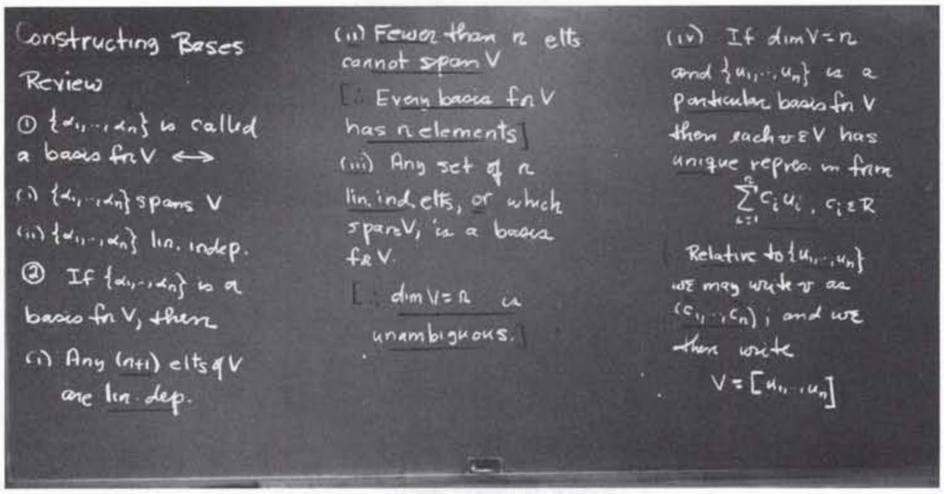
Unit 3: Additional Comments on Dimension

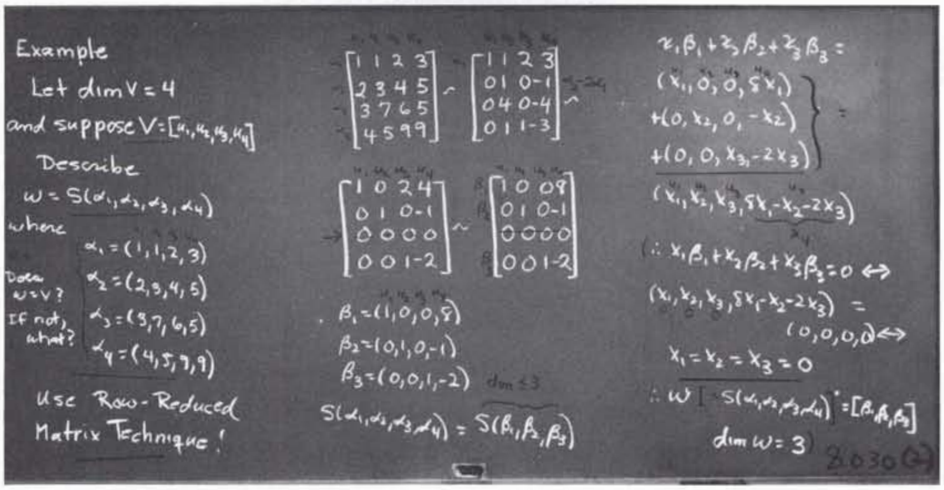
1. Overview

   In most respects this Unit could have been included as a subtopic of the previous one, but we have elected to include it as a separate unit in order to give you another chance for grasping the "big picture".

