

1.6.7 continued

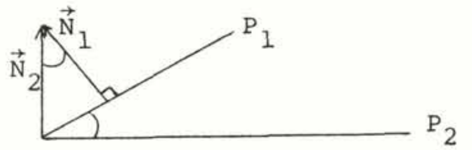
and

$$\frac{19}{3} + 7(9) - 5\left(\frac{20}{3}\right) = \frac{19}{3} + 63 - \frac{100}{3} = 63 - \frac{81}{3} = 63 - 27$$

$$= 36.$$

1.6.8

a. The angle between the two planes is equal to the angle between their normals. (From a "side view"



.)

Now,

$\vec{A} = 3\vec{i} + 2\vec{j} + 6\vec{k}$ is normal to one plane

while

$\vec{B} = 2\vec{i} + 2\vec{j} - \vec{k}$ is normal to the other.

Now the angle σ between $\vec{A}$ and $\vec{B}$ is determined by

$\vec{A} \cdot \vec{B} = |A||B| \cos\sigma$

Therefore, $(3,2,6)\cdot(2,2,-1) = |7||3| \cos\sigma$

or $6 + 4 - 6 = 21 \cos\sigma$.

Therefore, $\cos\sigma = \frac{4}{21}$

$$\sigma = \cos^{-1} \frac{4}{21}.$$

b. The line we seek lies in $3x + 2y + 6z = 8$. Hence it is perpendicular to $3\vec{i} + 2\vec{j} + 6\vec{k}$. It also lies in $2x + 2y - z = 5$, so it is also perpendicular to $2\vec{i} + 2\vec{j} - \vec{k}$.

1.6.8 continued

Now one vector perpendicular to both $3\vec{i} + 2\vec{j} + 6\vec{k}$ and $2\vec{i} + 2\vec{j} - \vec{k}$ is their cross product

$$\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & 6 \\ 2 & 2 & -1 \end{vmatrix}$$

$$= -14\vec{i} + 15\vec{j} + 2\vec{k}.$$

Therefore, $-14\vec{i} + 15\vec{j} + 2\vec{k}$ is parallel to the line we seek.

To find a point on the line we may take any solution of

$$\begin{cases} 3x + 2y + 6z = 8 \\ 2x + 2y - z = 5 . \end{cases}$$

Letting, for example, $z = 0$ we obtain

$$\left.\begin{matrix} 3x + 2y = 8 \\ 2x + 2y = 5 \end{matrix}\right\} \quad \text{therefore, } x = 3, \; y = -\tfrac{1}{2} .$$

Therefore, $(3,-\frac{1}{2},0)$ is on our line and our line is parallel to $-14\vec{i} + 15\vec{j} + 2\vec{k}$.

Therefore, the equation of our line is

$$\frac{x-3}{-14} = \frac{y+\frac{1}{2}}{15} = \frac{z-0}{2} \qquad (= t)$$

or, in parametric form:

$$\left.\begin{matrix} x = -14t + 3 \\ y = 15t - \frac{1}{2} \\ z = 2t. \end{matrix}\right\}$$

Check:

$$\underline{3x + 2y + 6z} = 3(-14t + 3) + 2(15t - \tfrac{1}{2}) + 6(2t)$$

$$= -42t + 9 + 30t - 1 + 12t$$

$$= \underline{8}$$

1.6.8 continued

and

$$\underline{2x + 2y - z} = 2(-14t + 3) + 2(15t - \tfrac{1}{2}) - (2t)$$

$$= -28t + 6 + 30t - 1 - 2t$$

$$= \underline{5}.$$

Quiz

1. Since $1 - 1 = 0$ and we have proven the theorem that $b(0) = 0$ for all numbers b, if follows that

$$(-1)(1 - 1) = (-1)(0) = 0. \quad (1)$$

Next, recalling that $1 - 1$ means $1 + (-1)$, we may use the distributive rule to conclude

$$(-1)(1 - 1) = (-1)(1 + [-1]) = (-1)(1) + (-1)(-1). \quad (2)$$

Using substitution [e.g., replacing $(-1)(1 - 1)$ in (2) by its value in (1)] we may conclude from (1) and (2) that

$$(-1)(1) + (-1)(-1) = 0. \quad (3)$$

By the property of 1 being a multiplicative identity (i.e., $b \times 1 = b$ for all numbers b), it follows that

$$(-1)(1) = -1. \quad (4)$$

Putting (4) into (3) yields

$$-1 + (-1)(-1) = 0 \quad (5)$$

whereupon adding 1 to both sides of (5) yields the desired result. More formally, by substitution using the equality in (5),

