Unit 3: Change of Variables in Multiple Integrals

5.3.1(L)

Introduction

The easiest case in which to visualize the role of the Jacobian determinant is when we have a linear mapping. That is, when we are given a mapping of the xy-plane into the uv-plane defined by

$$\left.\begin{aligned} u &= ax + by \\ v &= cx + dy \end{aligned}\right\} . \qquad (1)$$

We know that each rectangular region in the uv-plane is the image of a parallelogram in the xy-plane.* (In particular, the line $u = k$ is the image of the line $ax + by = k$ while the image of $v = k$ is the line $cx + dy = k$.) In other words, in this special case, the back-map of a rectangular region is exactly a parallelogram, rather than approximately a parallelogram.

a. Given the mapping $\underline{f}:E^2 \to E^2$ defined by $\underline{f}(x,y) = (u,v)$ where

$$\left.\begin{aligned} u &= 3x - 2y \\ v &= x + y \end{aligned}\right\} . \qquad (2)$$

We see that

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} = 3 -(-2) = 5.$$

[Note that the linear system defined by (1) always has $\frac{\partial(u,v)}{\partial(x,y)}$ as its determinant of coefficient.]

At any rate, since

$$\frac{\partial(x,y)}{\partial(u,v)} = [\frac{\partial(u,v)}{\partial(x,y)}]^{-1}$$

*We are assuming, of course, that the mapping is 1-1; for otherwise the image is either a line or a point, in which case there is no back-map of the rectangular region.

5.3.1(L) continued

the fact that

$$\frac{\partial(u,v)}{\partial(x,y)} = 5$$

implies that

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{5}\,. \tag{3}$$

The technique of computing $\partial(u,v)/\partial(x,y)$ and inverting it to find $\partial(x,y)/\partial(u,v)$ is often our only recourse since it is not always possible, in a more general system

$$\begin{cases} u = u(x,y) \\ v = v(x,y)\ , \end{cases}$$

to solve for x and y explicitly in terms of u and v.

Of course, in the present example, it is rather easy to invert the system (2) to obtain

$$\left.\begin{aligned} x &= \tfrac{1}{5}u + \tfrac{2}{5}v \\ y &= -\tfrac{1}{5}u + \tfrac{3}{5}v \end{aligned}\right\} \tag{4}$$

and, from the system (4), we obtain directly

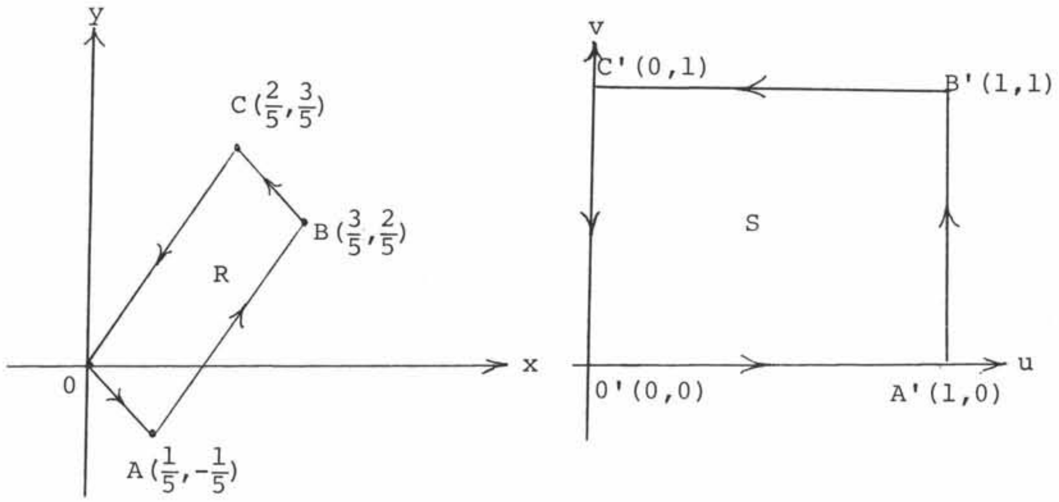
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{vmatrix} = \frac{3}{25} + \frac{2}{25} = \frac{1}{5}\,,$$

which agrees with our result in equation (3).

b. Let S be the unit square in the uv-plane with vertices at $0'(0,0)$, $A'(1,0)$, $B'(1,1)$, and $C'(0,1)$. Using the system (4) with (1) $u = 0$, $v = 0$, (2) $u = 1$, $v = 0$, (3) $u = 1$, $v = 1$, and (4) $u = 0$, $v = 1$, we locate the points $0(0,0)$, $A(\frac{1}{5}, -\frac{1}{5})$, $B(\frac{3}{5}, \frac{2}{5})$, and $C(\frac{2}{5}, \frac{3}{5})$ in the xy-plane whose images with respect to f are 0', A', B', and C', respectively.

5.3.1(L) continued

Pictorially,



(Figure 1)

The back-map of S is the parallelogram R* with vertices 0, A, B, and C.

The area of R is given by $|\vec{OA} \times \vec{OC}|$, or

$$A_R = |(\tfrac{1}{5}\vec{i} - \tfrac{1}{5}\vec{j}) \times (\tfrac{2}{5}\vec{i} + \tfrac{3}{5}\vec{j})|$$

$$= \left|\tfrac{3}{25}(\vec{i} \times \vec{j}) - \tfrac{2}{25}(\vec{j} \times \vec{i})\right|$$

$$= \left|\tfrac{3}{25}\vec{k} - \tfrac{2}{25}(-\vec{k})\right| = \tfrac{1}{5}\,. \qquad (5)$$

*While we could check this in more computational detail, note that the linearity of our mapping guarantees that $\underline{f}^{-1}(S)[=R]$ has straight line boundaries with vertices at O,A,B, and C. Then, since $\underline{f}^{-1}$ is 1-1, the interior of S maps into the interior of R under $\underline{f}$. Finally, R is a parallelogram since

$\vec{OA} + \vec{OC} = (\frac{1}{5}, -\frac{1}{5}) + (\frac{2}{5}, \frac{3}{5}) = (\frac{3}{5}, \frac{2}{5}) = \vec{OB}$.

5.3.1(L) continued

That is, $\frac{1}{5}$ is the necessary "scaling factor" to convert the area of S into the area of R.

Note:
If $\underline{f}$ maps E^2 onto E^2 in a 1-1 manner, and if S_1 and S_2 are two congruent but different regions in the uv-plane, it is quite possible that $R_1 = \underline{f}^{-1}(S_1)$ and $R_2 = \underline{f}^{-1}(S_2)$ are not congruent. That is, the properties of $\underline{f}$ are local and as a result it is not enough merely to know the shape of a region under investigation. In the case of a linear change of variables, i.e., in a system of equations such as in equation (1), if we let S denote any rectangle whose boundaries are the parallel lines $u = k_1$ and $u = k_2$, and $v = k_3$ and $v = k_4$, where k_1, k_2, k_3 and k_4 are constants, then the back-map of S is congruent to the back-map of any region congruent to S. In other words, had we wished to be more precise in part (b), we should have let S denote the rectangle with vertices at $0'(u_0,v_0)$, $A'(u_0 + \Delta u, v_0)$, $B'(u_0 + \Delta u, v_0 + \Delta v)$, $C'(u_0, v_0 + \Delta v_0)$. Then, with $R = \underline{f}^{-1}(S)$ we could have shown that in this more general case

$$A_R = \frac{1}{5} A_S,$$

but the special case treated in part (b) is sufficient for our purpose.

a. The main aim of this part of the exercise is to help you get a feeling for the meaning of a negative "scaling factor". In the same way that we may visualize a negative number as a length with a different orientation (sense), we may interpret a negative scaling-factor as meaning that the mapping $\underline{f}$ reverses the orientation of an element of area. Rather than go on in too abstract a vein here, let us illustrate our remarks by solving the given problem.

In part (b) we say that the parallelogram R with vertices at $0(0,0)$, $A(\frac{1}{5},-\frac{1}{5})$, $B(\frac{3}{5},\frac{2}{5})$, and $C(\frac{2}{5},\frac{3}{5})$ was mapped onto the unit square S.

