CALCULUS REVISITED
PART 2
A Self-Study Course

SUPPLEMENTARY NOTES

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7   Applications of Linear Algebra to Non-Linear Functions

AN INTRODUCTION TO MATHEMATICAL STRUCTURE

A

Introduction

In recent times, there has been considerable emphasis placed on the concept of mathematical <u>structure</u>. One motivation for this is that it often happens that two apparently different topics are based on the same rules. Thus, if we assume that we accept only those consequences which follow inescapably from the rules, then as soon as two different "models" obey the same rules it follows that something that is a consequence in one model will be an inescapable consequence in the other model. In other words, once we have proven a result in one model, the proof automatically holds in the other. This affords us a remarkable short cut in studying topics which have the same structure as preciously-studied topics. For this reason alone (although there are others which we shall see as our course unfolds), it would be worthwhile for us to study mathematical structure.

In case our discussion seems devoid of any practical application, let us review what we have said in terms of a situation which has occured in Part 1 of our course. Specifically, let us revisit the study of area.

We mentioned that it was easy to define area, subjectively, as the amount of space contained in a region. The trouble was that this definition from a computational point of view, gave us preciously little with which to work. So we set out to find a more practical way of computing area by getting a more objective definition which still agreed with what we believed to be true intuitively (subjectively). To this end, we essentially imposed three rules on area, rules which were based on properties that we felt certain applied to area.

(1) The area of a rectangle is the product of its base and height.

(2) If one region is contained within another, the area of the contained region cannot exceed that of the containing region.

(3) If a region is subdivided into a union of mutually-exclusive parts, then the area of the region is equal to the sum of the areas of the constituent parts.

Once these three rules were imposed, we applied nothing but accepted principles of mathematics to deduce inescapable conclusions. We not only relived the ancient Greek experience of computing areas; we also showed that the acceptance of the three rules led to a truly marvelous relationship between areas and differential calculus which culminated in the two fundamental theorems of integral calculus.

But, the real advantage of this _structural_ approach was yet to come! In particular, when we decided to study volume, we found that with appropriate changes in vocabulary we could get the same three rules of area to apply for volume as well. In fact, our only change in the second and third rules was to replace the _word_ "area" by the _word_ "volume." Namely,

(2') If one region is contained in another, the volume of the con- tained region cannot exceed that of the containing region.

(3') If a region is subdivided into a union of mutually-exclusive parts then the volume of the region is equal to the sum of the volumes of the constituent parts.

As for our first rule, we not only replace "area" by "volume" but we also replace "rectangle" by "cylinder." We then obtain

(1') The volume of a cylinder is the product of (the area of) its base and height.

The key point was then to remember that rules were relationships be- tween terms and that if the terms changed but the relationships didn't, the _structure_ was unaltered. We were able to prove many results about volume "instantaneously," so to speak, merely by recopying the corresponding proof for areas. In fact, this is why we referred to volume in our lecture as "3-dimensional area."

Equally important in this study was the discovery that if the rules are different, the structures may vary. In this respect, we showed that we had to be a bit wary when we studied arclength because, in this case, we could _not_ hand down the same three rules just by re- placing "area" by "length." In particular, had we tried this with (2), we would have obtained

(2") If one region is contained within another, the length (perimeter) of the contained region cannot exceed that of the containing region.

It happens, as we showed, that this need not be true. In many cases, the region with _lesser_ area may have a _greater_ perimeter.

In terms of a trivial but, hopefully, informative example, had we agreed to accept (2") as a rule, we should have been forced to accept such inescapable consequences as: the circumference of a circle of radius R is indeterminate in the sense that it can be made to exceed any given number. Namely, given that number we inscribe a region in the circle whose perimeter exceeds that number. Since the region is contained in the circle, (2") forces us to conclude that the length of the circle (circumference) exceeds the length of the inscribed region!

