Topic 8

The Two-Noded Truss Element— Updated Lagrangian Formulation

Contents:

- **Derivation of updated Lagrangian truss element displacement and strain-displacement matrices from continuum mechanics equations**
- **Assumption of large displacements and rotations but small strains**
- **Physical explanation of the matrices obtained directly by application of the principle of virtual work**
- **Effect of geometric (nonlinear strain) stiffness matrix**
- **Example analysis: Prestressed cable**

Textbook: Section 6.3.1

Examples: 6.15, 6.16

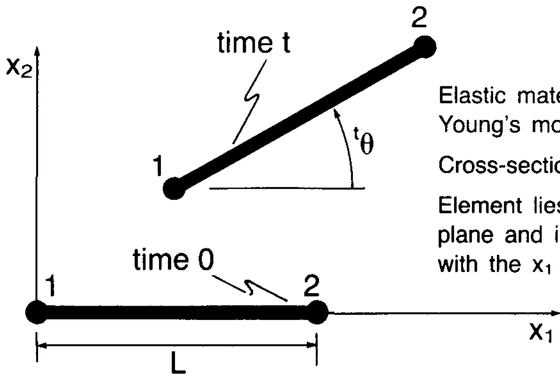
TRUSS ELEMENT DERIVATION

A truss element is a structural member which incorporates the following assumptions:

- Stresses are transmitted only in the direction normal to the cross-section.
- The stress is constant over the cross-section.
- The cross-sectional area remains constant during deformations.

Transparency 8-1

We consider the large rotation–small strain finite element formulation for a straight truss element with constant cross-sectional area.



Elastic material with Young's modulus E

Cross-sectional area A

Element lies in the $x_1 - x_2$ plane and is initially aligned with the x_1 axis.

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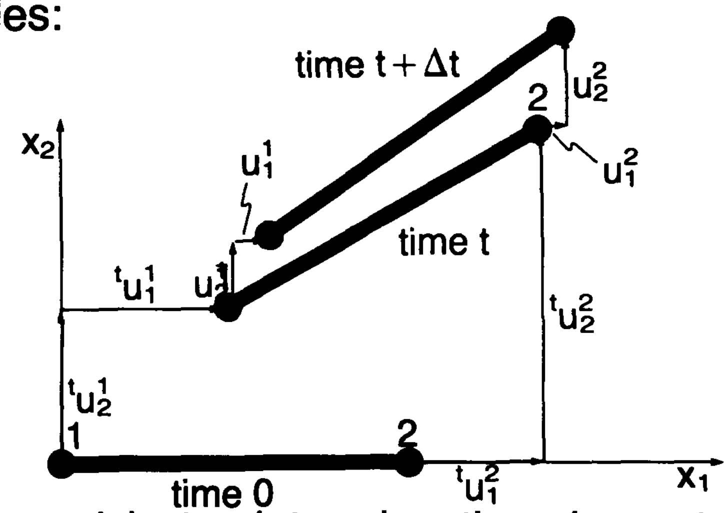
Transparency 8-3

The deformations of the element are specified by the displacements of its nodes:



Our goal is to determine the element deformations at time t+Δt.

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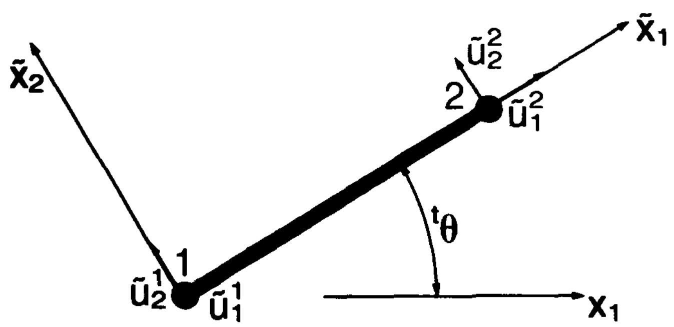
<u>Updated Lagrangian formulation:</u>

The derivation is simplified if we consider a coordinate system aligned with the truss element at time t.