5.3.1(L) continued

What happens to R under the mapping $\underline{g}$ defined by $\underline{g}(x,y) = (u,v)$ where

$$\begin{cases} u = x + y \\ v = 3x - 2y \end{cases} \quad ?$$

Well, $\underline{g}(0) = \underline{g}(0,0) = (0,0) = 0'$

$\underline{g}(A) = \underline{g}(\frac{1}{5}, -\frac{1}{5}) = (\frac{1}{5} + [-\frac{1}{5}], \frac{3}{5} - [-\frac{2}{5}]) = (0,1) = C'$

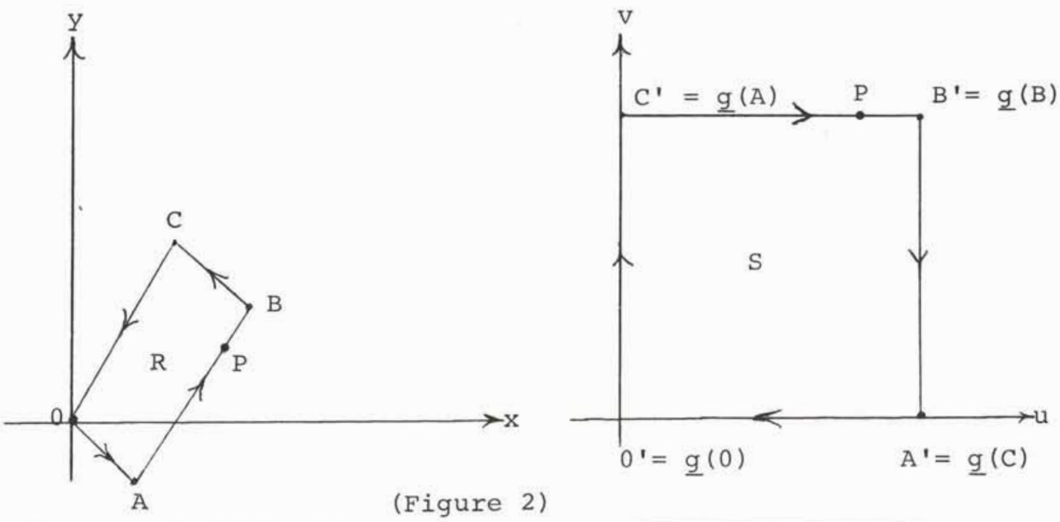
$\underline{g}(B) = \underline{g}(\frac{3}{5}, \frac{2}{5}) = (\frac{3}{5} + \frac{2}{5}, \frac{9}{5} - \frac{4}{5}) = (1,1) = B'$

$\underline{g}(C) = \underline{g}(\frac{2}{5}, \frac{3}{5}) = (\frac{2}{5} + \frac{3}{5}, \frac{6}{5} - \frac{6}{5}) = (1,0) = A'$.

In other words $\underline{f}$ and $\underline{g}$ map R onto S (or, conversely $\underline{f}^{-1}$ and $\underline{g}^{-1}$ map S onto R) but with opposite orientation.

Pictorially (Figure 1) we saw in part (b) that as P traced the boundary of R in the counter-clockwise direction, i.e. so that the region R appeared on our left as we traversed its boundary, $\underline{f}$(P) traversed the boundary of S with the same sense (orientation), i.e., counter-clockwise.

Now, look what happens to P under $\underline{g}$. As P varies continuously from 0 to A, $\underline{g}$(P) varies continuously from 0' to C', etc. so that the graph of the mapping is given by,



(Figure 2)

5.3.1(L) continued

From Figure 2 we see that the point P traverses the boundary of R with a sense opposite to that with which $\underline{f}(P)$ traverses the boundary of S.

This reversal of sense accounts for why the scaling-factor (i.e., the Jacobian determinant) is negative.

An interesting way to avoid the negative sign, if you so desire, is to observe that when one interchanges two rows of a square matrix, the determinant changes sign. Rather than prove this in general, let us observe in the 2 by 2 case that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}$$

since

$$ad - bc = -(bc - ad).$$

Thus when the change of variables

$$\begin{cases} u = f(x,y) \\ v = g(x,y) \end{cases}$$

produces a negative Jacobian determinant, the change of variables

$$\begin{cases} u = g(x,y) \\ v = f(x,y) \end{cases}$$

(i.e., interchanging the roles of u and v) will produce a positive Jacobian determinant.

5.3.2

a. Since $\underline{f}(3,4) = (1,0)$ and $\underline{f}(5,6) = (0,1)$ we have by linearity that

$$\left.\begin{array}{l} \underline{f}(3u,\ 4u) = (u,0) \\ \underline{f}(5v,\ 6v) = (0,v) \end{array}\right\} \quad (1)$$

5.3.2 continued

[i.e., $u\underline{f}(\alpha) = \underline{f}(u\alpha)$ if $\underline{f}$ is linear]

From (1)

$$\underline{f}(3u, 4u) + \underline{f}(5v, 6v) = (u,0) + (0,v) = (u,v). \tag{2}$$

Again, by linearity [i.e., $\underline{f}(\alpha) + \underline{f}(\beta) = \underline{f}(\alpha+\beta)$],

$$\underline{f}(3u, 4u) + \underline{f}(5v, 6v) = \underline{f}(3u + 5v, 4u + 6v), \tag{3}$$

so substituting (3) into (2) yields

$$\underline{f}(3u + 5v, 4u + 6v) = (u,v). \tag{4}$$

Hence, since $\underline{f}(x,y) = (u,v)$, it follows from (4) that

$$\left.\begin{aligned} x &= 3u + 5v \\ y &= 4u + 6v \end{aligned}\right\}. \tag{5}$$

b. From (5), we see that if S is the square with vertices 0'(0,0), A'(1,0), B'(1,1), and C'(0,1) in the uv-plane, then $\underline{f}^{-1}(S) = R$ is the parallelogram in the xy-plane with vertices at 0(0,0), A(3,4), B(8,10), and C(5,6).

c. The area of R is given by

$$\begin{aligned} A_R &= |\vec{0A} \times \vec{0C}| \\ &= |(3\vec{i} + 4\vec{j}) \times (5\vec{i} + 6\vec{j})| \\ &= |18\vec{k} - 20\vec{k}| \\ &= 2 \end{aligned}$$

while $A_S = 1$. Hence, $A_R = 2A_S$, the fact that

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 3 & 5 \\ 4 & 6 \end{vmatrix} = 18 - 20 = -2$$

implies that the point P traverses the boundary of R in the opposite sense from which $\underline{f}(P)$ traverses the boundary of S.

5.3.2 continued

d. In this case, we desire that

$\underline{f}(5,6) = (1,0)$

and

$\underline{f}(3,4) = (0,1)$

so that

$\underline{f}(5u, 6u) = (u,0)$

and

$\underline{f}(3v, 4v) = (0,v)$.

Hence,

$\underline{f}(5u + 3v, 6u + 4v) = (u,v)$

so that

$$\left.\begin{aligned} x &= 5u + 3v \\ y &= 6u + 4v \end{aligned}\right\} \qquad (6)$$

(obviously,the systems (4) and (6) may be inverted so that u and v are expressed in terms of x and y, but this is not necessary for the results we are interested in obtaining).

5.3.3(L)

A preliminary aim of this exercise is to give us a bit of insight into the case in which the integrand of our double integral can be put into the special form f(u)g(v). In this case, we say that the variables are separable. In other words, when the variables are separable we mean that the integrand consists of the product of two functions, one of which is a function of one of the variables alone and the other a function of only the other variable.

5.3.3(L) continued

The point is that when the variables are separable and the limits of integration are all constants, we may view the double integral as the product of two single definite integrals. In other words, while it is not generally true that a double integral is the product of two single integrals, the fact remains that:

$$\int_c^d \int_a^b f(u)g(v)\,dv\,du^* = \left[\int_c^d f(u)\,du\right]\left[\int_a^b g(v)\,dv\right]. \qquad (1)$$

One way of verifying equation (1) is to observe that since f(u) is a constant when we integrate with respect to v, we have

$$\int_c^d \left[\int_a^b f(u)g(v)\,dv\right] du = \int_c^d \left[f(u) \int_a^b g(v)\,dv\right] du \qquad (2)$$

and we next observe that $\int_a^b g(v)\,dv$, being a constant, can be taken outside the integral involving du that is,

$$\int_c^d \left[f(u) \int_d^b g(v)\,dv\right] du = \left\{\int_a^b g(v)\,dv\right\}\left[\int_c^d f(u)\,du\right]. \qquad (3)$$

Combining the results of (2) and (3) yields equation (1).