More concretely, to "prove" that the circumference of a circle whose radius is 1 exceeds 10 (and this must be false since we "know" the circumference is 2π or a little more than 6 but less than 7), we take a piece of string whose length is, say, 11 inches and curl it around inside the circle. Pictorially,

11" piece of string is wrapped around inside a circle whose radius is 1".

Then, from (2"), we conclude that the circumference of the circle exceeds 11 inches. Notice that while the result is indeed false, it is still an inescapable consequence of the rules we have accepted.

In any event, this did not mean that we could not study arclength. Indeed, we went on to study it in rather great detail. What was important was that whenever there was a property that was true for either area or volume and which was a consequence of (2) [or (2')] we had to exercise caution in the study of length since (2") was not a realistic rule to accept. In an abstract "game" one need not require that the rules be realistic, but in the "game of life" where we try to measure and define reality, it is quite natural that we would insist on "realistic" rules.

This, in turn, leads to the important concept of distinguishing between truth and validity, where, by validity, we mean that the conclusion follows inescapably from the rules, without regard to the truth (or falsity) of the rules, and this too, will be discussed in much greater detail later in the chapter.

To conclude our introduction, we must now explain, at least from a motivational point of view, why we elected to introduce this material at the present time. The answer is quite simple. In this course, we shall be primarily concerned with the study of functions of several (i.e., more than one) variables. We have already studied functions of a single (one) variable, and, in the context, we introduced such terms as absolute value, limits, continuity, derivative, etc. These concepts will also occur (perhaps in an altered form) in the study of several variables. Our point is that if the structure (the rules and definitions) are the same for these concepts

as they were in the study of calculus of a single variable then we may
carry over the previous structure virtually <u>verbatim</u>. In this way, we
not only have a short cut for studying these "new" concepts but we
also have the advantage of seeing more clearly the structure upon which
everything is based.

Before we get on with this idea, it is important that the concept of
structure along with the companion concept of truth and validity be
understood in their own rights.   The remainder of this chapter is
devoted to this purpose.

The "Game" of Mathematics

In our introduction to mathematical structure, we have employed words
like "definitions and rules" and "inescapable consequences" as though
we were dealing with a "game" rather than a mathematical concept. The
analogy is deliberate. For it is our claim that, not only mathematics,
but any topic in the curriculum can be viewed as a game provided we
define a game in the most general terms.

To see how we should define a game, let us ask ourselves what it is
that <u>all</u> games, no matter how different they may seem to be, have in
common. In other words, how can we abstractly (meaning without refer-
ence to any particular game) define a game so that every game is cover-
ed by our definition? The answer that we shall use, for purposes of
this course, is that a game is any system consisting of definitions,
rules, and objectives, where the objectives are carried out as <u>inescap-</u>
<u>able</u> consequences of the definitions and the rules by means of strategy.
Paraphrasing this in terms of a diagram, we have
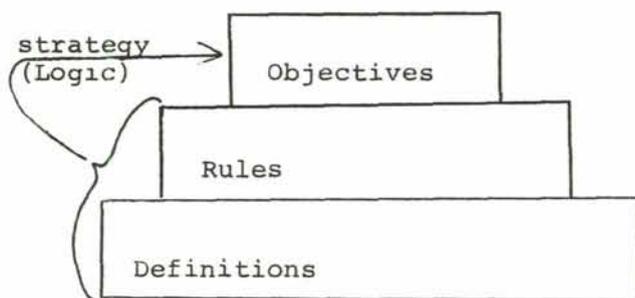
The interesting thing from our point of view is that this definition
of a game does indeed make almost any study a game. That is, in any
study, we define certain concepts, impose certain rules (usually

dictated by our experience), and the objective is then to see what conclusions follow inescapably from our rules.

