PROFESSOR: Ladies and gentlemen, welcome to lecture number 11. In this lecture, I would like to continue discussing with you the solution of the dynamic equilibrium equations. We discussed the solution of dynamic equilibrium equations already in lecture number 10. In that lecture, we consider direct integration methods.

In this lecture, I would like to consider with you, discuss with you the mode superposition analysis. The basic idea in the mode superposition analysis is the transformation from the U displacements into a set of new displacements, X.

P is an n by n matrix, which is nonsingular. If P is nonsingular, we should recognize that the vector U of lengths n is uniquely given when we have the vector X, also of lengths n and vice versa. In other words, if we have the vector X, we can calculate U. Or if we have U, we can calculate X. P, of course, has to be a nonsingular matrix. We call this a transformation matrix. And we call the X displacement, that I will be talking about, the generalized displacement.

Now, if we use this transformation, we find that if we substitute this transformation into this equation for U, and we postulate that P shall not be a function of time. So that when we find the derivative of U, with respect to time-- let me denote it by a dot here-- we simply obtain that U dot is equal to P times X dot. And similarly, U double dot is equal to P times X double dot.

If we then substitute for U, U dot, and U double dot-- in other words, for the displacements, velocities, and accelerations-- into this equation, we obtain this equation, provided-- and now, there's one additional step-- after the substitution for the displacements, velocities, and accelerations, we also premultiply the total equation by a P transposed. In other words, what we do is we take a P transposed here, and we are substituting for these vectors here from that relation here.

The P transposed, of course, gives us, also, a P transposed R on the right-hand side. So R curl here is defined as shown down here. Notice that M curl, C curl, and

K curl are given as shown here. And they involve, of course, the M, C, and K matrices. And this P here comes via this substitution into these vectors. This P transposed comes from that P transposed here.

Now, what have we been doing so far? Basically, what we have been doing is we have changed bases. We have come from the displacement defined by the finite elements-- by the finite element nodal points-- to generalized displacements. The generalized displacements now govern the equilibrium of the system.

Now, this transformation is effective if we find that our M curl, C curl, and K curl have a better structure than the matrices, M, C, and K. And, of course, if it is simpler to find the P matrix or a P matrix.

Let's look at a very simple case. If, for example, M curl, C curl, and K curl would all be diagonal matrices, then the solution of this set of equations would be very simple. In fact, this is our objective. It is our objective to come up with a P matrix, which when used in this transformation, gives us an M curl, C curl, and K curl matrix that are all diagonal. And, of course, in addition we want to find the P matrix with a minimum amount of effort.

Once we have diagonal matrices here on the left-hand side-- in other words, M curl, C curl, and K curl-- then, we can use our direct time integration schemes, the Newmark method, or the essential difference method. Usually we use the Newmark method. Or we might even be able to solve the decoupled equations. They're decoupled because M curl, CK curl, are diagonal. We might even solve these equations in closed form.

The question then is how do we find P or [UNINTELLIGIBLE] P? We want to find it cheaply. It shall be economical to calculate P. And we want to, of course, have that M curl, C curl, K curl are ideally diagonal matrices.

Well, the P matrix can be constructed-- or a P matrix can be constructed, effectively, by looking at the free vibration equilibrium equations of the system. If we look at these equilibrium equations here, in which we neglect damping and we neglect the

right-hand side forces-- we're looking at the equilibrium equations of the dynamic system without damping and without forces applied. And if we look for a solution to this system, then, we can assume a solution of this form. U, of course, being our displacement vector. Phi being all the vector that is independent now of time. And here is the time dependency, sin, omega, omega is a radiant per second circular frequency. t, of course, being the time. And t 0 is a time shift.

If we make this assumption, substitute into here-- when we do substitute, of course, remember we have to take the second derivative of this function here with respect to time because we're taking the second derivative of U here. And that gives us a minus omega squared sin, omega t minus t 0, substituting from here into there. We immediately get this equation because we can cancel out sin omega t minus t 0 on both sides. This is the resulting equation.