Written in the rotated coordinate system, the equation of the principle of virtual work is

$$\int_{{}^tV} {}^{t+\Delta t}_{\ t}\tilde{S}_{ij}\, \delta\, {}^{t+\Delta t}_{\ t}\tilde{\varepsilon}_{ij}\, {}^t dV = {}^{t+\Delta t}\tilde{\mathcal{R}}$$

As we recall, this may be linearized to obtain

$$\int_{{}^tV} {}_t\tilde{C}_{ijrs}\, {}_t\tilde{e}_{rs}\, \delta_t\tilde{e}_{ij}\, {}^t dV + \int_{{}^tV} {}^t\tilde{\tau}_{ij}\, \delta_t\tilde{\eta}_{ij}\, {}^t dV = {}^{t+\Delta t}\tilde{\mathcal{R}} - \int_{{}^tV} {}^t\tilde{\tau}_{ij}\, \delta_t\tilde{e}_{ij}\, {}^t dV$$

Transparency 8-5

Because the only non-zero stress component is ${}^t\tilde{\tau}_{11}$, the linearized equation of motion simplifies to

$$\int_{{}^tV} {}_t\tilde{C}_{1111}\, {}_t\tilde{e}_{11}\, \delta_t\tilde{e}_{11}\, {}^t dV + \int_{{}^tV} {}^t\tilde{\tau}_{11}\, \delta_t\tilde{\eta}_{11}\, {}^t dV = {}^{t+\Delta t}\tilde{\mathcal{R}} - \int_{{}^tV} {}^t\tilde{\tau}_{11}\, \delta_t\tilde{e}_{11}\, {}^t dV$$

Notice that we need only consider one component of the strain tensor.

Transparency 8-6

Transparency 8-7

We also notice that:

$${}_{t}\tilde{C}_{1111} = E$$

$${}^{t}\tilde{\tau}_{11} = \frac{{}^{t}P}{A}$$

$${}^{t}V = A\,L$$

The stress and strain states are constant along the truss.

Hence the equation of motion becomes

$$(EA)\,{}_{t}\tilde{e}_{11}\,\delta_{t}\tilde{e}_{11}\,L + {}^{t}P\,\delta_{t}\tilde{\eta}_{11}\,L = {}^{t+\Delta t}\tilde{\mathcal{R}} - {}^{t}P\,\delta_{t}\tilde{e}_{11}\,L$$

Transparency 8-8

To proceed, we must express the strain increments in terms of the (rotated) displacement increments:

$${}_{t}\tilde{e}_{11} = {}^{t}_{t}\underline{\tilde{B}}_{L}\,\underline{\hat{u}}\,,$$

$$\delta_{t}\tilde{\eta}_{11} = \underbrace{(\delta\underline{\hat{u}}^{T}\,{}^{t}_{t}\underline{B}^{T}_{NL})({}^{t}_{t}\underline{B}_{NL}\,\underline{\hat{u}})}$$

This form is analogous to the form used in the two-dimensional element formulation.

where

$$\underline{\hat{u}} = \begin{bmatrix} \tilde{u}_1^1 \\ \tilde{u}_2^1 \\ \tilde{u}_1^2 \\ \tilde{u}_2^2 \end{bmatrix}$$

Since ${}_t\tilde{\varepsilon}_{11} = {}_t\tilde{u}_{1,1} + \frac{1}{2}\left(({}_t\tilde{u}_{1,1})^2 + ({}_t\tilde{u}_{2,1})^2\right)$,
we recognize

$$ {}_t\tilde{e}_{11} = {}_t\tilde{u}_{1,1} $$

$$ {}_t\tilde{\eta}_{11} = \frac{1}{2}\left(({}_t\tilde{u}_{1,1})^2 + ({}_t\tilde{u}_{2,1})^2\right) $$

and

$$ \begin{aligned} \delta_t\tilde{\eta}_{11} &= \delta_t\tilde{u}_{1,1}\ {}_t\tilde{u}_{1,1} + \delta_t\tilde{u}_{2,1}\ {}_t\tilde{u}_{2,1} \\ &= \underbrace{\begin{bmatrix} \delta_t\tilde{u}_{1,1} & \delta_t\tilde{u}_{2,1} \end{bmatrix} \begin{bmatrix} {}_t\tilde{u}_{1,1} \\ {}_t\tilde{u}_{2,1} \end{bmatrix}}_{\text{matrix form}} \end{aligned} $$

Transparency 8-9

We can now write the displacement derivatives in terms of the displacements (this is simple because all quantities are constant along the truss). For example,

$$ {}_t\tilde{u}_{1,1} = \frac{\partial \tilde{u}_1}{\partial\, {}^t\tilde{x}_1} = \frac{\Delta \tilde{u}_1}{\Delta\, {}^t x_1} = \frac{\tilde{u}_1^2 - \tilde{u}_1^1}{L} $$

