

Unit 3: Additional Comments on Dimension

1. Overview

In most respects this Unit could have been included as a subtopic of the previous one, but we have elected to include it as a separate unit in order to give you another chance for grasping the "big picture".

2. Lecture 3.030

**Constructing Bases**

**Review**

①  $\{u_1, \dots, u_n\}$  is called a basis for  $V \iff$

(i)  $\{u_1, \dots, u_n\}$  spans  $V$   
 (ii)  $\{u_1, \dots, u_n\}$  lin. indep.

② If  $\{u_1, \dots, u_n\}$  is a basis for  $V$ , then

(i) Any  $(n+1)$  elts of  $V$  are lin. dep.

(ii) Fewer than  $n$  elts cannot span  $V$   
 [  $\therefore$  Every basis for  $V$  has  $n$  elements ]

(iii) Any set of  $n$  lin. ind. elts, or which spans  $V$ , is a basis for  $V$ .  
 [  $\therefore \dim V = n$  is unambiguous. ]

(iv) If  $\dim V = n$  and  $\{u_1, \dots, u_n\}$  is a particular basis for  $V$  then each  $v \in V$  has unique rep'n in form  $\sum_{i=1}^n c_i u_i$ ,  $c_i \in \mathbb{R}$   
 Relative to  $\{u_1, \dots, u_n\}$  we may write  $v$  as  $(c_1, \dots, c_n)$ ; and we then write  $V = [u_1, \dots, u_n]$

a.

**Example**

Let  $\dim V = 4$   
 and suppose  $V = [u_1, u_2, u_3, u_4]$

Describe  $W = S(u_1, u_2, u_3, u_4)$  where

$u_1 = (1, 1, 2, 3)$   
 $u_2 = (2, 3, 4, 5)$   
 $u_3 = (3, 7, 6, 5)$   
 $u_4 = (4, 5, 7, 9)$

Does  $W = V$ ?  
 If not, what?

Use Row-Reduced Matrix Technique!

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 3 & 7 & 6 & 5 \\ 4 & 5 & 7 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 4 & 0 & -4 \\ 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$\beta_1 = (1, 0, 0, 6)$   
 $\beta_2 = (0, 1, 0, -1)$   
 $\beta_3 = (0, 0, 1, -2)$   $\dim = 3$

$S(u_1, u_2, u_3, u_4) = S(\beta_1, \beta_2, \beta_3)$

$x_1 \beta_1 + x_2 \beta_2 + x_3 \beta_3 =$   
 $(x_1, 0, 0, 6x_1)$   
 $+ (0, x_2, 0, -x_2)$   
 $+ (0, 0, x_3, -2x_3)$   
 $(x_1, x_2, x_3, 6x_1 - x_2 - 2x_3)$

$\therefore x_1 \beta_1 + x_2 \beta_2 + x_3 \beta_3 = 0 \iff$   
 $(x_1, x_2, x_3, 6x_1 - x_2 - 2x_3) = (0, 0, 0, 0) \iff$   
 $x_1 = x_2 = x_3 = 0$

$\therefore W = S(u_1, u_2, u_3, u_4) = [ \beta_1, \beta_2, \beta_3 ]$   
 $\dim W = 3$  3.030 (2)

b.

Study Guide  
 Block 3: Selected Topics in Linear Algebra  
 Unit 3: Additional Comments on Dimension

$\{\beta_1, \beta_2, \beta_3\}$  is a  
 "natural" basis for  $W$   
 $v \in (x_1, x_2, x_3, x_4) \in W \Leftrightarrow$   
 $(x_1, x_2, x_3, x_4) = x_1\beta_1 + x_2\beta_2 + x_3\beta_3$   
 Equivalently,  $\Leftrightarrow$   
 $x_4 = 8x_1 - x_2 - 2x_3$