With this in mind, we see that the change of variables

$$\left.\begin{aligned} u &= \frac{-x + 3y}{5} \\ v &= \frac{2x - y}{5} \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} u &= -\frac{1}{5}x + \frac{3}{5}y \\ v &= \frac{2}{5}x - \frac{1}{5}y \end{aligned}\right\} \qquad (4)$$

converts the given integral into one in which the variables are separable. That is, if we define $\underline{f}$ by $\underline{f}(x,y) = (u,v)$ where u and v are as given in (4) we have

* Recall that this means

$$\int_c^d \left[\int_a^b f(u)g(v)\,dv\right] du$$

5.3.3(L) continued

$$\iint_R e^{-\frac{x+3y}{5}} \cos^2\left(\frac{2x-y}{5}\right) dA_R = \iint_S e^u \cos^2 v \, \frac{\partial(x,y)}{\partial(u,v)} \, dA_S \qquad (5)$$

where $S = \underline{f}(R)$.

Notice from (5) that we cannot be sure that the variables are separable in the right hand integral for arbitrary changes of variables*, but since $\partial(x,y)/\partial(u,v)$ is <u>constant</u> for a <u>linear</u> change of variables, everything is fine! In fact, from (4) we have

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*For example, given

$$\iint_R e^{x^2-y^2} \cos 2xy \, dA_R$$

the change of variables

$$\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$$

leads to

$$\iint_S e^u \cos v \, \frac{\partial(x,y)}{\partial(u,v)} \, dA_S,$$

but now

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2 + y^2) = 4\sqrt{u^2 + v^2},$$

so that

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{4\sqrt{u^2 + v^2}}.$$

Thus, the transformed integral is

$$\iint_S e^u \cos v \left(\frac{4}{\sqrt{u^2 + v^2}}\right) dA_S,$$

and the variables are not "separated".

5.3.3(L) continued

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -\frac{1}{5} & \frac{3}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{vmatrix} = \frac{1}{25} - \frac{6}{25} = -\frac{1}{5},$$

so that

$$\frac{\partial(x,y)}{\partial(u,v)} = -5. \qquad (6)$$

Substituting (6) into (5) we have

$$\iint_R e^{-\frac{x+3y}{5}} \cos^2\left(\frac{2x-y}{5}\right) dA_R = -5 \iint_S e^u \cos^2 v \; dA_S. \qquad (7)$$

The minus sign in (7) merely means that as a point P traverses the boundary of S with the opposite sense (orientation). In other words, using (4) we have

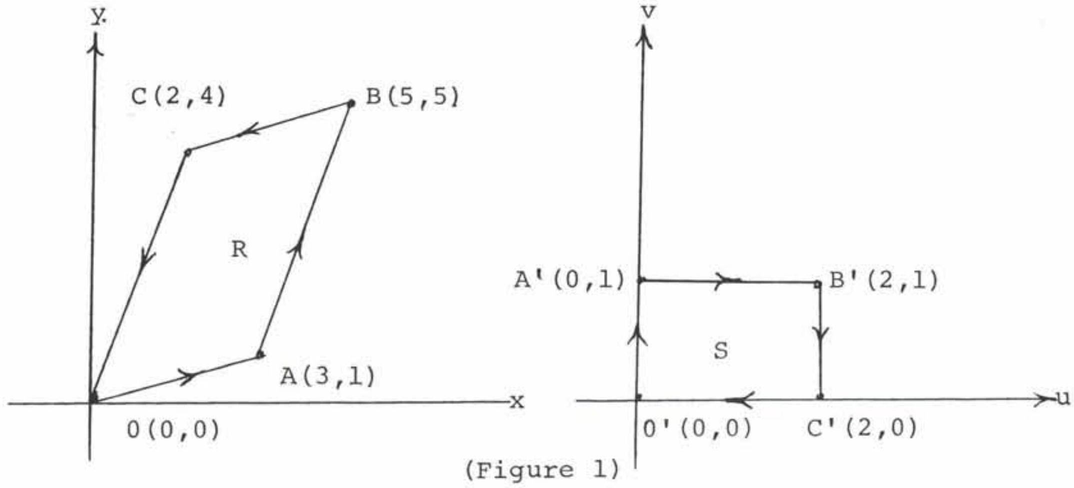
$\underline{f}(0) = \underline{f}(0,0) = (0,0) = 0'$

$\underline{f}(A) = \underline{f}(3,1) = \left[\frac{-3+3(1)}{5}, \frac{2(3)-1}{5}\right] = (0,1) = A'$

$\underline{f}(B) = \underline{f}(5,5) = \left[\frac{-5+3(5)}{5}, \frac{2(5)-5}{5}\right] = (2,1) = B'$

$\underline{f}(C) = \underline{f}(2,4) = \left[\frac{-2+3(4)}{5}, \frac{2(2)-4}{5}\right] = (2,0) = C'$,

or, pictorially,



(Figure 1)

5.3.3(L) continued

Indeed, since $e^{-\frac{x+3y}{5}}\cos^2\left(\frac{2x-y}{5}\right)$ and $e^u\cos^2 v$ are non-negative for all values of x,y,u, and v, it should be clear that the minus sign in (7) refers only to the change of sense indicated in Figure 1, and is not to be interpreted algebraically.*

What is even more significant about Figure 1, however, is the "coincidence" that the change of variables given by (4) also transforms our integral into one with constant limits of integration. Had this not happened the transformed integral would have still been a bit awkward to handle, even though the variables were separated, the limits would introduce the other variable again. In other words, in working with multiple integrals our change of variables must not only simplify the integrand but the limits of integration as well. This is one important reason why changing variables in a multiple integral is such a difficult procedure compared with the procedure used in a single integral. That is, when we transform

$$\int_a^b f(x)\,dx \text{ into } \int_{u=g^{-1}(a)}^{g^{-1}(b)} f(g(u))\frac{dx}{du}\,du$$

by the 1-1 mapping $x = g(u)$, all we have to do is concentrate on the integrand since the image of the interval $[a,b]$ is the interval $[g^{-1}(a), g^{-1}(b)]$ (or $[g^{-1}(b), g^{-1}(a)]$ if g is decreasing). The luxury of integration in the case of a single variable is that our domain is 1-dimensional and consequently the "shape" is very limited (i.e., it must be an interval).

At any rate, returning to the given problem we have from (7) and Figure 1,

$$\iint_R e^{-\frac{x+3y}{5}}\cos^2\left(\frac{2x-y}{5}\right)dA_R = 5\int_0^1\int_0^2 e^u\cos^2 v\,du\,dv$$

*See note at end of exercise.

5.3.3(L) continued

$$= 5[\int_0^1 \cos^2 v(\int_0^2 e^u du)dv$$

$$= 5[\int_0^1 \cos^2 v\ dv][\int_0^2 e^u du]. \qquad (8)$$

Since

$$\int_0^1 \cos^2 v dv = \int_0^1 \frac{1 + \cos 2v}{2} dv = \frac{1}{2}v + \frac{1}{4}\sin 2v\Big|_{u=0}^{1}$$

$$= \frac{1}{2} + \frac{1}{4} \sin 2$$

and

$$\int_0^2 e^u du = e^u \Big|_{v=0}^{2} = e^2 - e^0 = e^2 - 1,$$

we see from (8) that

$$\iint_R e^{-\frac{x+3y}{5}} \cos^2(\frac{2x-y}{5})\ dA_R = 5(\frac{1}{2} + \frac{1}{4} \sin 2)(e^2 - 1). \qquad (9)$$

Notice in (9) that the expression sin 2 refers to the number 2, so that if we want an angular interpretation it means sin (2 radians) $\approx$ sin 114° $\approx$ 0.91. Hence $(\frac{1}{2} + \frac{1}{4}\sin 2) \approx 0.5 + 0.23 = 0.73$. Then since $e^2 \approx 7.39$, $e^2 - 1$ 6.39, so that

$$5(\frac{1}{2} + \frac{1}{4} \sin 2)(e^2-1) \approx 5(0.73)(6.39) \approx 23.$$

In summary, then

$$\iint_R e^{-\frac{x+3y}{5}} \cos^2(\frac{2x-y}{5})dA_R \approx 23.$$

5.3.3(L) continued

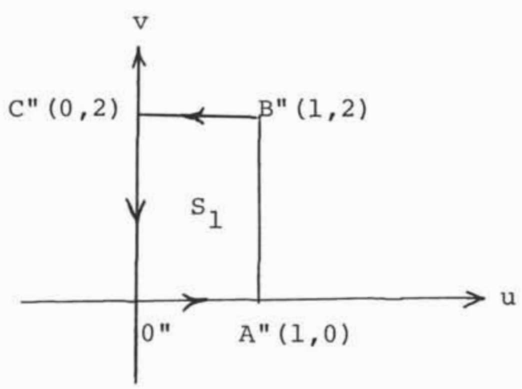
Note:

As we mentioned at the end of Exercise 5.3.1, we could avoid the negative sign in equation (6) by interchanging the roles of u and v in equation (4).