So what we now want is to solve for phi and omega squared. This is the generalized eigenvalue problem. The generalized eigenvalue problem because M is a matrix other than the identity matrix in general. And we know that they are n solutions to this problem if K and M are n by n matrices. There are n solutions to this problem.

We call these solutions the eigensolutions. And we call omega 1 squared, I should say, and phi 1 the first eigenpair. This is the second eigenpair, this is the nth eigenpair. There are n such eigenpairs. Each of these eigenpairs satisfies this equation here. Therefore, these are a solution-- we call it an eigensolution-- to this equation.

And in addition, we also can prove-- we can show-- that the phi I vectors-- these vectors here-- are M orthogonal. M orthogonal because phi I transposed M phi j-- for example, phi 1 transposed M phi 2 is equal to 0. And phi 1 transposed M phi j-- sorry phi 1 transposed M phi 1 is equal to 1. And because it's equal to 1, they're not only M orthogonal, they're indeed even M ortho normal.

What we find is that if we look at this eigenproblem, and we notice that a certain vector satisfies this eigenproblem, then also a multiple of that vector satisfies the eigenproblem. And we determine this constant, this undetermined constant, in such

a way as to have phi I transposed M phi I equal to 1.

So these are the eigenpairs and the eigenvectors, phi 1 to phi n, satisfy this relation. The eigenvalues, omega 1 squared to omega n squared can be ordered in this way where we find that we might have multiple eigenvalues. Omega 1 might be equal to omega 2. But if we do, we still have only a total number of n such eigenvalues.

I will discuss in the next lecture with you the solution techniques that we're using to calculate these eigenpairs. Let's for the moment assume that we have calculated them. And we want to use them.

Well, if we now define the following matrix, phi capital phi. It's a capital phi because we see crossbars top and bottom here. If we define this matrix as phi 1 to phi n-- in other words, we're taking the first eigenvector, and list it as a first column of this matrix here. The second eigenvector, we list as the second column in this matrix and so on.

So that we find, of course, that this matrix here is an n by n matrix. And if we define a diagonal matrix, omega squared. This is a capital omega. Omega squared, being equal to 0s in the off diagonal elements. But the diagonal elements being omega 1 squared to omega n squared.

Then, we can write the solutions to the eigenproblem. The solutions to the eigenproblem-- and I'm still talking about this eigenproblem problem here. We can write the solutions to this e eigenproblem in the following form.

Notice that this omega squared here has to be on the back of the capital phi. In other words, this is the solution to the problem, K phi equals omega squared M phi, where the omega squared is a scalar and appears here in front. We usually write it in front. Of course, we can put it also at the back because it's a scalar. We can put in front or the back.

However, when we have listed the solutions to this problem in this way, then, we have to put the capital omega squared at the back of M times phi. You can verify that by simply multiplying out the right-hand side.

We also have this relationship here, which is the M ortho normality, the phi ortho normality with M. In other words, this here says nothing else and phi I transposed M phi j is equal to delta I j. We call this the chronica delta. And this chronica delta is equal to 1 for I being equal to j. And it's 0 otherwise.

So we are having this relationship here. And if we use this relationship, and we are premultiplying this equation by phi transposed to capital phi transposed now, both sides. We are finding that we are getting here the identity matrix. And therefore, we're having that this left-hand side is equal to omega squared. This matrix here.

And this is an important fact. In other words, we find that the vectors-- the eigenvectors-- satisfy these two relationships. Now, if we look back at what we started off with in our multiple position analysis discussion, we wanted a matrix that we call P, which is such that P transposed M, P ideally is a diagonal matrix.

Well, we have no such a matrix. If we take phi being equal to P. Then, phi transposed M phi is in fact a diagonal matrix. In fact, a very nice diagonal matrix, the identity matrix, which is a matrix that is diagonal. It has just once on the diagonal.