First we define certain terms, but even this is not as easy as it may sound. For example, in arithmetic, it is clear that the basic "playing piece" is a number, but how shall we define a number? Certainly, we all know what a number is, but any attempt to define a number objectively seems to lead to the circular reasoning property that sooner or later our best definition of number contains the concept of number in the definition. This is true everywhere, not just in mathematics. From a nontechnical point of view, imagine that we are back at the dawn of consciousness and we are trying to invent our first language. Clearly the first object that we decide to name cannot be defined in terms of other named objects because there are none. While we can quibble as to what words will be "undefinable" and what words won't be, the fact is that certain concepts are too elementary (meaning basic, not simple) to be defined other than by circular reasoning. These concepts are called primitives, and in arithmetic, number is a primitive. In geometry, examples of primitives would be point and line.

Once we have our primitives, we may then define other terms in terms of our primitives. For example isosceles isnot primitive. We call a triangle isosceles if it has two sides of equal length. Thus, isosceles can be defined in terms of triangles and lengths. We shall discuss any additional examples as they may occur in the context of our course.

Next, we invent rules (which for some reason are always referred to as axioms or postulates) which tell us how the various terms in our "game" are related. These rules may be motivated by what we believe to be true in real-life, but even this is not mandatory. What is important is that, for example, since number is a primitive concept, any attempt to define, say, equality of numbers, or the sum of two numbers, or the product of two numbers, might itself be subjective.

To avoid this problem, we agree to use only certain specific "facts" about these concepts, which we introduce into the game as rules. In this way, we can make an objective study of a subjective concept. If this seems difficult, notice that we have already done this when we studied area. The concept of "the amount of space" was subjective, but the three axioms were quite objective, and these axioms were all we used in deriving other properties of area.

Our objectives were then inescapable conclusions based on our definitions and rules. Notice that our aim is not to ask whether a conclusion is true but rather whether it is an inescapable consequence of

our definitions and rules. For example, long before the time of Euclid, practical men knew (and used the fact) that the base angles of an isosceles triangle were equal. The contribution of Euclid was that he showed that this result could be deduced from "rules of the game." That is, in terms of our game idea, a conclusion is only a conjecture until we show that it is an inescapable consequence of our definitions and rules. The process by which we decide that the conclusion is inescapable is known as a proof (which corresponds to the "strategy" part of the game). It is this proof together with the conjectured conclusion that becomes known as a theorem or proposition.

It is of interest to note that the words "axiom," "postulate," "proof," and "theorem" inevitably suggest the typical high school geometry course. And, in fact, the geometry proof is an excellent example of mathematical structure, clearly seen. What the "new math" in our schools now emphasizes, however, is that we can make a similar "game" out of the other mathematical topics. Just as in plane geometry, we can apply a statement-reason format in any branch of mathematics to proceed logically to an inescapable conclusion from a collection of definitions, rules, and hypotheses.*

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Truth and Validity

Thus far in our discussion of mathematical structure, we have avoided any discussion of truth and validity. There is an important, if somewhat subtle, difference between these two terms which we shall examine in this section.

Before we do this, however, perhaps it would be wise to try to describe the two concepts informally. To begin with, truth involves a subjective value judgement concerning particular statements. As such, truth is a rather nebulous thing. It is controversial in the sense that different people will make different judgements (for example, is it true

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*The meaning of hypothesis occupies a special role within the structure of a game. In every game, when we study strategy we consider what we should do if a particular event occurs. The particular event need not actually happen; all that we want to be sure of is that if it does happen, we know what to do. In this context, we refer to such an event as a hypothesis. For example, when we say "the base angles of an isosceles triangle are equal" the fact that the triangle is isosceles is called the hypothesis, for certainly, given a triangle at random, there is no rule of the game that says it must be isosceles. In still other words, if the student asks how we know the triangle is isosceles, we simply tell him that we were told so, or it was given information, or it was the hypothesis in this example.

that a particular painting is beautiful?). Moreover, truth is also subject to change. That is, what is believed to be true at one time may be believed to be false later.