For example, with

$$\left.\begin{aligned} u &= \frac{2}{5}x - \frac{1}{5}y \\ v &= -\frac{1}{5}x + \frac{3}{5}y \end{aligned}\right\} \qquad (10)$$

we see from (10) that A(3,1) is mapped onto A"(1,0); B(5,5) onto B"(1,2); and C(2,4) onto C"(0,2). So that (10) maps R onto S_1 where,



(Figure 2)

Comparing Figure 2 with Figure 1, we see that S_1 has the opposite sense of S (hence, the same sense as R).

In any event, had we elected to use (10) rather than (4), we would have obtained

$$\frac{\partial(x,y)}{\partial(u,v)} = 5,$$

whereupon

$$\iint_R e^{-\frac{x+3y}{5}} \cos^2\left(\frac{2x-y}{5}\right) dA_r = \iint_{S_1} e^v \cos u(5)\, dA_{S_1}$$

5.3.3(L) continued

$$= 5 \int_0^1 \int_0^2 e^v \cos^2 u \, dv du$$

$$= 5 \int_0^2 e^v dv \int_0^1 \cos^2 u du$$

$$= 5(e^2 - 1) \int_0^1 \frac{1 + \cos 2u}{2} du$$

$$= 5(e^2 - 1)\left(\frac{1}{2} + \frac{1}{4} \sin 2\right),$$

which agrees with equation (9).

5.3.4

Letting

$$\left.\begin{aligned} u &= x - y \\ v &= x + y \end{aligned}\right\} \qquad (1)$$

we obtain

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2.$$

Hence

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}.$$

Therefore,

$$\iint_R (x - y)^6 e^{x+y} \, dA_R = \iint_S u^6 e^v \frac{1}{2} \, dA_S \qquad (2)$$

where $S = \underline{f}(R)$.

5.3.4 continued

Since $\underline{f}(x,y) = (u,v) = (x - y, x + y)$, we have

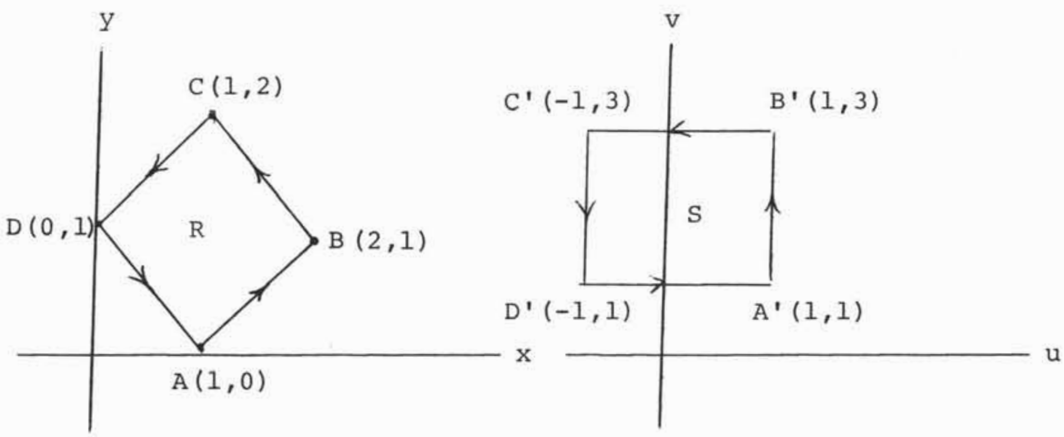
$\underline{f}(A) = \underline{f}(1,0) = (1 - 0, 1 + 0) = (1,1) = A'$

$\underline{f}(B) = \underline{f}(2,1) = (2 - 1, 2 + 1) = (1,3) = B'$

$\underline{f}(C) = \underline{f}(1,2) = (1 - 2, 1 + 2) = (-1, 3) = C'$

$\underline{f}(D) = \underline{f}(0,1) = (0 - 1, 0 + 1) = (-1, 1) = D'$

so that pictorially,



(Figure 1)

Using Figure 1, equation (2) becomes

$$\iint_R (x - y)^6 e^{x+y} dA_R = \frac{1}{2} \int_1^3 \int_{-1}^1 u^6 e^v \, du \, dv$$

$$= \frac{1}{2}\left[\int_1^3 e^v dv\right]\left[\int_{-1}^1 u^6 du\right]$$

$$= \frac{1}{2}[e^3 - e]\left[\frac{1}{7}u^7 \Big|_{u=-1}^{1}\right]$$

5.3.4 continued

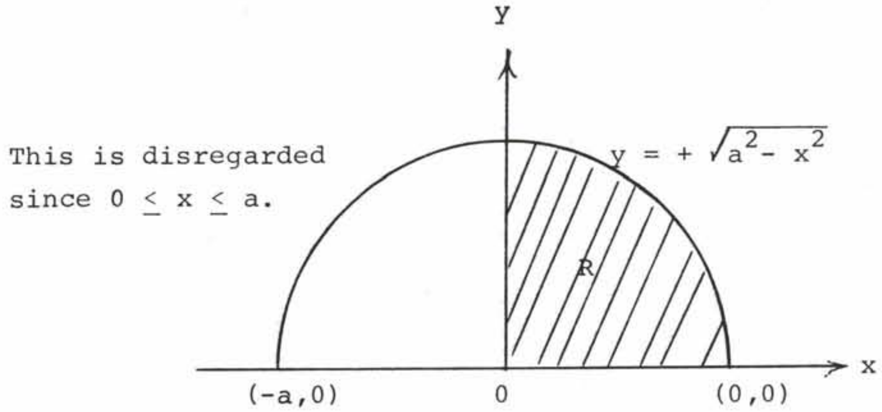
$= \frac{1}{2} e(e^2 - 1)[\frac{2}{7}]$

$= \frac{e(e^2 - 1)}{7}$

≈ 2.5

5.3.5(L)

a. The region R defined by the limits of integration is the disc $x^2 + y^2 \leq a^2$, in the first quadrant. That is, for a fixed x where $0 \leq x \leq a$, y varies from $y = 0$ to $y = \sqrt{a^2 - x^2}$. Recall that $y = \sqrt{a^2 - x^2}$ implies $y^2 = a^2 - x^2$ or $x^2 + y^2 = a^2$, and we use only the upper half of this circle because $y = \sqrt{a^2 - x^2}$ means $y = +\sqrt{a^2 - x^2}$. Pictorially



(Figure 1)

Applying polar coordinates to R, we have that for a fixed θ, where $0 \leq \theta \leq \frac{\pi}{2}$, r varies from 0 to a, and since the element of area in polar coordinates is r drdθ, we obtain

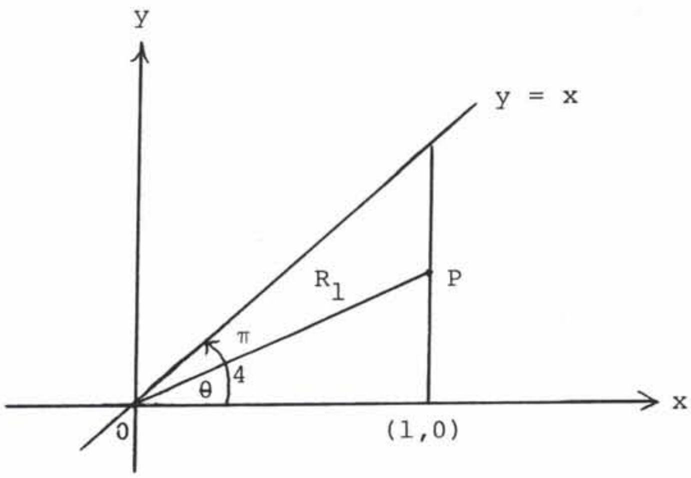
$$\int_0^a \int_0^{\sqrt{a^2 - x^2}} e^{-(x^2 + y^2)}\, dydx = \int_0^{\frac{\pi}{2}} \int_0^a e^{-r^2} r\, drd\theta \qquad (1)$$

5.3.5(L) continued

From (1) we see the key reason for trying polar coordinates. Namely, while e^{-x^2} and e^{-y^2} do not have elementary anti-derivatives, re^{-r^2} does have one. Thus, the "extra" factor of r obtained by switching to polar coordinates simplifies our integrand.

