

Unit 3: Change of Variables in Multiple Integrals

1. Overview

Just as in the study of calculus of a single variable, it will often be to our advantage to be able to perform a multiple integration by means of a change of variables (substitution). In this unit we shall show how this is accomplished. The key idea is very much the same as it was in the case of a single real variable, but complications set in when we deal with more than 1-dimensional space. (Geometrically, this can be explained by observing that it is more complicated to keep track of how a region in either 2-space or 3-space is transformed by a mapping than it was in 1-space since in 1-space the image of an interval is again an interval.)

At any rate, the lecture will try to make this point seem a bit easier to digest. In essence, the lecture will present the main idea in a conveniently oversimplified form. Following the lecture, you will read a short note that will generalize the result of the lecture, and after that you will read about polar coordinates in the text.

The point is that while polar coordinates provide us with an important application of a situation in which one would want to make a change of variables, it must also be remembered that there are other times that we might want to make changes of variables that are unrelated to polar coordinates. The approach in this unit is meant to extend the results of the text beyond the application to polar coordinates.

Finally the exercises have been selected to present you with a rather wide variety of situations in which you would want to make a change of variables (sometimes in terms of polar coordinates and other times a more general change) in order to evaluate a particular double integral.

Study Guide
 Block 5: Multiple Integration
 Unit 3: Change of Variables in Multiple Integrals

2. Lecture 5.030

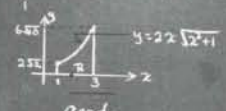
Multiple Integration and the Jacobian

Review


To compute $\int_3^{10} 2x\sqrt{x^2+1} dx$ we let $u = x^2+1$ $\Rightarrow x = \sqrt{u-1}$
 $\therefore du = 2x dx$
 $\therefore \int_3^{10} 2x\sqrt{x^2+1} dx = \int_2^{10} \sqrt{u} du$

Note that $\int_2^{10} 2\sqrt{u-1} \sqrt{u} du$ is incorrect!

Pictorially $\int_3^{10} 2x\sqrt{x^2+1} dx = A_R$, where

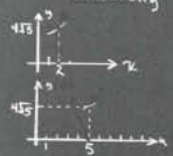


and $\int_2^{10} \sqrt{u-1} \sqrt{u} du = A_S$, where



Notice that for the u corresponding to a given x , $2x\sqrt{x^2+1} = 2\sqrt{u-1}\sqrt{u}$ since $x = \sqrt{u-1}$

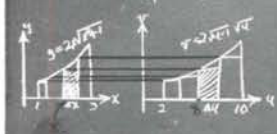
Pictorially



In terms of mappings R is mapped onto S by $\begin{cases} u = x^2+1 \\ v = y \end{cases}$

a.

Since $\begin{cases} u = x^2+1 \\ v = y \end{cases}$ maps R onto S in a 1-1 manner, each ΔA_S "matches" one and only one ΔA_R , i.e.,



For small Δu , $\frac{\Delta x}{\Delta u} \approx \frac{dx}{du}$
 $\therefore \Delta x \approx \frac{dx}{du} \Delta u$
 $\therefore \Delta A_R = y_0 \Delta x \approx y_0 \frac{dx}{du} \Delta u = \frac{dx}{du} \Delta A_S$

More generally, $\Delta A_R = \lim_{\max \Delta u \rightarrow 0} \sum \Delta A_R$ "scaling" factor $= \lim_{\max \Delta u \rightarrow 0} \sum \frac{dx}{du} \Delta A_S$


Note that $\Delta A_S = y_0 \Delta u = 2\sqrt{u_0-1} \Delta u$
 $\frac{dx}{du} \Big|_{u=y_0} = \frac{1}{2\sqrt{u_0-1}}$
 $\therefore \frac{dx}{du} \Delta A_S = \sqrt{u_0} \Delta u$
 $\therefore A_R = \int_2^{10} \sqrt{u} du$

Suppose \mathcal{L} maps R onto S in 1-1 manner.

b.