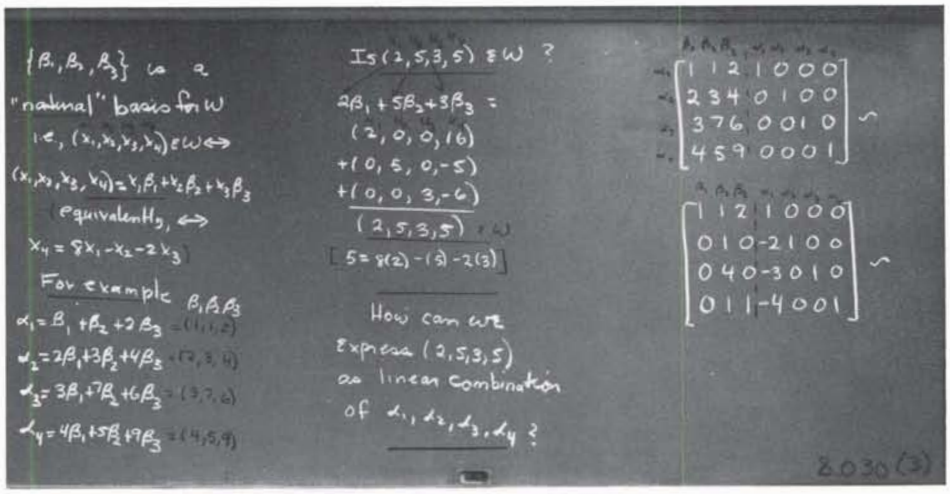
2. Lecture 3.030



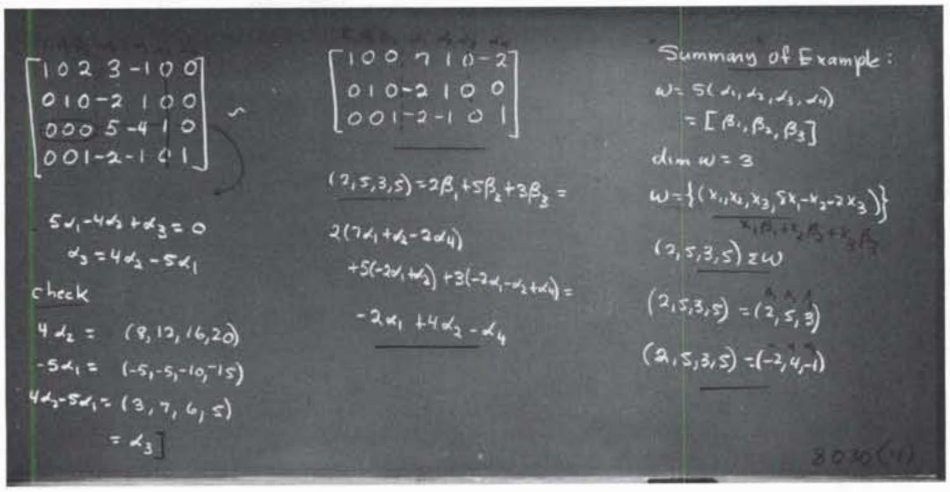
a.



b.


"natural" basis for W
(equivalently,
For example
Is (2,5,3,5) ε W ?
How can we express (2,5,3,5) as linear combination
8030(3)

c.


check
Summary of Example:
dim W = 3
8030(4)

d.

3. Exercises :

3.3.1(L)

Let dim V = 4 and assume that $[u_1, u_2, u_3, u_4]$ is the coordinate system being used for denoting the elements of V as 4-tuples. Let W be the subspace of V generated by $\alpha_1 = (1, 1, 3,4)$, $\alpha_2 = (2,3,7,9)$, $\alpha_3 = (3,-2,4,7)$, $\alpha_4 = (4,-5,3,7)$, and $\alpha_5 = (4,5, 14,9)$.

a. Find the dimension of W.

b. Express x_4 as a linear combination of x_1, x_2, and x_3 if it is known that $(x_1,x_2,x_3,x_4) \varepsilon W$.

c. Find vectors $\beta_1,\beta_2,\beta_3 \varepsilon W$ such that $(x_1,x_2,x_3,x_4) \varepsilon W \leftrightarrow (x_1,x_2,x_3,x_4) = x_1\beta_1 + x_2\beta_2 + x_3\beta_3$. Then express α_1,α_2, and α_5 as linear combinations of β_1,β_2, and β_3.

3.3.2

Let $V = [u_1,u_2,u_3]$ and define $\alpha_1,\alpha_2,\alpha_3$ by $\alpha_1 = (5,2,7)$, $\alpha_2 = (-3,4,1)$, and $\alpha_3(-1,-2,-3)$. Let $W = S(\alpha_1,\alpha_2,\alpha_3)$.

a. Show that dim W = 2.

b. Find a linear combination of $\alpha_1,\alpha_2,\alpha_3$ which is zero even though no coefficients are zero.

c. Show that α_1 may be written in infinitely many different ways as a linear combination of α_1,α_2, and α_3.

3.3.3(L)

Show that $W = \{f: f''(x) - 4f(x) \equiv 0\}$ is a subspace of the space of continuous functions.

3.3.4(L)

Let $V = [u_1,u_2,u_3,u_4]$ and let S be the subspace of V generated by $\alpha_1 = (1,1,2,3)$, $\alpha_2 = (2,3,5,7)$ and $\alpha_3 = (2,1,3,5)$.

(continued on next page)

3.3.4(L) continued

a. If $(x_1,x_2,x_3,x_4) \varepsilon S$, how are x_3 and x_4 related to x_1 and x_2?

b. With V as above, let T be the subspace of V generated by $\alpha_4 = (1,2,2,3)$, $\alpha_5 = (2,5,4,7)$, and $\alpha_6 = (3,7,7,8)$.

   If $(x_1,x_2,x_3,x_4) \varepsilon T$, how is x_4 expressed in terms of x_1, x_2, and x_3?

c. Describe the subspace $S \cap T$.

d. Describe the subspace $S + T = \{s + t: s \varepsilon S \text{ and } t \varepsilon T\}$.

e. Verify that in this example, $\dim(S + T) = \dim S + \dim T - \dim S \cap T$.

3.3.5

Let $V = [u_1,u_2,u_3,u_4,u_5]$. Let S be the subspace spanned by (1,1,2,3,3), (2,3,4,5,7), and (3,4,7,8,8), and let T be the subspace spanned by (1,1,1,3,5), (1,2,3,2,2), and (2,3,3,7,8).

a. Find the dimension of S.

b. Find the dimension of T.

c. Find the dimension of $S \cap T$, and, in particular, find a row reduced basis for $S \cap T$.

d. Find the dimension of S + T and in particular show how x_1, x_2, x_3, x_4, and x_5 must be related if $(x_1,x_2,x_3,x_4,x_5) \varepsilon S + T$.

e. Again verify that $\dim(S + T) = \dim S + \dim T - \dim S \cap T$.

3.3.6 (optional)

Our main aim in this exercise is to show how one constructs a basis for S + T by starting with a basis for $S \cap T$. In the course of this construction, we manage to prove that if S and T are subspaces of a finite dimensional space V, then $\dim(S + T) = \dim S + \dim T - \dim (S \cap T)$.

Use the result of the previous exercise to obtain a basis for $S \cap T$ and then show how this may be augmented by the given basis vectors for S to form a new basis for S. Apply a similar approach to find a new basis for T and then explain why

$$\dim (S + T) = \dim S + \dim T - \dim (S \cap T).$$

Resource: Calculus Revisited: Complex Variables, Differential Equations, and Linear Algebra
Prof. Herbert Gross