Similarly, we wondered here that P transposed KP is a diagonal matrix. Well, we have achieved that by this phi matrix. So the phi matrix is, indeed, a particular P matrix that we can use.

So what we are then saying is let us use a multiple position analysis this particular P matrix. The P matrix, which is equal to the phi matrix and phi storing the eigenvectors of the generalized eigenproblem.

Well, substituting from here, as before, into the governing dynamic equilibrium equations, we directly obtain this equation. However, we notice now that this matrix here, in general, is not yet a diagonal matrix. We call that our C curl. And our C curl has not been diagonalized yet. We will have to deal with that matrix still a little later.

How do we solve these equations? Well, if we solve these equations, we have to also recognize, of course, that we have to include the initial conditions to the

problem. And the initial conditions mean that the initial displacements are defined and the initial velocities are defined.

These, of course, are the initial conditions that are used in the direct integration solution of the governing dynamic equilibrium equations. Well, if we are now performing the solution of this set of equations, we will have to transform these initial conditions to initial conditions on the generalized displacements. And that is achieved via these relations here.

Let us see how we derive these. Well, if we look at this equation here, and if we notice that, of course, this equation must be applicable 4 times 0. We can simply put a 0 there. And we put a 0 there.

Then, of course, we also recognize that we can premultiply this equation with this relation here. And if we do that on the right-hand side, we notice that phi transposed M times that phi is the identity matrix.

And so we have this relation directly obtained. The left-hand side in this equation is the right-hand side in this equation. And the right-hand side here is simply this 0 x now. The same holds also for the initial velocity. We can simply put dots on top of the U here and of the X. And therefore, we have this relation here.

So the solution of this set of equations and with these initial conditions subject to these initial conditions, gives us a solution to of the dynamic response. Now, I should mention here that we still have to deal with the damping matrix.

The damping matrix here needs particular attention. We do not have, in general, a diagonal matrix here. And we will want to construct specific damping matrices with which we can deal, effectively. And yet, of course, these matrices must make, physically, sense.

Let us look now, however, first at the case where we neglect damping. When we neglect damping, we have this set of equations and individual equations of this form. Notice that these are individual equations because this is a vector. This is a diagonal matrix. And on the right-hand side, we simply have a vector off load, of

course, time dependent.

So we can look at the first row in this matrix relation. And the first row in that matrix relation will be this equation here with I equal to 1. The I-th row in this relation here is the I-th equation here. In other words, I subscripts on the [UNINTELLIGIBLE].

And, in general therefore, we want to solve n such equations. I going from 1 to n, with this being the load vector and, of course, with the initial conditions. These initial conditions are obtained by simply looking at this equation here and extracting the I-th row from each of these two relations.

So this is then the problem. Notice the following that if we solve all n equations with these initial conditions, then, if we use, for example, the Newmark direct integration method on these equations, we would obtain exactly the same solution as if we had solved the fully coupled equations. The fully coupled equations-- let me write them down once more here-- plus KU equals R.

In other words, whether we apply the Newmark time integration scheme, a direct integration scheme, to this set of equations with the initial conditions on U and U dot or apply the Newmark direct time integration scheme to this set of equations, or rather this one here with these initial conditions, we obtain exactly the same response provided we use the same time step, delta t in both integrations.

There has been no assumption so far. All we have done is a transformation from the U displacements to the X displacements. And this transformation is effective if we can find the phi matrix that we're using in the transformation very cheaply, very economically. Well, the effectiveness, really, of multiple position goes beyond what I have just described.

The important point in multiple position analysis is that we do not need to solve all of these equations. That we do not need to solve and look at all of the decoupled equations. But let us look first once also at how we can solve the decoupled equations now exactly. Where we can solve them exactly for example using a Duhamel integral formulation as shown here. X i of t is given via this relation here.

And we would, of course, substitute our ri of tau where we now assume that this part here is given, directly.