It is in the latter context that one can begin to sense the idea of validity. That is, do we change our minds about the truth of a statement? In many cases it is that someone presents us with evidence that we hadn't considered before. In other words, part of our concept of truth seems to involve not just the statement involved, but the reasoning by which we arrived at the statement. In short, we have certain "evidence" that we believe to be true and we then ask on the basis of the evidence whether the conclusion is justifiable. In still other words, we want to know whether the conclusion follows inescapably from our assumed beliefs.

This is what validity is all about. It is the study of determining whether a statement follows as an inescapable consequence of other statements. As far as terminology is concerned, the statement being tested is called the conclusion, and the assumed statements are called the premises (or simply the assumptions). The process of testing whether the conclusion follows inescapably from the premises is called the argument. In this context, truth is used to describe the premises and the conclusion, while validity is used to describe the argument. If the conclusion follows inescapably from the assumptions, the argument is called valid regardless of the truth of the premises or the conclusion, and if the conclusion doesn't follow inescapably from the premises then the argument is called invalid, again regardless of the truth of either the premises or the conclusion.

The main idea is that truth and validity are entirely different concepts, related, however, by the basic belief that if the premises are true and the argument is valid than the conclusion must also be true. We must, however, be careful not to interpret our last statement too liberally. It is possible that an argument can be valid and the conclusion be true, even though the premises are false (this is why no number of experiments ever prove a theory to be correct; the correct results could be occurring despite wrong assumptions). By way of an example, the true statement all bears are animals follows inescapably from the assumptions that all bears are trees and all trees are animals; yet each of the assumptions is false. Again, emphasizing the correct connection between truth and validity in a valid argument we cannot obtain a false conclusion from true premises.

Hopefully, the next diagram summarizes the two concepts succinctly.

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┌─────────────────┐                              ┌──────────────┐
│ Premises        │                              │              │
│ which may       │                              │              │
│ consist of      │          argument            │  Conclusion  │
│ definitions,    │──────────────────────────────│              │
│ rules, and      │             ↑                │              │
│ hypotheses.     │             │                │              │
└─────────────────┘             │                └──────────────┘
         ↑                      │                       ↑
      Truth                 Validity                  Truth
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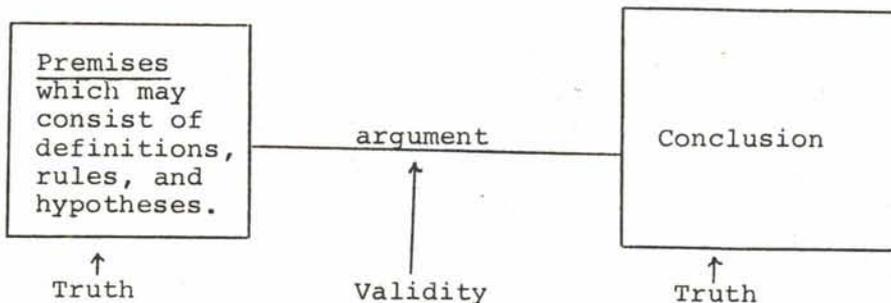
The point is that we can have many combinations involving truth and
validity. In fact, in a valid argument we can have any combination
of truth and falsity for the assumptions and the conclusions, except
as we have mentioned, in a valid argument the conslusion must be true
as soon as the assumptions are.

Rather than continue in this expository tone, perhaps it would be bet-
ter at this time to illustrate these ideas in terms of some actual ex-
amples. To this end, we have:

(1)   All Bostonians are New Yorkers.
   All New Yorkers are Texans.
      All Bostonians are Texans.

In this case, our argument is called valid since the conclusion does
follow inescapably from the assumptions, even though the conclusion is
false. (To think of this from another point of view, imagine a game
wherein there are three types of "pieces" called Bostonians, New Yorkers,
and Texans. If we impose as rules of our game that in this game, all
Bostonians are New Yorkers, and that all New Yorkers are Texans, then
it is an inescapable conclusion in this game that all Bostonians are
Texans.)