At the same time, however, it should be pointed out that for most regions other than R, the change to polar coordinates might not have been too helpful. For example if R_1 is the region bounded by the three lines y = x, y = 0, and x = 1, then in polar coordinates R_1 is described by saying that for θ such that $0 \leq \theta \leq \pi/4$ r, varies from 0 to sec θ.

Again, pictorially,



(Figure 2)

For a given θ, r varies from 0 to P. P is on the line x = 1, but in polar coordinates this line has the equation r cos θ=1 or r = sec θ.

In this case,

$$\int_0^1 \int_0^x e^{-(x^2+y^2)} dydx = \int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} e^{-r^2} rdr\, d\theta \qquad (2)$$

5.3.5(L) continued

and even though

$$\int e^{-r^2} r dr = -\frac{1}{2} e^{-r^2}$$

we see that

$$\int_0^{\sec\theta} e^{-r^2} r dr = -\frac{1}{2} e^{-\sec^2\theta} - [-\frac{1}{2} e^0] = \frac{1}{2}(1 - e^{-\sec^2\theta}).$$

Therefore,

$$\int_0^1 \int_0^x e^{-(x^2+y^2)} dydx = \frac{1}{2} \int_0^{\frac{\pi}{4}} (1 - e^{-\sec^2\theta}) d\theta \qquad (3)$$

and the integral on the right side of (3) is hardly an improvement over the original double integral (as far as direct integration is concerned).

Thus, we see once again how any change of variables depends both on the region and the integrand for its success.

In any event, returning to (1) we have

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)} dydx = \int_0^{\frac{\pi}{2}} -\frac{1}{2} e^{-r^2} \Bigg|_{r=0}^{a} d\theta$$

$$= \int_0^{\frac{\pi}{2}} [-\frac{1}{2} e^{-a^2} - (-\frac{1}{2} e^{-0})] d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 - e^{-a^2}) d\theta,$$

5.3.5(L) continued

and since $(1 - e^{-a^2})$ is a constant, we conclude that

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)}\, dydx = \frac{1}{2}(1 - e^{-a^2}) \int_0^{\frac{\pi}{2}} d\theta$$

$$= \frac{\pi}{4}(1 - e^{-a^2}). \quad (4)$$

b. If we let $a\to\infty$ on the right side of (4), we obtain

$$\lim_{a\to\infty} \frac{\pi}{4}(1 - e^{-a^2}) = \frac{\pi}{4}. \quad (5)$$

On the other hand as $a \to \infty$, our region R becomes the entire first quadrant of the xy-plane (i.e., it is the circular disc whose radium increases without bound).

In other words,

$$\lim_{a\to\infty} \left[\int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)} dydx\right] = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dydx. \quad (6)$$

Hence, by taking the limit in (4) as $a\to\infty$, and using the result of (5) and (6), we have

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dydx = \frac{\pi}{4}. \quad (7)$$

Mechanically, one may obtain (7) directly using polar coordinates by observing that in polar coordinates the first quadrant is characterized by saying that for any θ such that $0 \leq \theta \leq \frac{\pi}{2}$, r varies from a to ∞ so that

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dydx = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} rdrd\theta = \int_0^{\frac{\pi}{2}} - \frac{1}{2} e^{-r^2}\Big|_{r=0}^{\infty} d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{4}.$$

5.3.5(L) continued

Our technique in deriving (7) merely formalizes this idea. Namely, saying that r varies from 0 to ∞ really means r varies from 0 to a, and we then let $a \to \infty$.

c. Since $e^{-(x^2+y^2)} = e^{-x^2}e^{-y^2}$, the variables are separable, so that

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dydx = \left[\int_0^\infty e^{-x^2}dx\right]\left[\int_0^\infty e^{-y^2}dy\right] \tag{8}$$

(where the usual caution is taken to make sure that each of these improper integrals is convergent).

We next notice that since x and y are merely variables of integration (and hence do not appear once the definite integral is evaluated),

$$\int_0^\infty e^{-x^2}dx = \int_0^\infty e^{-y^2}dy.$$

Therefore since

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dydx = \frac{\pi}{4}$$

[by equation (7)], and

$$\left[\int_0^\infty e^{-x^2}dx\right]\left[\int_0^\infty e^{-y^2}dy\right] = \left[\int_0^\infty e^{-x^2}dx\right]^2,$$

we obtain from (8) that

$$\left[\int_0^\infty e^{-x^2}dx\right]^2 = \frac{\pi}{4},$$

so that

$$\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi} \tag{9}$$

5.3.5(L) continued

The derivation of (9) shows how multiple integrals with a change of variables may be used to evaluate rather difficult (improper) single integrals. It should also be noted that while for teaching reasons we gave the parts of this exercise in the given order, in real-life the procedure is usually the opposite order. That is, we most likely would be given the integral

$$\int_0^\infty e^{-x^2}dx,$$

after which we would have had to have the ingenuity to observe that

$$\int_0^\infty e^{-x^2}dx = \left[\int_0^\infty e^{-x^2}dx \int_0^\infty e^{-y^2}dy\right]^{1/2}$$

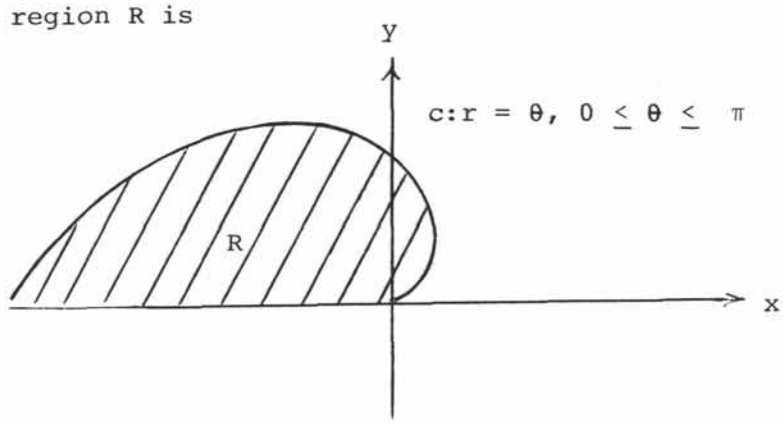
$$= \left[\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dydx\right]^{1/2}$$

$$= \left[\int_0^\infty\int_0^\infty e^{-r^2}rdrd\theta\right]^{1/2}$$

etc.

5.3.6

Our region R is



5.3.6 continued

For a given θ, $0 \leq \theta \leq \pi$, r varies continuously from r = 0 to r = θ, hence:

$$\iint_R \sqrt{x^2 + y^2}\, dA_R = \int_0^{\pi}\int_0^{\theta} r(r\,dr\,d\theta)$$

$$= \int_0^{\pi} \frac{1}{3} r^3 \Big|_{r=0}^{\theta} d\theta$$

$$= \int_0^{\pi} \frac{1}{3} \theta^3 d\theta$$

$$= \frac{1}{12} \theta^4 \Big|_{\theta=0}^{\pi}$$

$$= \frac{\pi^4}{12} \ (\approx 8.3).$$

(See next exercise for a further discussion of how we handle the integrand if R has a different shape.)