Key Point

For "small" Δu and Δv , area of back-map of ΔA_S is area of \square , i.e.,



Back-map of ΔA_S yields ΔA_R , and $\Delta A_R \approx |\vec{AB} \times \vec{AC}|$

$\vec{AB} = \Delta x_1 \vec{i} + \Delta y_1 \vec{j}$
 $\vec{AC} = \Delta x_2 \vec{i} + \Delta y_2 \vec{j}$
 $\vec{AB} \times \vec{AC} = (\Delta x_1 \Delta y_2 - \Delta x_2 \Delta y_1) \vec{k}$

$\Delta x_1 = x_{u_0} \Delta u + x_{v_0} \Delta v$ since $v = v_0$
 $\Delta y_1 = y_{u_0} \Delta u + y_{v_0} \Delta v$ on ΔR
 $\Delta x_2 = x_{u_0} \Delta u + x_{v_0} \Delta v$ since $u = u_0$
 $\Delta y_2 = y_{u_0} \Delta u + y_{v_0} \Delta v$ on ΔS

$\therefore \Delta x_1 \Delta y_2 = (x_{u_0} \Delta u)(y_{v_0} \Delta v)$
 $= x_{u_0} y_{v_0} \Delta u \Delta v$
 $\Delta x_2 \Delta y_1 = x_{v_0} y_{u_0} \Delta u \Delta v$

$\therefore |\vec{AB} \times \vec{AC}| = |(x_{u_0} y_{v_0} - x_{v_0} y_{u_0}) \Delta u \Delta v| = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta A_S$

$\therefore \Delta A_R \approx \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta A_S$

$A_R = \iint_R dx dy = \iint_S \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv$ may be "messy"

c.

3. Note on the Lecture:

We would now like to extend the result of our lecture to cover the more general integral $\int_R g(x,y) dA_R$.

If $\underline{f}: R \rightarrow S$ is defined by $\underline{f}(x,y) = (u,v)$ where

$$u = u(x,y)$$

$$v = v(x,y)$$

is invertible (so that we may view x and y as functions of u and v), then

$$\iint_R g(x,y) dA_R = \iint_S g(x(u,v), y(u,v)) \frac{\partial(x,y)}{\partial(u,v)}^* dA_S. \quad (1)$$

Notice that in the special case $g(x,y) \equiv 1$ equation (1) is the result described in Lecture 5.030.

The derivation of equation (1) is analogous to the derivation of $\int_a^b g(x) dx = \int_{u=x^{-1}(a)}^{u=x^{-1}(b)} g(x(u)) \frac{dx}{du} du$.

That is, we may think of

$$\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$$

as the 3-dimensional**system

*If \underline{f} is given by $\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$, it is easy to compute $\partial(u,v)/\partial(x,y)$ explicitly, but it may be difficult to express x and y explicitly in terms of u and v . In this case, it is best to invoke our earlier result that $\partial(x,y)/\partial(u,v) = [\partial(u,v)/\partial(x,y)]^{-1}$.

**Recall in our lecture that when we let $u = u(x)$ in $\int_a^b g(x) dx$, we viewed $u = u(x)$ as the 2-dimensional system

$$\begin{cases} u = u(x) \\ v = y \end{cases}$$

$$\begin{aligned}u &= u(x,y) \\v &= v(x,y) \\w &= z\end{aligned}\tag{2}$$

We then see that the mapping defined by (2) preserves the height of a rectangular parallelepiped, but the rectangular base R_1 is mapped into a region S_1 , which is essentially a parallelogram for sufficiently small Δx and Δy , and for such choices of Δx and Δy

$$A_{R_1} \approx \frac{\partial(x,y)}{\partial(u,v)} A_{S_1}.$$

In other words the result described in the lecture extends to $\iint_R g(x,y) dA_R$ where $\partial(x,y)/\partial(u,v)$ is the "correction factor" for obtaining A_{R_1} in terms of A_{S_1} .

In summary, it is equation (1) that we really wanted to develop, and the equation developed in our lecture was the special case of equation (1) when $g(x,y)$ was identically equal to 1 on R . Notice that in the development of equation (1) the crucial point was that the correction factor depended on the region R not on the function g . Consequently, the demonstration given in the lecture is the same as it would have been had we tackled the more general $\iint_R g(x,y) dA_R$, but the presence of $g(x,y)$ might have proved to be a distraction that would have taken our attention away from the fact that the Jacobian was introduced to "correct" only the area of the region R .

4. Read Thomas, Section 16.4.

5. Exercises

5.3.1(L)

a. Given the mapping $\underline{f}: E^2 \rightarrow E^2$ defined by $\underline{f}(x,y) = (u,v)$ where

$$\begin{cases}u = 3x - 2y \\v = x + y\end{cases},$$

(continued on next page)

5.3.1(L) continued

Compute $\frac{\partial(x,y)}{\partial(u,v)}$.