(2)   All Frenchmen are European.
      All Germans are European.
      All Frenchmen are Germans.

In this case our conclusion is false and the argument is invalid (i.e.,
not valid) since it does not follow inescapably from our assumptions.

(3)   All Parisians are Europeans.
      All Frenchmen are European.
      All Parisians are Frenchmen.

In this case the assumptions and the conclusion are all true, yet the argument is invalid since the truth of the conclusion does not follow merely from the truth of the assumptions. In fact, structurally, (2) and (3) have the same form. Namely, the first set is a subset of the second set, the third set is also a subset of the second set - from which it need not be true that the first set is a subset of the third set. Pictorially,
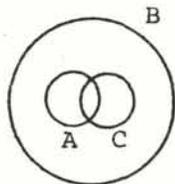
All A's are B's
All C's are B's } but it is false that all A's are C's.

The fact that the conclusion may be true is not as important as that it need not be true. Mathematically speaking, our aim is to draw inescapable conclusions, and in such a quest we must demand that our rules of logic be restricted to those which are always true.

(4) All Parisians are Frenchmen.
     All Frenchmen are Europeans.
     All Parisians are Europeans.

In this case our conclusion is true and the argument is valid. In terms of a more symbolic approach, both (1) and (4) have the form: All A's are B's, All B's are C's. Hence all A's are C's. Again, in terms of a picture,
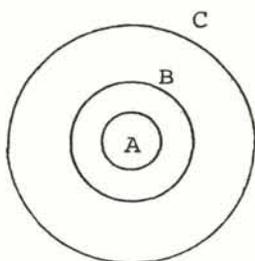
By this time it should begin to appear obvious that the distinction between truth and validity is of paramount importance in any scientific investigation (social science as well as physical science). To make sure that you have adequate opportunity to understand this distinction, we have supplied some exercises in order that you may practice. Other than this, we shall not explore further, in this course, the philosophical

implications of what is meant by truth, nor shall we introduce an
in-depth course in logic so that we may better understand the full
meaning of inescapable (that is, as simple as it might seem, the
concept of "inescapable" is quite sophisticated, for we must come
to grips with that subtle distinction between that which is truly
inescapable and that which is not inescapable but we don't know how
to avoid the conclusion). These topics are indeed important, but
for the purposes of this course we shall assume that our previous
remarks are sufficient.

We would, however, like to close this section with one more observa-
tion; an observation which is particularly pertinent to the idea of
structure.

Looking at mathematics in the light of what we have said about truth
and validity, we can now make a distinction between "pure" and "applied"
mathematics. If the axioms and postulates happen to be based on what
we believe to be reality, the resulting structure is referred to as
applied (i.e., practical) mathematics. If the rules are merely con-
sistent but do not correspond to reality, then we say that the struc-
ture is pure (or abstract) mathematics.

In making this distinction, however, we should remember that, struc-
turally, pure and applied mathematics are identical. And, in fact,
a model that seemingly bears no relationship to the real world at
present may turn out to be a "realistic" model in the future. A
classical example of this is Lobachevsky's geometry which was pure
math from its invention in 1829 until 1915 at which time Einstein
noticed that it served as a realistic model for his theory of
relativity.

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Algebra Revisited

Surprising as it may seem, the traditional sequence of high school
algebra courses may be viewed as a very elegant example of the game
idea. Indeed, algebra may justifiably be called the game of arithmetic.
To begin this game of arithmetic, let us assume that we are familiar
with the real number system wherein we will take as our primitives
equality, addition, and multiplication (omitting subtraction and
division which are not primitive since they can be defined as the
inverses of addition and multiplication respectively).

As we mentioned in Section B, to keep our game objective, we do not
give specific verbal definitions of these primitive concepts but

rather list the rules which we will use to relate these concepts so that we have an objective starting point from which to deduce conclusions. In somewhat oversimplified form, the idea is that we make up the rules and the other "players" agree to accept them. Once they agree to this, we are not allowed to invoke any "facts" in the game <u>other than those which follow inescapably from the accepted definitions and rules</u>.