$$1 + [-1 + (-1)(-1)] = 1 + 0. \quad (6)$$

By virtue of 0 being the additive identity

$$1 + 0 = 1, \quad (7)$$

while by the associativity of addition

$$1 + [-1 + (-1)(-1)] = [1 + (-1)] + (-1)(-1). \quad (8)$$

1. continued

By the additive inverse rule $1 + (-1) = 0$, whence (8) becomes

$$1 + [-1 + (-1)(-1)] = 0 + (-1)(-1). \quad (9)$$

Since $0 + (-1)(-1) = (-1)(-1) + 0 = (-1)(-1)$, we may rewrite (9) as

$$1 + [-1 + (-1)(-1)] = (-1)(-1). \quad (10)$$

Putting the results of (7) and (10) into (6) yields

$$(-1)(-1) = 1.$$

2. Since $\frac{dy}{dx} = 3x^2 + 1$, the slope of the line tangent to our curve at (1,2) is given at $\left.\frac{dy}{dx}\right|_{x=1} = 4$. Then, since the normal vector is at right angles to the tangent vector (by definition of normal), its slope must be the reciprocal of the slope of the tangent vector. Hence, the required normal has $-\frac{1}{4}$ as its slope. In Cartesian coordinates, the slope of a vector is the quotient of its $\vec{j}$-component divided by its $\vec{i}$-component. Therefore, one normal vector would be

$$\vec{N} = 4\vec{i} - \vec{j}.$$

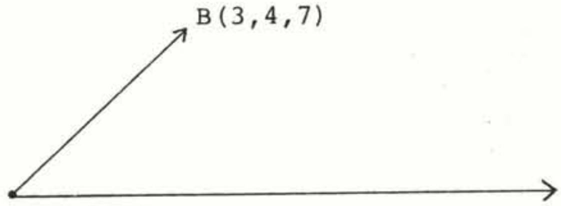
Since $|\vec{N}| = \sqrt{(4)^2 + (-1)^2} = \sqrt{17}$, we see that a unit normal is given by

$$\vec{u}_N = \frac{\vec{N}}{|\vec{N}|} = \frac{1}{\sqrt{17}}(4\vec{i} - \vec{j}).$$

Finally, since $4\vec{i} - \vec{j}$ and $-4\vec{i} + \vec{j}$ have the same direction and magnitude but opposite sense, we see that there are <u>two</u> unit normals given by

$$\vec{u}_N = \pm\frac{1}{\sqrt{17}}(4\vec{i} - \vec{j}).$$

3.



$$\left.\begin{aligned} \vec{AB} &= 2\vec{i} + \vec{j} + 2\vec{k} \\ \vec{AC} &= -2\vec{i} - 3\vec{j} - 6\vec{k} \end{aligned}\right\} \quad (1)$$

Now $\vec{AB} + \vec{AC}$ is the vector which is the diagonal of the parallelogram determined by A, B, and C. In general, such a diagonal does not bisect the vertex angle. In fact it bisects the angle if and only if the parallelogram is a rhombus (all four sides have equal length).

So consider instead the parallelogram determined by $|\vec{AC}|\vec{AB}$ and $|\vec{AB}|\vec{AC}$. This is a rhombus (since each vector has $|\vec{AC}|\,|\vec{AB}| = |\vec{AB}|\,|\vec{AC}|$ as its magnitude).

Therefore,

$$|\vec{AC}|\vec{AB} + |\vec{AB}|\vec{AC}$$

bisects $\measuredangle BAC$. (Notice that $|\vec{AC}|\vec{AB}$ is parallel to $\vec{AB}$ and $|\vec{AB}|\vec{AC}$ is parallel to $\vec{AC}$; hence, in both paralellograms, $\measuredangle BAC$ is a vertex angle.)

From (1)

$$|\vec{AB}| = \sqrt{2^2 + 1^2 + 2^2} = 3$$

$$|\vec{AC}| = \sqrt{(-2)^2 + (-3)^2 + (-6)^2} = 7.$$

Hence one bisecting vector is

$$7\vec{AB} + 3\vec{AC} =$$

$$7(2\vec{i} + \vec{j} + 2\vec{k}) + 3(-2\vec{i} - 3\vec{j} - 6\vec{k}) =$$

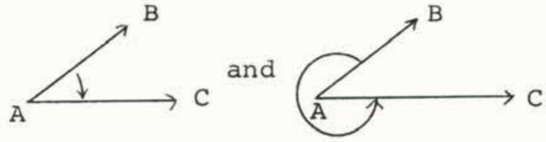
$$8\vec{i} - 2\vec{j} - 4\vec{k}.$$

3. continued

Hence, any scalar multiple of $4\vec{i} - \vec{j} - 2\vec{k}$ is a bisecting vector. Since $|4\vec{i} - \vec{j} - 2\vec{k}| = \sqrt{16 + 1 + 4} = \sqrt{21}$, a unit vector which bisects ∢BAC is given by