In a computer program, very frequently r i would only be given as a segment of straight lines. And if that is the case, then, of course, we might have to still evaluate all of this here numerically if the segment is very complicated, complex, like an earthquake analysis. ri might be varying very rapidly. And then, we might still want to evaluate this, numerically.

However in general, we can evaluate here this relationship analytically. And we don't need to use numerical integration. Having got xi, of course, we get our U of t as shown here by our transforming back.

Well, the important point that I now would like to get back to you and discuss with you for a little while is that we do not need to use or to solve all of these equations in multiple position analysis. In fact, multiple position analysis is only effective when we do not need to look at all of these decoupled equations. If we have to solve all decoupled equations, then, we would have to calculate, course, first all eigenvalues and eigenvectors, which can be an enormous expense. And then, the multiple position analysis would not be effective.

I would like to repeat, however, if we do solve all decoupled equations, using a specific time integration scheme with a specific data t time step. Then, if we use the same time integration scheme on decoupled equations with the same time step, we must get exactly the same response. Then there has been no assumption so far.

Well, however, if we then do neglect some of the equations here. We do not solve some of the equations. Then, of course, our response will be different that we are predicting via this approach when we compare with the response predicted by solving these equations.

Well, let us look once at this graph again, which we already looked at briefly in the last lecture. And I would like to explain to you now why it is not necessary to solve all of the decoupled equations in multiple position analysis. Why it is not, in general, not

necessary?

Well, if we look at a single oscillator equation, as given here. I include a damp in here, of course, side can be equal to 0. Where P is the driving frequency, omega is the frequency of the oscillator itself.

And if we plot the maximum dynamic response observed, which we define if the dynamic load factor when we divide that maximum dynamic response by the static response, we get this set of curves. Let us look at say the curved side equal to 0 a little bit closer. If we do that, and if we solve this equation with side equals 0. This term now not being there. For certain values of P over omega, we are getting this curve here and that curve here.

What do we notice? Well, we notice that when P over omega is equal to 0, we are getting the static response. Or P over omega very small, we're getting the static response. This means, of course, that omega, the system frequency, is much larger than the applied frequency, which means really that this oscillator is extremely stiff. Well, when compared to the frequency of load application.

So then, we are in this range. And we notice that say P over omega in this range here, smaller than a certain value, and this is here 0.5 say, smaller than 0.2. We would see that basically, the maximum dynamic response is really just nothing else than the static response.

So we are talking in this region here really about static response. And in this region here, we are talking about a truly dynamic response. Now what does that mean? Well, if we look at our decoupled equations here, it means that if i consist of frequencies, basically, which when measured on the system of the oscillator frequency are very small, we, basically, would have a static solution. You would, basically, have a static solution for this equilibrium equation.

A truly dynamic solution will only be obtained when P, in other words, the frequency of the load divided by omega, falls into this range here. Well, this is what we are using in most superposition analysis.

We only need to consider those frequencies of the system. And these are the frequencies of the systems, the omegas, which are truly excited by the dynamic loads. We only need to look at those decoupled equilibrium equations for which we have that the frequencies in the decoupled equations are truly excited by the dynamic loads.

Now, I only talked about the P value here. Of course, in an actual analysis, we would find that ri of P contains a number of frequencies, a spectrum of frequencies. Well, what we have to do is look at the highest frequency that is contained in ri. And we measure that highest frequency on the frequencies of the system.

If that highest frequency divided by the frequency of the system is falling into this range here, we are talking about a static response, basically. And we do not need to consider those equilibrium equations in are multiple position analysis.

Well, so what we are then saying is let us all solve these equations not from i equals 1 to n but from i equals 1 to P. And then, instead of obtaining the true response, the "exact" response, we are obtaining a response U P. i equals 1 to P. Phi i, x i gives us U P, where U P is an approximation now to the U displacements. And the U displacements, once again, are the solution to this system of equations.