For example  $\beta_1, \beta_2, \beta_3$   
 $d_1 = \beta_1 + \beta_2 + 2\beta_3 = (1, 1, 2)$   
 $d_2 = 2\beta_1 + 3\beta_2 + 4\beta_3 = (2, 3, 4)$   
 $d_3 = 3\beta_1 + 7\beta_2 + 6\beta_3 = (3, 7, 6)$   
 $d_4 = 4\beta_1 + 5\beta_2 + 9\beta_3 = (4, 5, 9)$

$I_5(2, 5, 3, 5) \in W?$   
 $2\beta_1 + 5\beta_2 + 3\beta_3 =$   
 $(2, 0, 0, 16)$   
 $+ (0, 5, 0, -5)$   
 $+ (0, 0, 3, -6)$   
 $(2, 5, 3, 5) = W$   
 $5 = 8(2) - (5) - 2(3)$

How can we  
 express  $(2, 5, 3, 5)$   
 as linear combination  
 of  $d_1, d_2, d_3, d_4$ ?

$\begin{matrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 0 \\ 3 & 7 & 6 & 0 & 0 \\ 4 & 5 & 9 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{matrix}$

8030(3)

c.

$\begin{bmatrix} 1 & 0 & 2 & 3 & -1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 & -4 & 1 & 0 \\ 0 & 0 & 1 & -2 & -1 & 0 & 1 \end{bmatrix}$

$5d_1 - 4d_2 + d_3 = 0$   
 $d_3 = 4d_2 - 5d_1$

check

$4d_2 = (8, 12, 16, 20)$   
 $-5d_1 = (-5, -5, -10, -15)$   
 $4d_2 - 5d_1 = (3, 7, 6, 5)$   
 $= d_3$

$\begin{matrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ \begin{bmatrix} 1 & 0 & 0 & 7 & 1 & 0 & -2 \\ 0 & 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & -1 & 0 & 1 \end{bmatrix} \end{matrix}$

$(2, 5, 3, 5) = 2\beta_1 + 5\beta_2 + 3\beta_3 =$   
 $2(7d_1 + d_2 - 2d_4)$   
 $+ 5(-2d_1 + d_2) + 3(-2d_1 - d_2 + d_4) =$   
 $-2d_1 + 4d_2 - d_4$

Summary of Example:  
 $W = \text{span}\{d_1, d_2, d_3, d_4\}$   
 $= [\beta_1, \beta_2, \beta_3]$   
 $\dim W = 3$   
 $W = \{(x_1, x_2, x_3, 8x_1 - x_2 - 2x_3)\}$   
 $(2, 5, 3, 5) \in W$   
 $(2, 5, 3, 5) = (2, 5, 3)$   
 $(2, 5, 3, 5) = (-2, 4, -1)$

8030(4)

d.

3. Exercises :

3.3.1(L)

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Let  $\dim V = 4$  and assume that  $[u_1, u_2, u_3, u_4]$  is the coordinate system being used for denoting the elements of  $V$  as 4-tuples.

Let  $W$  be the subspace of  $V$  generated by  $\alpha_1 = (1, 1, 3, 4)$ ,  $\alpha_2 = (2, 3, 7, 9)$ ,  $\alpha_3 = (3, -2, 4, 7)$ ,  $\alpha_4 = (4, -5, 3, 7)$ , and  $\alpha_5 = (4, 5, 14, 9)$ .

- Find the dimension of  $W$ .
- Express  $x_4$  as a linear combination of  $x_1, x_2,$  and  $x_3$  if it is known that  $(x_1, x_2, x_3, x_4) \in W$ .
- Find vectors  $\beta_1, \beta_2, \beta_3 \in W$  such that  $(x_1, x_2, x_3, x_4) \in W \Leftrightarrow (x_1, x_2, x_3, x_4) = x_1\beta_1 + x_2\beta_2 + x_3\beta_3$ . Then express  $\alpha_1, \alpha_2,$  and  $\alpha_5$  as linear combinations of  $\beta_1, \beta_2,$  and  $\beta_3$ .