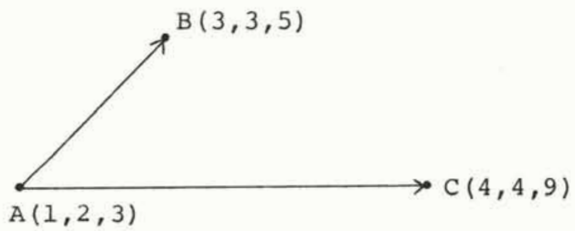
$$\pm \frac{1}{\sqrt{21}} (4\vec{i} - \vec{j} - 2\vec{k}).$$

(In this type of problem, it is often customary to reject one of the signs [though this is not mandatory]. The reason is that ∢BAC, unless some convention is made, is ambiguous since it does not distinguish between



However, we shall not quibble about this distinction here and we shall accept either of the two angles.)

4.



We have

$$\left.\begin{array}{l} \vec{AB} = 2\vec{i} + \vec{j} + 2\vec{k} \\ \vec{AC} = 3\vec{i} + 2\vec{j} + 6\vec{k} \end{array}\right\} .$$

Therefore, $\left.\begin{array}{l} |\vec{AB}| = 3 \\ |\vec{AC}| = 7. \end{array}\right\}$

4. continued

a. We have that

$$\vec{AB} \cdot \vec{AC} = |\vec{AB}||\vec{AC}| \cos \angle(\vec{AC}, \vec{AB}) .$$

Therefore, $(2,1,2)\cdot(3,2,6) = (3)(7) \cos \sphericalangle BAC$.

Therefore, $(2)(3) + (1)(2) + (2)(6) = 21 \cos \sphericalangle BAC$.

Therefore, $\cos \sphericalangle BAC = \frac{6 + 2 + 12}{21} = \frac{20}{21}$.

Therefore, $\sphericalangle BAC = \cos^{-1} \frac{20}{21} \approx 18°$.

b. A normal to P is given by

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 2 \\ 3 & 2 & 6 \end{vmatrix} = (6 - 4)\vec{i} - \vec{j}(12 - 6) + \vec{k}(4 - 3)$$

$$= 2\vec{i} - 6\vec{j} + \vec{k}.$$

Since (1,2,3) is in P[(3,3,5) and (4,4,9) could have been chosen as well], the equation for P is

$$2(x - 1) - 6(y - 2) + 1(z - 3) = 0.$$

[I.e., recall that $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ is the equation of the plane passing through (x_0, y_0, z_0) and having $A\vec{i} + B\vec{j} + C\vec{k}$ as a normal vector.]

The equation of P is

$$2x - 6y + z + 7 = 0.$$

[Check

$2(1) - 6(2) + (3) + 7 = 0$

$2(3) - 6(3) + (5) + 7 = 0$

$2(4) - 6(4) + (9) + 7 = 0$.]

4. continued

c. $|\vec{AB} \times \vec{AC}|$ is the area of the parallelogram which has $\vec{AB}$ and $\vec{AC}$ as consecutive sides. Since the area of the triangle is half that of the parallelogram, we have

$$\text{Area of } \Delta ABC = \frac{1}{2}|\vec{AB} \times \vec{AC}|$$

$$= \frac{1}{2}|2\vec{i} - 6\vec{j} + \vec{k}| \qquad \text{[from b.]}$$

$$= \frac{1}{2}\sqrt{(2)^2 + (-6)^2 + (1)^2}$$

$$= \frac{\sqrt{41}}{2}.$$

d. The key here is that if $A(a_1,a_2,a_3)$, $B(b_1,b_2,b_3)$, and $C(c_1,c_2,c_3)$ are the given points then the medians of ΔABC intersect at the point $\left(\frac{a_1+b_1+c_1}{3}, \frac{a_2+b_2+c_2}{3}, \frac{a_3+b_3+c_3}{3}\right)$. In our present example we find that the medians intersect at
$\left(\frac{1+3+4}{3}, \frac{2+3+4}{3}, \frac{3+5+9}{3}\right) = \left(\frac{8}{3}, 3, \frac{17}{3}\right)$.