How can we find out the error? Well, if U P is an approximate solution to the system of equations that we want to solve, then, let us substitute back into that system of equations. Let me write it down once more here. We have MU double dot plus KU equals R that we want to solve.

Well, we can also write that, of course, MU double dot plus KU minus R. And this shall be 0. So if we have an approximate solution to this set of equations, why not just substitute that approximate solution in here and see how large the right-hand side is. If it's exactly 0, well, then we have that our U P is, indeed, very close to U. And if it's exactly 0 because U P would be equal to U.

We will not find this to be 0. We will never find it to be 0 because in a computer, we would, of course, use finite digit arithmetic to evaluate the left-hand side. And we

would have at least round off. But what we want is at the right-hand side calculated, which we now might call some value shall be close to 0. And that means really what we want to measure is this part here.

Now what have I written down here? I have here epsilon P of t being this vector here. This is a vector. And we can call this an out of balance load vector divided by the applied load vector. Notice that what I'm doing here, I take the norm of this vector because I want to get a single number. I want to get a single number here, which is representative of all the elements in that vector.

The same applies here. This norm is calculated by taking each value in this vector, squaring that value, adding the squares of the values up, and then, taking a square root out of that final value.

In other words, if we write it down here, quickly, what we do is if we have a vector a, and we want to find this norm. What we do is we take a 1 squared plus a 2 squared, et cetera, add all the individual entries up of that vector here-- squares of the entries up-- and then, take the square root out of it. And that gives us a single number, which is representative of all the elements in that vector. Notice that if one element is large, well then, this value must be large, too, because that one element will make this value large here.

Well, if epsilon t is close to 0, assuming that r of t is not 0, of course. Otherwise, you couldn't use this. You would have to use some equivalent approach. If epsilon P is close to 0, then, we know that our equilibrium equation has been solved quite accurately.

In general, we might find that it's not close to 0. And what we want to do is a static correction. This static correction takes into account the fact that we have neglected the higher modes. And in the higher modes, of course, we know we have basically static response. that's why we did not include those equations in the multiple position analysis.

So what do we do in that static correction? We recognize that R could be written in

this form. This RI is exactly the value that I defined earlier. M phi i is a vector. You see, I can take M and multiply it by phi i, and I get a vector.

Now they are in such a vectors. We might not have calculated all n. But you can say that indeed, in theory there are n. So we can write this down.

We also then can calculate the delta R, which is that amount of the load vector, which has not been included in the multiple position analysis. How do we obtain it? Well, we know R is the total load vector. And if we now sum instead of i equals 1 to n, only from i equals 1 to P, we are getting this part here.

But this part is exactly what we have included in the multiple position analysis already. So we are left with a vector on the left-hand side, which we have not included in the multiple position analysis. And we did not include it because we knew at most, we can have static response, corresponding to these loads.

Let us now calculate that static response. That's done right down here. Now, this static response has to be added to the response which we predicted in the multiple position analysis. Let us now go to the case with damping included. So far we neglected that.

Now, with damping included, we have this part here, too. Notice phi transpose C phi, in general, is not a diagonal matrix. And that's why I considered first the case without damping. However, if we have proportionate damping, and in structural analysis, frequently, all that is necessary is to include proportionate damping. Which we postulate to be of this form. 2 omega i xi i. Delta ij is again the chronica delta, which is equal to 1 when i is equal to j and is equal to 0 otherwise.

So if we are postulating this, where omega i, of course, is given. That is the free vibration frequency, which is stored in this matrix here. xi i is a value, which we are not given yet. We have not discussed it yet. But if we postulate that this matrix here, basically, or each entry in that matrix can be written in this way. Then, indeed, we would have decoupled equations.

It turns out that in practice, as I mentioned already, this is all that is necessary to

include damping appropriately. Well, if we know these xi i values, then, we can also construct a C matrix with the xi i values given. The xi i values are frequently obtained from experiments. In other words, the structure is taken, its vibrated, it's excited into certain vibration modes. And from these experiments then, we obtain the xi i value.