Thus, our game of arithmetic might begin with the following five rules* which we hope will give the necessary degree of objectivity.

E-1: For any number b, b = b. (The <u>Reflexive</u> Rule)

E-2: For any numbers a and b, if a = b then b = a. (The <u>Symmetry</u> Rule)

E-3: Given any three numbers a, b, and c, if a = b and b = c, then a = c. (The <u>Transitive</u> Rule)

Let us pause briefly to make a few remarks about our first three rules concerning equality. Equality is a <u>relation</u> just as "is the brother of," "is less than," etc. are also relations. Not every relation is reflexive. Stated in abstract terms: if we write aRb to abbreviate "a is related to b by the relation R," reflexive means that aRa is a true statement. If we let R denote "is the brother of," then aRa need not be true since a person is not his own brother, or if we let R denote "is less than" then for any number b, bRb is false, since no number is less than itself.

In a similar way, not every relation is symmetric. Again if R denotes the relation "is the brother of," if aRb is true it <u>need</u> not follow that bRa is true. For example, if it is true that John is the brother of Mary, this does not imply (we hope) that Mary is the brother of John! If R denotes "is less than," notice that as soon as aRb is true, bRa is false (since if the first is less than the second, the second is <u>greater</u> than the first).

Finally, not every relation is transitive. For example, if R denotes "is the father of" then if both aRb and bRc are true, aRc <u>must</u> be false. Namely, if a is the father of b and b is the father of c, then a is the grandfather of c, <u>not</u> the <u>father</u> of c.

It should be observed that, for a given relation, the properties of being reflexive, symmetric, and transitive are independent.

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*Some texts refer to "properties" rather than "rules." Our interpretation will be that in the context of a game, we shall use the term "rules." If we then find the "real-life" model which obeys our rules of the game, then in referring to the model we will call our rules <u>properties</u> of the model.

That is, a given relation might have none, any one, any two, or all
three of the properties. For example, "is less than" is neither reflex-
ive nor symmetric but it is transitive (i.e., if the first is less than
the second and the second is less than the third, then the first is
less than the third).

Any relation that is reflexive, transitive, and symmetric is called
an equivalence relation. Thus, equality is an equivalence relation.
Other examples of equivalence relations are "is the same height as"
and, in geometry, "is congruent to." In terms of the structure of
our game, then, if we use "=" to denote any equivalence relation, then
E-1, E-2, and E-3 are properties for that model.

The key point of interest about equivalence relations is that the
usual rule of substitution as learned in high school algebra applies.
Namely, in its most abstract form, if R is an equivalence relation
and aRb is true then we may replace a by b and vice versa with respect
to R. For example, if R denotes "is the same height as" and if aRb
is true then a and b are equivalent (may be substituted for one another)
as far as height is concerned. Of course a and b might not be equi-
valent with respect to other relations, such as "has the same color
hair." For example, a and b could have the same height but different
color hair.

At any rate, since equality is an equivalence relation, we add to our
list of rules

E-4: If a = b, then we may interchange a and b at will in any
      relation involving equality*. (The Substitution Rule)

Finally, to exclude any "middle ground" (in fact, in logic we refer to
this as the rule of the excluded middle), we introduce

E-5: Exactly one of the following statements must be true (1) a = b,
      (2) It is false that a = b. If (2) is true then we write a $\neq$ b.
      (The Rule of Dichotomy)

All we ask now is that each player agree to accept these five rules
governing equality. Aside from this, we ask him to accept no further
assumptions, nor do we ask him why he accepts our rules. On the other

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*For example, when we say 3 + 2 = 5, we certainly do not mean that the
symbols 3 + 2 = 5 look alike. Rather, what we mean is that any problem
to which the number represented by the symbol 3 + 2 is the correct
answer, also has the number represented by 5 as the right answer.

hand, during the playing of our game, we must never invoke any properties of equality other than those stated in our rules unless they can be shown to follow inescapably from these rules.