- b. Explain the sense in which $\frac{\partial(x,y)}{\partial(u,v)}$ is a "scaling factor".
- c. Explain the meaning of the negative "scaling factor" $\frac{\partial(x,y)}{\partial(u,v)}$ where

$$\begin{cases} u = x + y \\ v = 3x - 2y \end{cases}.$$

5.3.2

- a. Describe the linear mapping \underline{f} which maps $(3,4)$ into $(1,0)$ and $(5,6)$ into $(0,1)$.
- b. What region R is mapped by \underline{f} into the square S with vertices at $O'(0,0)$, $A'(1,0)$, $B'(1,1)$, and $C'(0,1)$?
- c. How does the area of R compare with the area of S ?
- d. Describe \underline{f} if $\underline{f}(5,6) = (1,0)$ and $\underline{f}(3,4) = (0,1)$.

5.3.3(L)

Use the change of variables

$$\begin{cases} u = \frac{-x + 3y}{5} \\ v = \frac{2x - y}{5} \end{cases}$$

to evaluate

$\iint_R e^{-\frac{x+3y}{5}} \cos^2\left(\frac{2x-y}{5}\right) dA_R$ where R is the parallelogram with vertices at $O(0,0)$, $A(3,1)$, $B(5,5)$ and $C(2,4)$.

5.3.4

Let R be the square with vertices at $A(1,0)$, $B(2,1)$, $C(1,2)$ and $D(0,1)$. Make the substitution

$$\begin{cases} u = x - y \\ v = x + y \end{cases}$$

to evaluate

$$\iint_R (x - y)^6 e^{x+y} dA_R.$$

5.3.5(L)

- a. Use polar coordinates to evaluate

$$\int_0^a \int_0^{\sqrt{a^2 - x^2}} e^{-(x^2 + y^2)} dy dx.$$

- b. What happens in part (a) if we let $a \rightarrow \infty$?

- c. Use the answer to part (b) to compute $\int_0^{\infty} e^{-x^2} dx$.

5.3.6

Compute

$$\iint_R \sqrt{x^2 + y^2} dA_R \text{ where } R \text{ is the region bounded by the } x\text{-axis}$$

and the curve C whose polar equation is $r = \theta$, $0 \leq \theta \leq \pi$.

5.3.7(L)

Let R be the region in the first quadrant bounded between the curves $x^2 - y^2 = a$, $x^2 - y^2 = b$, $2xy = c$, and $2xy = d$ (where $0 < a < b$ and $0 < c < d$).

- a. Discuss the effect on $\iint_R g(x,y) dA_R$ brought about by the change of variables, $u = x^2 - y^2$, $v = 2xy$, and in particular, discuss the most general form of $g(x,y)$ which results in the variables (continued on next page)
-

5.3.7(L) continued

being separated in the transformed integral.

- b. Use part (a) to evaluate $\iint_R (x^2 + y^2) e^{x^2 - y^2} \cosh(2xy) dA_R$.

5.3.8

Let R be the region in the first quadrant bounded between the curves $x^2 - y^2 = a$, $x^2 - y^2 = b$, $x^2 + y^2 = c$, $x^2 + y^2 = d$ (where $0 < a < b < c < d$). In a manner similar to that used in the preceding exercise discuss $\iint_R f(x, y) dA_R$ under the change of variables

$$\begin{aligned} u &= x^2 + y^2 \\ v &= x^2 - y^2. \end{aligned}$$

In particular evaluate $\iint_R 8xy dA_R$

5.3.9 (optional)

Let R be the triangular region bounded by the lines $y = x$, $x = 1$, and the x -axis. Discuss what happens to the integral $\iint_R f(x, y) dA_R$ under the change of variables $u = x$ and $v = y/x$.

(The aim of this exercise is to show that the mapping need not be 1-1 for our change of variable technique to work. What is important is not so much that $\partial(u, v)/\partial(x, y) \neq 0$ but rather that the set of points at which $\partial(u, v)/\partial(x, y) = 0$ is not "too great!")

Evaluate $\iint_R \sqrt{x^2 + y^2} dA_R$.

5.3.10 (optional)

Show that the technique for interchanging the order of integration in a double integral (discussed in the previous unit) is a special case of change of variables. In particular, show that the change of order is accomplished by the substitution

$$\left. \begin{aligned} u &= y \\ v &= x \end{aligned} \right\}.$$

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Resource: Calculus Revisited: Multivariable Calculus
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