3.3.2

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Let  $V = [u_1, u_2, u_3]$  and define  $\alpha_1, \alpha_2, \alpha_3$  by  $\alpha_1 = (5, 2, 7)$ ,  $\alpha_2 = (-3, 4, 1)$ , and  $\alpha_3 = (-1, -2, -3)$ . Let  $W = S(\alpha_1, \alpha_2, \alpha_3)$ .

- Show that  $\dim W = 2$ .
- Find a linear combination of  $\alpha_1, \alpha_2, \alpha_3$  which is zero even though no coefficients are zero.
- Show that  $\alpha_1$  may be written in infinitely many different ways as a linear combination of  $\alpha_1, \alpha_2,$  and  $\alpha_3$ .

3.3.3(L)

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Show that  $W = \{f: f''(x) - 4f(x) \equiv 0\}$  is a subspace of the space of continuous functions.

3.3.4(L)

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Let  $V = [u_1, u_2, u_3, u_4]$  and let  $S$  be the subspace of  $V$  generated by  $\alpha_1 = (1, 1, 2, 3)$ ,  $\alpha_2 = (2, 3, 5, 7)$  and  $\alpha_3 = (2, 1, 3, 5)$ .

(continued on next page)



3.3.4(L) continued

- If  $(x_1, x_2, x_3, x_4) \in S$ , how are  $x_3$  and  $x_4$  related to  $x_1$  and  $x_2$ ?
- With  $V$  as above, let  $T$  be the subspace of  $V$  generated by  $\alpha_4 = (1, 2, 2, 3)$ ,  $\alpha_5 = (2, 5, 4, 7)$ , and  $\alpha_6 = (3, 7, 7, 8)$ .  
If  $(x_1, x_2, x_3, x_4) \in T$ , how is  $x_4$  expressed in terms of  $x_1$ ,  $x_2$ , and  $x_3$ ?
- Describe the subspace  $S \cap T$ .
- Describe the subspace  $S + T = \{s + t : s \in S \text{ and } t \in T\}$ .
- Verify that in this example,  $\dim(S + T) = \dim S + \dim T - \dim S \cap T$ .

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3.3.5

Let  $V = [u_1, u_2, u_3, u_4, u_5]$ . Let  $S$  be the subspace spanned by  $(1, 1, 2, 3, 3)$ ,  $(2, 3, 4, 5, 7)$ , and  $(3, 4, 7, 8, 8)$ , and let  $T$  be the subspace spanned by  $(1, 1, 1, 3, 5)$ ,  $(1, 2, 3, 2, 2)$ , and  $(2, 3, 3, 7, 8)$ .

- Find the dimension of  $S$ .
- Find the dimension of  $T$ .
- Find the dimension of  $S \cap T$ , and, in particular, find a row reduced basis for  $S \cap T$ .
- Find the dimension of  $S + T$  and in particular show how  $x_1$ ,  $x_2$ ,  $x_3$ ,  $x_4$ , and  $x_5$  must be related if  $(x_1, x_2, x_3, x_4, x_5) \in S + T$ .
- Again verify that  $\dim(S + T) = \dim S + \dim T - \dim S \cap T$ .

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3.3.6 (optional)

Our main aim in this exercise is to show how one constructs a basis for  $S + T$  by starting with a basis for  $S \cap T$ . In the course of this construction, we manage to prove that if  $S$  and  $T$  are subspaces of a finite dimensional space  $V$ , then  $\dim(S + T) = \dim S + \dim T - \dim(S \cap T)$ .

Use the result of the previous exercise to obtain a basis for  $S \cap T$  and then show how this may be augmented by the given basis vectors for  $S$  to form a new basis for  $S$ . Apply a similar approach to find a new basis for  $T$  and then explain why

$$\dim(S + T) = \dim S + \dim T - \dim(S \cap T).$$

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