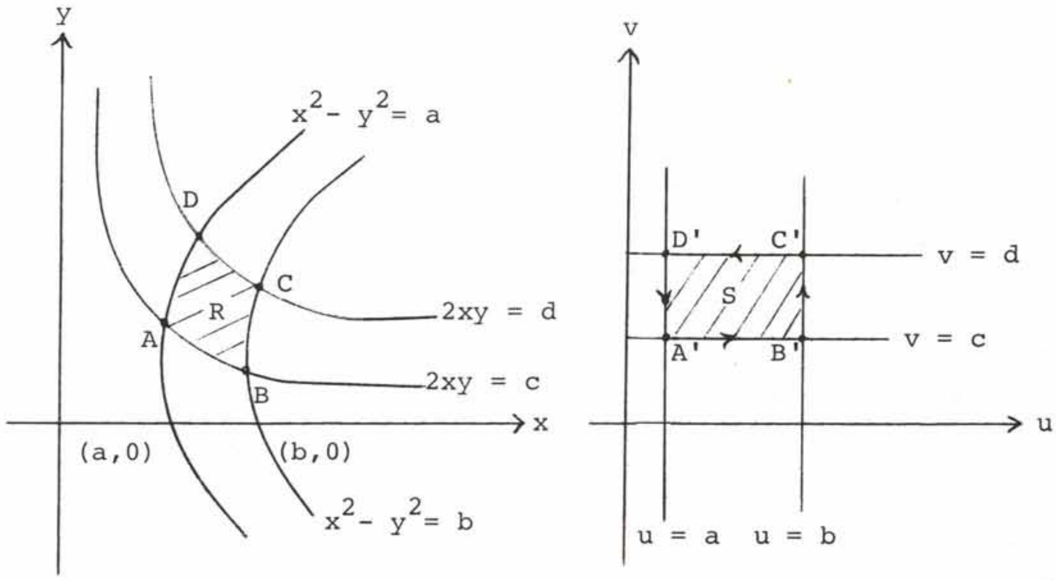
5.3.7(L)

a. An "obvious" change of variables which maps R into a rectangle whose sides are parallel to the u and v axes is

$$\left.\begin{aligned} u &= x^2 - y^2 \\ v &= 2xy \end{aligned}\right\}. \qquad (1)$$

Clearly, we see from (1) that if $\underline{f}$ is defined by $\underline{f}(x,y) = (u,v)$ then $x^2 - y^2 = a$ is mapped into the line u = a, $x^2 - y^2 = b$ is mapped into the line u = b, 2xy = c is mapped into the line v = c, and 2xy = d is mapped into the line v = d. In other words, $\underline{f}(R) = S$, where

5.3.7(L) continued



Notice that while $\underline{f}$ is not 1-1 (recall that the Jacobian of $\underline{f}$ vanished at the origin, and, in particular, $\underline{f}(x,y) = \underline{f}(-x,-y)$), $\underline{f}$ is 1-1 when its domain is restricted to R*.

Thus, we may talk about the back-map of S (that is, there are two regions, R and R', in the xy-plane that are mapped into S by $\underline{f}$; R' is the region in the third quadrant which is congruent to R) since the domain of $\underline{f}$ is restricted to R and $\underline{f}$ is 1-1 on R. The main point, however, is that the geometry for finding the area of R is not as straight-forward as that for finding the area of circular regions. In other words, polar coordinates are a rather special case in the sense that we know the geometry of circles sufficiently well that we prefer to divide circular regions directly into polar elements of area rather than to view the problem as a mapping of the xy-plane into the $r\theta$-plane.

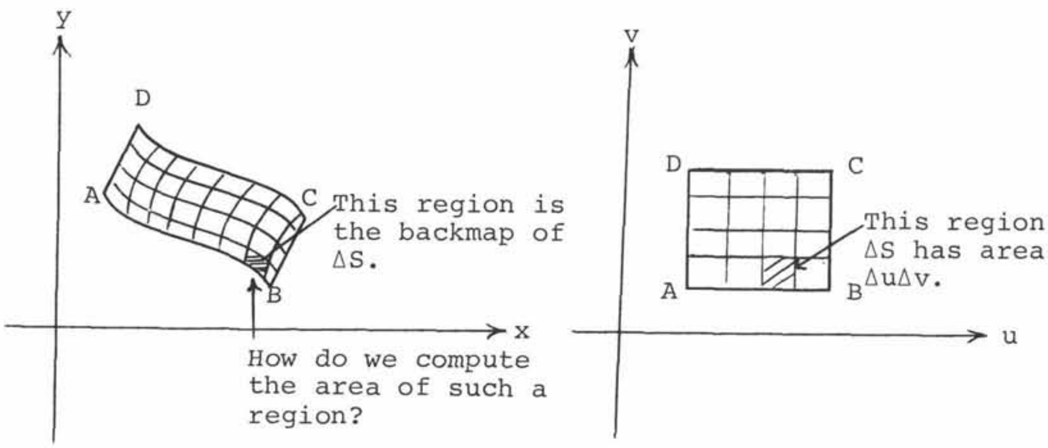
In the present exercise, if we divide R into the elements of area induced as a back-map from the corresponding rectangular

*Again, recall that a function is changed when its domain is changed. What we really mean is that the "new" function $\underline{f}_R$ (called the restriction of $\underline{f}$ to R) whose domain is R and such that $\underline{f}_R(x,y) = \underline{f}(x,y)$, for all $(x,y) \in R$ is 1-1. This follows from the fact that $\underline{f}$ is 1-1 on the boundary of R and

$\frac{\partial(u,v)}{\partial(x,y)} \neq 0$ inside R.

5.3.7(L) continued

elements of area of S, we wind up with hyperbolic elements of area, for which the geometry is not familiar to us. Pictorially,



In summary, when we view the region R under a change of variables given by

$$\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$$

there are two equivalent, but conceptually different, ways of looking at it. On the one hand, we may imagine that R is left unchanged but the elements of area of R are computed in terms of the new variables u and v (which is how we usually think of polar coordinates).* On the other hand, we may view the change of variables as mapping R into a region S in the uv-plane and we then find the area of R. Which of the two methods we use depends on the change of variables as well as on the region R. The dependence on R stems from the fact that S is the image of R and certainly the shape of S will depend on the shape of R. If the shape of S is not to our advantage, then clearly there is no advantage of mapping R into S.

In any event, if we now use the mapping defined by equation (1), our general theory yields

*See also the optional exercise 5.3.10.

5.3.7(L) continued

$$\iint_R g(x,y)\,dA_R = \int_a^b\int_c^d g(x(u,v),\ y(u,v)\ \frac{\partial(x,y)}{\partial(u,v)}\ dvdu. \qquad (2)$$

Now $g(x(u,v),\ y(u,v))$ is a function of u and v, say $h(u,v)$, and from (1)

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2+\ y^2).$$

Therefore,

$$\frac{\partial(x,y)}{\partial(u,v)} = [\frac{\partial(u,v)}{\partial(x,y)}]^{-1} = \frac{1}{4(x^2+\ y^2)}$$

[which is unequal to 0 since $x^2+\ y^2 = 0 \leftrightarrow x = y = 0$, and $(0,0) \neq R$]

Putting these results into (2) we obtain

$$\iint_R g(x,y)\ dA_R = \frac{1}{4}\int_a^b\int_c^d \frac{h(u,v)}{x^2+\ y^2}\ dvdu. \qquad (3)$$

Our final simplification of the integral on the right side of equation (3) involves using equation (1) to express $x^2+\ y^2$ in terms of u and v. As a quick review, recall that

$$\begin{cases} u^2 = x^4 - 2x^2y^2 + y^4 \\ x^2 = \qquad 4x^2y^2 \end{cases}$$

and therefore,

$$u^2 + v^2 = (x^2 + y^2)^2,$$

so that

$$x^2 + y^2 = \sqrt{u^2+\ v^2}$$

5.3.7(L) continued

(and we _must_ use the positive root here since $x^2 + y^2$ can't be negative).

Therefore, equation (3) takes the final form

$$\iint_R g(x,y)\,dA_R = \frac{1}{4}\int_a^b\int_c^d \frac{h(u,v)}{\sqrt{u^2+v^2}}\,dvdu. \tag{4}$$

Equation (4) now gives us a very strong hint as to what type of integrands, $g(x,y)$, will be converted into separation of variables by the mapping defined in equation (1). Namely if

$$h(u,v) = \sqrt{u^2+v^2}\,k(u)m(v) \tag{5}$$

then equation (4) becomes

$$\iint_R g(x,y)\,dA_R = \frac{1}{4}\int_a^b\int_c^d k(u)m(v)\,dvdu$$

$$= \frac{1}{4}\left[\int_a^b k(u)\,du\right]\left[\int_c^d m(v)\,dv\right].$$

Recalling that $h(u,v) = g(x,y)$* etc., we may convert equation (5) into the language of x and y by writing

$$g(x,y) = (x^2+y^2)k(x^2-y^2)m(2xy). \tag{6}$$

Summed up, then, equation (6) tells us that the most general integrand $g(x,y)$, which is converted into variables separable on the rectangle S defined by the mapping in equation (1) is one of the form

(x^2+y^2) x (any function of x^2-y^2) x (any function of 2xy).