Let us now proceed to impose a few rules of the game on addition. First of all, we know that the sum of two numbers is a number, so perhaps a good rule to invoke is

A-1:   If and b are numbers so also is a + b   (The Rule of <u>Closure</u>)

A-2:   If a and b are numbers a + b = b + a.   (The Cummutative Rule for Addition)

A-3:   If a, b, and c are numbers then a + (b + c) = (a + b) + c.   (The Associative Rule for Addition)

Again, before going further with our rules, let's make sure that we understand the impact of the first three rules. To begin with, in mathematical structure we often assume that when we combine "like" things we get "like" things. The point is that this is not always true. For example, if we are dealing with the set of odd numbers, we must observe that the sum of two such numbers is not a member of the set. Namely, the sum of any pair of odd numbers is always an even number. In other words, the rule of closure states that when elements of a set are "combined" by the given operation, the resulting element is again a member of the same set. Notice that closure depends both on the set and the operation. For example, the set odd numbers is not closed with respect to addition but it is closed with respect to multiplication, since the product of two odd numbers is always an odd number.

In any event, if we have a rule which tells us how to combine two elements of S so as always to obtain an element of S, we call such a rule a <u>binary operation on S</u>. Thus, addition is a binary operation on the set of real numbers.* In still other words, the Rule of Closure is associated with the concept of a binary operation.**

As for A-2, notice how this differs from E-2. In particular, from A-1, both a + b and b + a are numbers. All A-2 states is that these two numbers are equal.

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*Notice how this differs from a relation which <u>compares</u> two elements rather than combines them to form another element.

**That is, merely combining elements of S to form an element isn't called a binary operation unless that element also always belongs to S.

From another point of view, all E-2 states is that <u>if</u> a + b = b + a then
b + a = a + b. It does not state that a + b and b + a must be equal.
Indeed, commutivity is not a property shared by every binary operation.
Quite in general, order does make a difference. Thus, while it is
true that a + b = b + a for all numbers a and b, it is not true, for
example, that a - b = b - a.

As for A-3, in more "plain English," this merely says that the binary
operation called addition does not depend on <u>voice inflection</u>, so to
speak. For example, an expression such as 2 x 3 + 4 is ambiguous as
it stands. On the one hand, it can be read as (2 x 3) + 4 which is
10; and on the other hand, it can be read as 2 x (3 + 4) which is 14.
In a similar way, 9 - 3 - 1 can be thought of as equalling either 5
or 7 depending on whether we "pronounce" it "(9 - 3) - 1" or
"9 - (3 - 1)." What associativity implies is that we do not need to
use parentheses, braces, brackets, etc. to distinguish between various
voice inflections. That is, a + b + c yields the same answer whether
we read it as (a + b) + c or as a + (b + c).

So far, our rules do not mention a single number by name. We now
single out a rather special number from the point of view of addition.
Namely,

A-4: There exists a number denoted by 0 such that a + 0 = a for all
      numbers, a. (The Rule of Additive Identity)

Zero is called the additive identity since, with respect to addition,
the addition of 0 does not change the "identity" of a number. In a
similar way, 1 would be called the multiplicative identity since mul-
tiplying by 1 does not change the value. We shall say more about this
a bit later.

Our final rule for addition is the one which makes the concept of sub-
traction available to us. Namely,

A-5: For each number a, there exists a number b such that a + b = 0.
      We usually denote b by -a. That is, a + (-a) = 0. (The Rule
      of Additive Inverse)

In other words, A-5 tells us that we can "undo" addition. A-5 allows
us to talk about subtraction now in the following way. Given two num-
bers a and b, by A-5, a number (-b) exists. We then agree to abbrevi-
ate a + (-b) (which we know is a number by A-1) by a - b.