So once we have the xi i value given, we can calculate the C matrix. And one way of calculating is to use the Caughey series. Here we have the Caughey series, which was proposed for calculating the C matrix, which satisfies this criteria. In other words, damping is proportional.

Well, if the psi i values are given, we can calculate the omega i, of course are also given. We can calculate using this equation, the constants a 0 to ap minus 1. Once we have these constants, we can substitute them into here and be coming up with a C matrix. And this C matrix here, indeed, satisfies the normality property shown here.

So, in other words, given the psi i values from experimental results, given the omega i values, we can calculate a C matrix that, in fact, gives us a diagonal matrix when we carry out phi i transposed C phi j.

A special case of this C matrix is Rayleigh damping, used abundantly in practice. Where we just include the first two terms, alpha and beta being constants now. Let us go through a particular example.

Let's say that we know the critical damping psi 1 is 0.02 and psi 2 is 0.10. In other words, 10% critical damping in the second mode and 2% critical damping in the first mode. The corresponding frequencies are 1 and 3.

We want to calculate our friend, beta. Well, what we do is use this C matrix in the relation, phi i transposed, C times phi i. And that should give us 2 omega psi i. Of course, we have two i values, i equals 1 and i equals 2. So we can apply this equation twice.

Well, when we calculate it out, we get this. Applying it twice, we directly obtain these

two equations here. And we solve these two equations for alpha and beta. Having alpha and beta, we can substitute now back into this equation here and have our C matrix.

This would be, for example, a C matrix that we could use in a direct integration analysis. You see, if we are performing a multiple position analysis, we might know, from experience, from experimental results, that psi 1 and psi 2 are of these two values.

And then, if we know that we have so much critical damping or so much damping in the first and second vibration mode, and we want to perform a step-by-step direct integration analysis, we would have to construct a C matrix. And this is the procedure to construct a C matrix.

Then, of course, we would have to ask ourselves well, knowing these psi values for omega 1 and omega 2, what are now the psi values that we are implicitly using for the higher modes when we employ this specific C matrix? And we use now, this equation here. This is the equation, which we already derived on this view graph here by looking at this relation. You are now using this equation because this equation must be applicable not just for psi 1 and psi 2 but for all psi values.

Well, then using this equation, we can solve for the psi values. And, of course, psi 1 and psi 2, substituting alpha and beta in here and omega 1 and omega 2 would, indeed, be just 2% and 10%, 0.02 and 0.10.

However, we can now use this equation to get the high xi values. And there's something to be observed, which is quite important, namely we notice that the beta value that we calculated is associated with the high frequencies, basically. And the alpha value goes away. Or this contribution goes away when we're looking at the high frequencies.

For high frequencies therefore, the beta value is a dominant factor. And for low frequencies, the alpha value is the dominant factor. In other words, once again, looking at this equation here, in the high frequency response, beta K, really, will

determine the damping. And in the low frequency, the response is really the alpha value that determines the damping.

Well, the response solution now would be performed in exactly the same way as before. As in the case of no damping, we solve now again P equations. P equations only. And we are doing that by looking at this equation and with, of course, ri being given as shown here and the initial conditions being given here.

Solving this equation here via numerical integration or exact integration, using, for something, Duhamel integral, we obtain the xi values as a function of time. We substitute into here to get our U P, which is an approximation to the solution of the dynamic equilibrium equations. The dynamic equilibrium equations now being MU double dot plus CU dot plus KU, being equal to r.

Notice we have a C matrix in here now. However, and this is an important point, we have a specific C matrix constructed such that the phi i transposed C phi j gives us a diagonal matrix with the diagonal elements being equal to 2 omega i, psi i. And one of these matrices, for example, would be the Rayleigh matrix. The matrix obtained using Rayleigh damping, rather.