---

*That is, $h(u,v) = g(x(u,v),\ y(u,v)) = g(x,y)$. In other words the "height" of the surface $z = g(x,y)$ above the point (x,y) is the same as the height of the surface $w = h(u,v)$ above the point (u,r) where (u,v) is the image of (x,y) with respect to the given mapping.

5.3.7(L) continued

b. Applying our results to the special case

$k(u) = e^u$
$m(v) = \cosh v$

we have

$$\iint_R (x^2 + y^2) e^{x^2 - y^2} \cos h\ 2xy\ dA_R = \frac{1}{4}\int_a^b \int_c^d e^u \cosh v\, dv\ du$$

$$= \frac{1}{4}\int_a^b e^u du \int_c^d \cosh v\, dv$$

$$= \frac{1}{4}(e^b - e^a)(\cosh d - \cosh c)$$

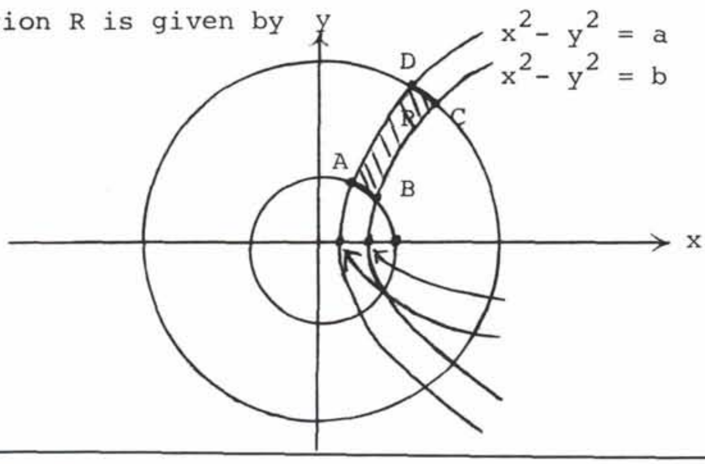
Note that while the mapping

$$\left.\begin{array}{l} u = x^2 - y^2 \\ v = 2xy \end{array}\right\}$$

always carries R into S, unless the integrand g(x,y) has the form described above, the new integral may be as difficult to evaluate as the original integral.

5.3.8

Our region R is given by

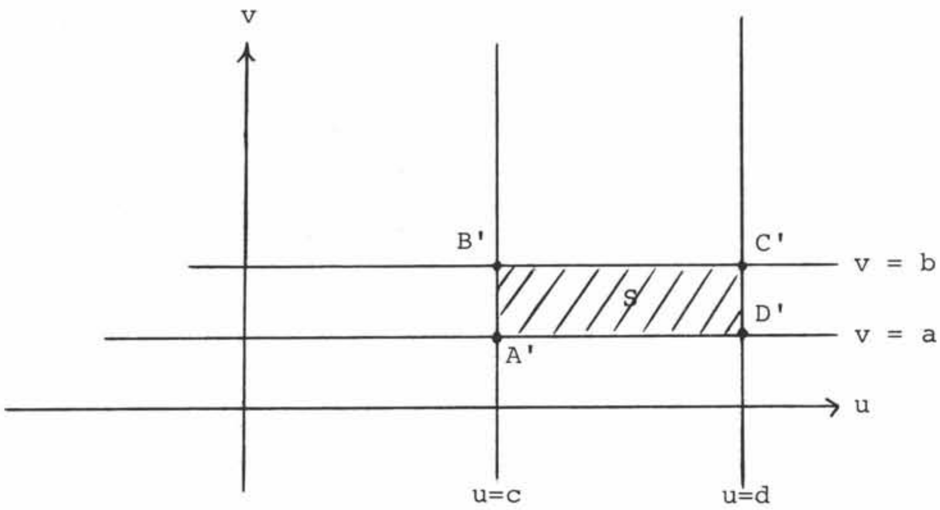


5.3.8 continued

and the image of R under the mapping

$$\left.\begin{aligned} u &= x^2 + y^2 \\ v &= x^2 - y^2 \end{aligned}\right\} \qquad (1)$$

is the region S, where



From (1),

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & 2y \\ 2x & -2y \end{vmatrix} = -8xy. \qquad (2)$$

The negative sign in (2) indicates that ABCD has the opposite sense of A'B'C'D', and we also conclude from (2) that

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{-1}{8xy}. \qquad (3)$$

(So that rather than (1), you might have preferred
$u = x^2 - y^2$
$v = x^2 + y^2$)

The fact that $x \neq 0$ and $y \neq 0$ for every $(x,y) \varepsilon R$ guarantees that $\partial(x,y)/\partial(u,v)$ exists at all points, and we conclude from (3) that

$$\iint_R f(x,y)\,dA_R = \int_c^d\int_a^b f\bigl(x(u,v),y(u,v)\bigr)\ \left[\frac{1}{8xy}\right]\,dvdu. \qquad (4)$$

5.3.8 continued

In particular if we let f(x,y) = 8xy, equation (4) yields

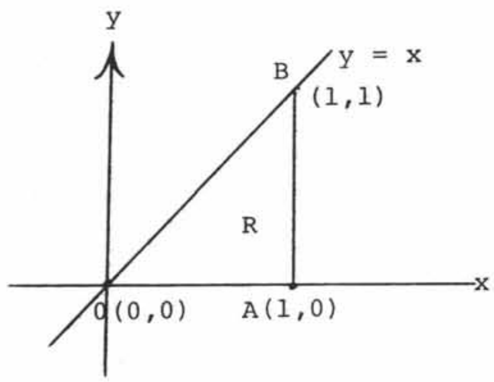
$$\iint_R 8xy\,dA_R = \int_c^d\int_a^b 8xy\left[\frac{1}{8xy}\right]dv\,du$$

$$= \int_c^d du\int_a^b dv$$

$$= (d - c)(b - a). \qquad (5)$$

5.3.9 (optional)

Our region R is defined by



Notice that the mapping given by

$$\left.\begin{array}{l} u = x \\ v = \frac{y}{x} \end{array}\right\} \qquad (1)$$

is not defined at 0(0,0) since in this case $v = \frac{0}{0}$.

The mapping is defined, however, on R with (0,0) deleted.

To see what the image of R with (0,0) deleted [i.e., R - (0,0)] is, let us observe that with y = x,

$\frac{y}{x} \equiv 1$ (if $x \neq 0$)

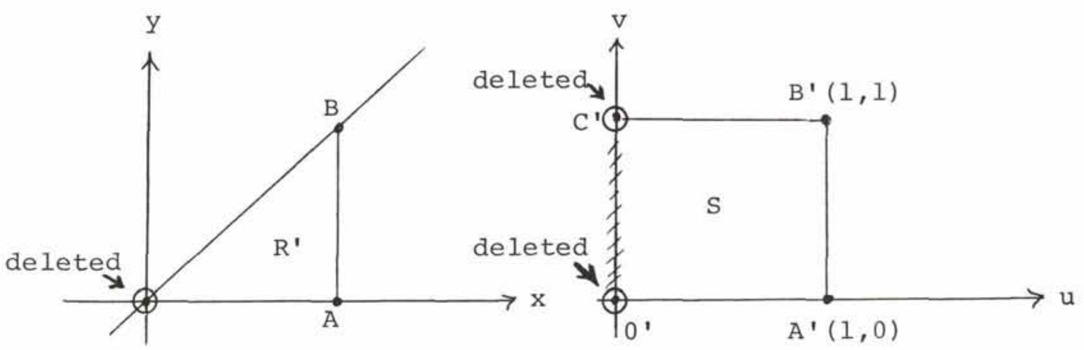
5.3.9 continued

so that from (1), $v = 0$. In particular the line OB (with (0,0) deleted) is mapped onto the line segment $v = 1$, with $0 \leq u \leq 1$.

Also from (1), $x = 1$ is mapped onto $u = 1$, so that the image of BA is $u = 1$ with v varying from 1 to 0 (i.e. $u = 1 \quad x = 1 \quad v = y$), and on BA y varies from 1 to 0.

Finally, $y = 0$ maps onto $v = 0$ (provided $x \neq 0$) since $v = \frac{y}{x} = \frac{0}{x}$. Therefore AO with 0 deleted corresponds to $v = 0$ as u varies from 1 to 0 (with 0 excluded).