e. Since $C(4,4,9)$ is on the line and $\vec{AB} = 2\vec{i} + \vec{j} + 2\vec{k}$ is parallel to the line, the equation of the line is

$$\frac{x-4}{2} = \frac{y-4}{1} = \frac{z-9}{2} \qquad (= t),$$

or, in parametric form

$$\frac{x-4}{2} = t, \; y - 4 = t, \; \frac{z-9}{2} = t.$$

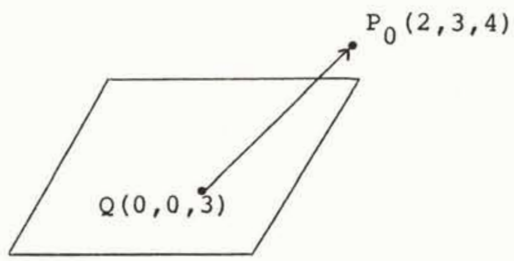
That is,

$$\left.\begin{aligned} x &= 2t + 4 \\ y &= t + 4 \\ z &= 2t + 9. \end{aligned}\right\}$$

4. continued

[The key here is that $\frac{x - x_0}{A} = \frac{y - y_0}{B} = \frac{z - z_0}{C}$ represents the equation of the line which passes through the point (x_0, y_0, z_0) and is parallel to $A\vec{i} + B\vec{j} + C\vec{k}$.]

5. There are, of course, infinitely many ways that we may choose a point in the plane. Our choice, quite arbitrarily, is to let $x = y = 0$, whence $4x + 5y + 2z = 6$ implies that $z = 3$. In other words $Q(0,0,3)$ is in the plane.



Thus, the distance from P_0 to the plane is the length of the projection of $\vec{P_0Q}$ in the direction of the normal to the plane and this in turn is given by $|\vec{P_0Q} \cdot \vec{u}_N|$, where $\vec{u}_N$ is a unit normal to the plane. Since the equation of the plane is $4x + 5y + 2z = 6$, a normal to the plane is $4\vec{i} + 5\vec{j} + 2\vec{k}$. Hence, a unit normal is

$$\frac{4\vec{i} + 5\vec{j} + 2\vec{k}}{|4\vec{i} + 5\vec{j} + 2\vec{k}|} = \frac{4\vec{i} + 5\vec{j} + 2\vec{k}}{\sqrt{45}} .$$

Therefore, the distance is

$$|\vec{P_0Q} \cdot \vec{u}_N| = \left|(2,3,4-3)\cdot\frac{(4,5,2)}{\sqrt{45}}\right| = \frac{8 + 15 + 2}{\sqrt{45}} = \frac{25}{\sqrt{45}} = \frac{25}{45}\sqrt{45}$$

$$= \frac{5}{9}\sqrt{45} = \frac{5}{3}\sqrt{5} .$$

The latter part of this problem also provides an alternative for solving the first part.

Since the line is perpendicular to the plane, it is parallel to $\vec{N}$ $(= 4\vec{i} + 5\vec{j} + 2\vec{k})$. Then, since $P_0(2,3,4)$ is on the line its equation is

5. continued

$$\frac{x-2}{4} = \frac{y-3}{5} = \frac{z-4}{2} \qquad (= t)$$

or, in parametric form

$$\left.\begin{array}{l} x = 4t + 2 \\ y = 5t + 3 \\ z = 2t + 4. \end{array}\right\} \qquad (1)$$

Thus, the required point R must satisfy the above system of equations (since R lies on this line) and it must also satisfy $4x + 5y + 2z = 6$ (since R also lies on the plane).

Hence, we must have that

$$4(4t + 2) + 5(5t + 3) + 2(2t + 4) = 6$$

or

$$45t = -25, \text{ or } t = \frac{-5}{9}.$$

Putting this value of t into (1) yields

$$x = 4(-\tfrac{5}{9}) + 2 = -\tfrac{2}{9}$$

$$y = 5(-\tfrac{5}{9}) + 3 = \tfrac{2}{9}$$

$$z = 2(-\tfrac{5}{9}) + 4 = \tfrac{26}{9}.$$

That is, R is given by $(-\frac{2}{9},\frac{2}{9},\frac{26}{9})$.

[As a check our answer to the first part of the problem should also be given by $|\vec{RP}_0|$.

Now $\vec{RP}_0 = (2,3,4) - (-\frac{2}{9},\frac{2}{9},\frac{26}{9})$

$= (2,3,4) + (\frac{2}{9},-\frac{2}{9},-\frac{26}{9})$

$= (\frac{20}{9},\frac{25}{9},\frac{10}{9})$.

5. continued

Therefore, $|\vec{RP}_0| = \frac{1}{9}\sqrt{400 + 625 + 100}$

$$= \frac{\sqrt{1125}}{9} = \frac{\sqrt{(225)(5)}}{9}$$

$$= \frac{15}{9}\sqrt{5}$$

$$= \frac{5}{3}\sqrt{5}$$

which checks with our previous result.]