The important point is, once again, that all we have been doing so far is to make a change of bases from the finite element coordinates to the generalized displacements, xi. That was the first step. The second step was that we are neglecting the modes P plus 1 to n by only solving the first P equations here and using this as an approximation to the actual solution.

The important point, of course, is now that we have to ask ourselves when is multiple position analysis really effective? Some considerations I have here summarized on the last view graph. The multiple position analysis is effective when the response lies in only a few modes.

What does this mean? It means that if we look at our eigenvalue problem, K phi equals omega squared M phi. We only need to solve this eigenvalue problem for a very few modes. In other words, P being much smaller than M, I will only have to

solve for phi 1 up to phi P and omega 1 squared up to omega P squared.

This is a very important point. I assumed here that we are taking the lowest frequencies, omega 1 to omega P. Of course, in general analysis, it might just be necessary to calculate certain frequencies, say omega 10 to omega 20. We only want to find the frequencies in a certain range because we know that the excitation frequency only lies in a particular range. That would be, for example, the case in vibration excitation analysis.

This then is a requirement for multiple position analysis, really, to be effective. When the response is obtained over many time intervals, then multiple position analysis also becomes effective. We should remember here that if we perform a direct integration analysis, and if that is an implicit direct integration analysis, using the Newmark method, for example.

Then, the number of operations that we talked about in the Newmark method was 1/2 nm squared. That is the initial triangular factorization that we have to perform. m being the half bandwidth. Plus 2 nm. n is the order, of course, of the system. m is the half bandwidth, once again. Times t, the number of time steps.

This is an initial expense that we have to perform, using the direct integration Newmark method, for example. The same holds for the Wilson Theta method. or the Houbolt method. This is an initial expense.

And then, we perform this number of operations for each time step. Now, if we have to run 5,000 time steps, then direct integration would be extremely, extremely expensive. And, in fact, you might not be able to afford it, using implicit direct integration.

Well, therefore, you can see directed by looking at this operation column. What should make the analysis more effective is to bring down the bandwidths. And, of course, this is exactly what we're doing in multiple position analysis. We are going from a bandwidth of m to a bandwidth of 0. You are having a diagonal matrix.

Remember, the bandwidths we defined to be the number of off diagonal elements

here that we are having. We did not include the diagonal. That is a half bandwidth. If you look at the total band here-- that total band here was equal to 2m plus 1.

So if we are reducing the bandwidths, as we do in multiple position analysis, then we can see immediately that this number of operations here, which is directly proportional to the number of times steps goes down. And that is exactly the objective in multiple position analysis that I talked about earlier. That is the first important objective. We want to reduce the bandwidths for the direct integration analysis.

The second objective, of course, is that we do not need to even consider all of the equations. So looking at n. We are reducing n, also. We are not looking anymore at n equations. We are looking at P equations. And we are reducing the bandwidths as I said earlier. m goes to 0. And n goes to P. Of course, this is an approximate count here.

So basically, you can see that in the solution using multiple position analysis, the number of operations in the time integration of the decoupled equations is relatively small, is little. And, in fact, this solution here can be obtained very, very effectively. The time integration of the decoupled equations is very effective in an actual practical analysis.

Where the expense lies in multiple position is really in the calculation of the required frequencies and motions. And, of course, this is exactly the subject of the next lecture where I want to discuss with you the effective solution techniques for calculating these vectors and frequencies. Multiple position analysis is effective, altogether only, when we can calculate these eigenvalues and eigenvectors in an effective way.

I should also finally, once again, emphasize that it may be important to calculate the error, epsilon pt that I referred to. Or to add a static correction to the predicted response in the multiple position analysis when P is, of course, much smaller than n. We have neglected the high frequency response here. That high frequency response is a static response, in general. And should be taken into account via say

static correction, as I have been discussing with you.

This is all I wanted to say in this lecture. Thank you very much for your attention.