Before continuing, it is crucial that we understand that, while the

ten rules listed thus far were motivated by our thinking of the real numbers as a model, once the rules are listed, we need no longer think of the model which motivated the rules. In other words, once the rules are accepted, we merely study the inescapable consequences of these rules. For example, let us show that the "cancellation law" (i.e., if $a + b = a + c$ then $b = c$) follows inescapably from our rules. First of all, A-5 tells us the existence of the number $(-a)$. Then since we are given that $a + b$ is equal to $a + c$ we may replace $(a + b)$ by $(a + c)$, which by A-1 are numbers, to obtain

$$-a + (a + b) = -a + (a + c). \tag{1}$$

Equation (1) is obtained from E-4.

By A-3 we know that $-a + (a + b) = (-a + a) + b$ while $-a + (a + c) = (-a + a) + c$. Again, by E-4, we substitute these equalities into (1) and obtain

$$(-a + a) + b = (-a + a) + c. \tag{2}$$

From A-5 we know that $a + (-a) = 0$, and from A-2 we know that $a + (-a) = (-a) + a$. Hence, by E-4 we may conclude that $-a + a$ is also equal to 0. Substituting this result into (2), yields

$$0 + b = 0 + c. \tag{3}$$

Since $0 + b = b + 0$ (by A-2) and $b + 0 = b$ (by A-4), we have by substitution (E-4) that $0 + b = b$. Similarly, $0 + c = c$. Using E-4 now, equation (3) becomes

$$b = c \tag{4}$$

and equation (4) is the desired result.

Let us point out that, to the uninitiated, the above proof might seem difficult, obscure, or unnecessary. A few clarifying remarks might be in order. First of all, as in most complex games, the strategy required to play the game of mathematics is complicated and will be mastered to *different degrees* by different players. For our immediate purposes, it is sufficient that the "player" appreciate the fact that the strategy used in our proof did show that the conclusion followed inescapably from the ten rules even if the player might not have been able to invent the strategy himself. (This is not too surprising.

After all, think of the other games wherein it is not uncommon for a player to be able to appreciate and comprehend the strategy of the master even though the player might not have been able to invent the strategy himself. In many cases, this difference is the difference between being "another player" and being a "pro.")

While there may be "intuitive" ways of visualizing the same result as we obtained, if the technique uitilizes properties of numbers other than those specified in our ten rules, then the result need not be true in models which obey only the ten rules. In other words, our approach guarantees that our conclusion is true in any model that has properties the ten rules specified in our game.

We would also like to point out that while our demonstration might not have made it that clear, our proof was modeled precisely after the statement-reason format of plane geometry. Had we wished to be more formal we could have written

Statement	Reason
(1) There exists a number -a	(1) A-5
(2) -a + (a + b) = -a + (a + c)	(2) E-4 (replacing a + b by a + c)
(3) -a + (a + b) = (-a + a) + b -a + (a + c) = (-a + a) + c	(3) A-3
(4) (-a + a) + b = (-a + a) + c	(4) Substituting (E-4) (3) into (2)
(5) -a + a = a + (-a)	(5) A-2
(6) a + (-a) = 0	(6) A-5
(7) -a + a = 0	(7) Substituting (E-4) (5) into (6)
(8) 0 + b = 0 + c	(8) Substituting (E-4) (7) into (4)
(9) b + 0 = 0 + b c + 0 = 0 + c	(9) A-2
(10) b + 0 = b c + 0 = c	(10) A-4
(11) 0 + b = b 0 + c = c	(11) Substituting (E-4) (9) into (10)
(12) b = c	(12) Substituting (11) into (8) q.e.d.

Usually, we are much less formal and write:

a+b=a+c→-a+(a+b)=-a+(a+c)→(-a+a)+b=(-a+a)+c→0+b=0+c→b=c.

The important point is that, one way or another, we must show that our conclusions follow inescapably from our assumptions. Other examples are left to the exercises.

Resource: Calculus Revisited: Multivariable Calculus
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