Pictorially then:



(Figure 1)

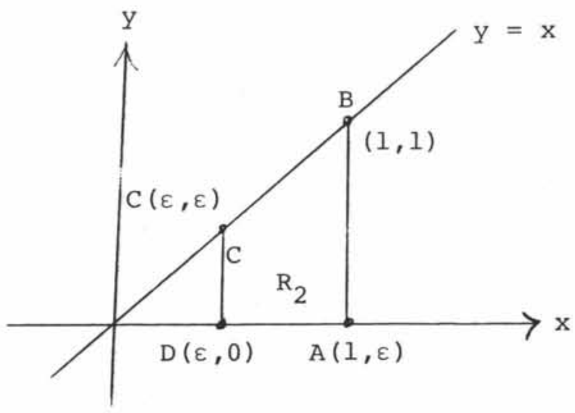
That is, letting $R' = R - (0,0)$, the image of R' is the unit square S with 0'C' deleted. Notice that 0'C' is part of the line $u = 0$, and $u = 0$ is where $\partial(u,v)/\partial(x,y) = 0$.

The key point is that the deleted points form sets of zero area. Namely in the xy-plane the deleted set is the point (0,0) while in the uv-plane the deleted set is the line segment 0'C'.

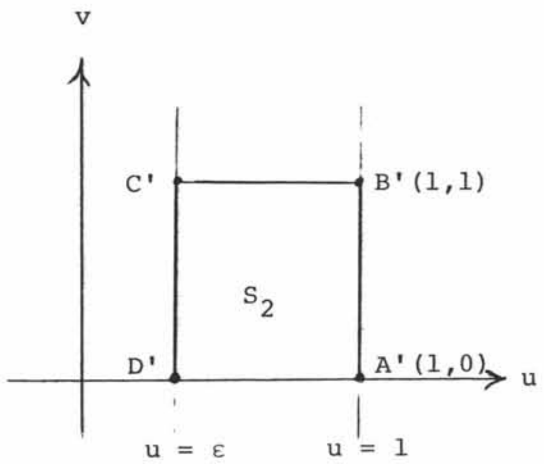
Since the double integral is essentially concerned only with area, the fact that things go wrong with the Jacobian on a area zero causes no change in the general result.

Had we wished to be more rigorous we could have defined R_2 by

5.3.9 continued



and observed that R_2 was mapped by (1) onto



Considered as a mapping from R_2 onto S_2, all is well since on S_2, u is never 0.

We would then <u>define</u>

$$\iint_R f(x,y)\,dA_R = \lim_{\varepsilon \to 0} \iint_{R_2} f(x,y)\,dA_{R_2}$$

and this would yield the proper result.

Notice that this problem occurs whenever we use polar coordinates on a region R which contains the origin (as in Exercises 5.3.5 and 5.3.6). We just did not notice it then since we usually do not

5.3.9 continued

view polar coordinates as mapping R onto S, but rather as another way to partition R.

The fact is, however, that

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

so that r = 0 is a "trouble spot". Indeed with r = 0, θ may assume any value since r = 0 denotes the origin regardless of the value of θ.

This difficulty, which doesn't seem to present itself when we view polar coordinates in the traditional manner, may be overlooked since the origin and its image (the line r = 0 in the rθ-plane) both are sets with zero area.

At any rate, if we now turn our attention to the original problem, we may conclude that

$$\iint_R \sqrt{x^2+y^2}\, dA_R = \int_0^1\int_0^1 [u\sqrt{1+v^2}]\, u\, dvdu$$

$$= \int_0^1 u^2 du \int_0^1 \sqrt{1+v^2}\, dv$$

just as we would have, had we neglected to notice that u = 0 was a "trouble-spot"!

5.3.10 (optional)

We have already emphasized how polar coordinates lend themselves to two interpretations of change of variables. It should also be clear that linear mappings may also be viewed in the same manner. That is, the substitutions u = ax + by and v = cx + dy are equivalent to partitioning our given region R into a grid of parallelograms whose sides belong to the families of straight lines

5.3.10 continued

ax + by = constant and cx + dy = constant. Since we know how to compute areas of parallelograms, it is not necessary, from a geometrical point of view, to map the xy-plane into the uv-plane in order to evaluate the double integral (although we may be more comfortable doing this).

With these remarks in mind, it is our feeling that the only remaining loose thread in terms of viewing double integrals in terms of a change of variables is that we have not yet used the change of variables idea to explain how one may view a change in the order of integration as a mapping of the xy-plane into the uv-plane. The aim of this exercise is to explain this.

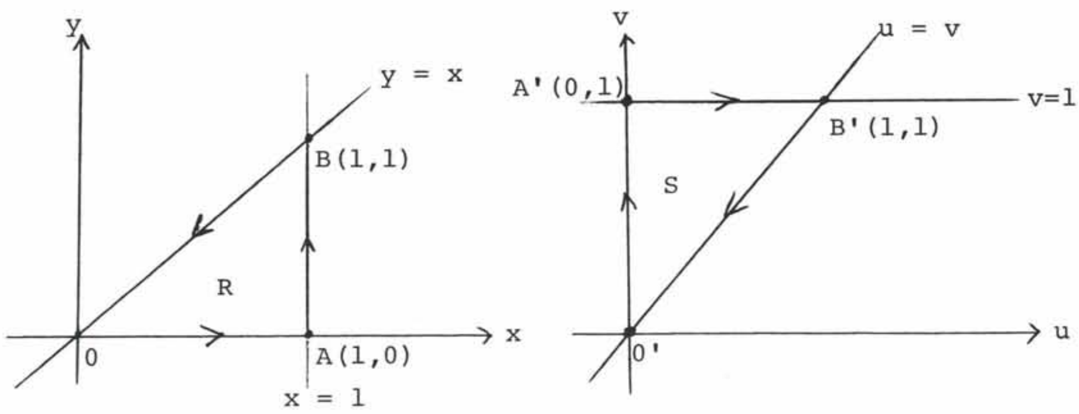
Notice that interchanging the order of integration is equivalent to interchanging the roles of the two variables in the integral. Since the u-axis is viewed as the image of the x-axis and the v-axis as the image of the y-axis, interchanging x and y is equivalent to the change of variables

$$\left.\begin{aligned} u &= y \\ v &= x \end{aligned}\right\} \qquad (1)$$

Using (1), we see that

$y = x \leftrightarrow u = v$
$x = 1 \leftrightarrow v = 1$
$y = 0 \leftrightarrow u = 0$

so that the mapping defined by (1) yields



5.3.10 continued

From (1) we also have that

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

(which agrees with our diagram since the mapping does give S the opposite orientation of R). Consequently,

$$\iint_R f(x,y)\,dydx = \iint_S f(v,u)\,dA_S \qquad (2)$$

and since we want our integral to be in the order dxdy (= dvdu) we write dA_S in the form dvdu, and obtain from (2) that

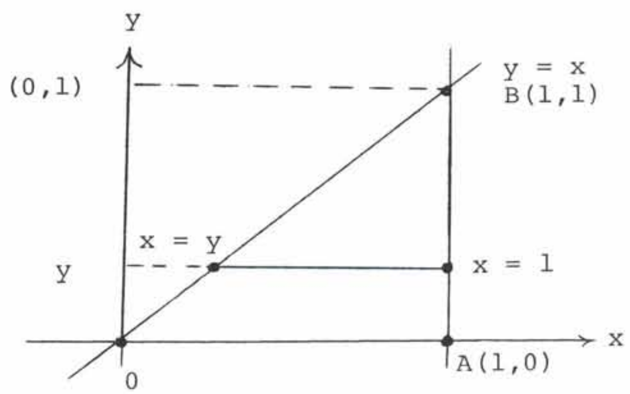
$$\int_0^1\int_0^x f(x,y)\,dydx = \int_0^1\int_u^1 f(v,u)\,dvdu.$$

The key point is that in the derivation of equation (3) we used a mapping of the xy-plane into the uv-plane. We <u>did not</u>, as in our earlier encounters with this type of problem, actually "slice up" R with horizontal strips rather than vertical strips.

If we now replace v by x and u by y in the right side of equation (3), we obtain

$$\int_0^1\int_0^2 f(x,y)\,dydx = \int_0^1\int_y^1 f(x,y)\,dxdy$$

and this agrees with our earlier method; i.e.,



5.3.10 continued

Thus, if we so desire, changing the order of integration can be accomplished by mapping (i.e., change of variables) techniques. Aside from the fact that such a result is nice from the point of view of making our theory complete, it is important to point out that in handling multiple integrals of higher dimension (in which we can no longer rely on a geometric model) all our transformations must be handled analytically (i.e., in terms of a change of variables).