Hence we obtain

$$ \begin{bmatrix} {}_t\tilde{u}_{1,1} \\ {}_t\tilde{u}_{2,1} \end{bmatrix} = \frac{1}{L} \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \tilde{u}_1^1 \\ \tilde{u}_2^1 \\ \tilde{u}_1^2 \\ \tilde{u}_2^2 \end{bmatrix}}_{\hat{\underline{\tilde{u}}}} $$

Transparency 8-10

Transparency 8-11

and

$$ {}_t\tilde{e}_{11} = \underbrace{\left(\frac{1}{L}\begin{bmatrix}-1 & 0 & 1 & 0\end{bmatrix}\right)}_{{}^{t}_{t}\tilde{\underline{B}}_L}\hat{\underline{u}} $$

$$ \delta_t\tilde{\eta}_{11} = \underbrace{\delta\hat{\underline{u}}^T \overbrace{\left(\frac{1}{L}\begin{bmatrix}-1 & 0\\ 0 & -1\\ 1 & 0\\ 0 & 1\end{bmatrix}\right)}^{{}^{t}_{t}\tilde{\underline{B}}^T_{NL}}}_{[\delta_t\tilde{u}_{1,1} \quad \delta_t\tilde{u}_{2,1}]} \underbrace{\overbrace{\left(\frac{1}{L}\begin{bmatrix}-1 & 0 & 1 & 0\\ 0 & -1 & 0 & 1\end{bmatrix}\right)}^{{}^{t}_{t}\tilde{\underline{B}}_{NL}}\hat{\underline{u}}}_{\begin{bmatrix}{}_t\tilde{u}_{1,1}\\ {}_t\tilde{u}_{2,1}\end{bmatrix}} $$

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Using these expressions,

$(EA)\ {}_t\tilde{e}_{11}\ \delta_t\tilde{e}_{11}\ L$

$$ \to\ \delta\hat{\underline{u}}^T \overbrace{\left(\frac{EA}{L}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\right)}^{{}^{t}_{t}\tilde{\underline{K}}_L}\hat{\underline{u}} $$

(setting successively each virtual nodal point displacement equal to unity)

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${}^tP\;\delta_t\tilde{\eta}_{11}\;L$

$$\delta\hat{\underline{u}}^T\left(\overbrace{\frac{{}^tP}{L}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\\ -1 & 0 & 1 & 0\\ 0 & -1 & 0 & 1\end{bmatrix}}^{{}^t_t\tilde{\underline{K}}_{NL}}\right)\hat{\underline{u}}$$

Transparency 8-14

and

${}^tP\;\delta_t\tilde{e}_{11}\;L$

$$\delta\hat{\underline{u}}^T\left(\overbrace{{}^tP\begin{bmatrix}-1\\ 0\\ 1\\ 0\end{bmatrix}}^{{}^t_t\tilde{\underline{F}}}\right)$$

Transparency 8-15

We have now obtained the required element matrices, expressed in the coordinate system aligned with the truss at time t.

To determine the element matrices in the stationary global coordinate system, we must express the rotated displacement increments $\underline{\hat{\tilde{u}}}$ in terms of the unrotated displacement increments $\underline{\hat{u}}$.

We can show that

$$\begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} = \begin{bmatrix} \cos{}^t\theta & \sin{}^t\theta \\ -\sin{}^t\theta & \cos{}^t\theta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Transparency 8-16

Hence

$$\underbrace{\begin{bmatrix} \tilde{u}_1^1 \\ \tilde{u}_2^1 \\ \tilde{u}_1^2 \\ \tilde{u}_2^2 \end{bmatrix}}_{\underline{\hat{\tilde{u}}}} = \underbrace{\begin{bmatrix} \cos{}^t\theta & \sin{}^t\theta & 0 & 0 \\ -\sin{}^t\theta & \cos{}^t\theta & 0 & 0 \\ 0 & 0 & \cos{}^t\theta & \sin{}^t\theta \\ 0 & 0 & -\sin{}^t\theta & \cos{}^t\theta \end{bmatrix}}_{\underline{T}} \underbrace{\begin{bmatrix} u_1^1 \\ u_2^1 \\ u_1^2 \\ u_2^2 \end{bmatrix}}_{\underline{\hat{u}}}$$

Transparency 8-17

Using this transformation in the equation of motion gives

$$\delta\hat{\underline{u}}^T \, {}^t_t\tilde{\underline{K}}_L \, \hat{\underline{u}} \rightarrow \delta\hat{\underline{u}}^T \underbrace{\underline{T}^T \, {}^t_t\tilde{\underline{K}}_L \, \underline{T}}_{{}^t_t\underline{K}_L} \hat{\underline{u}}$$

$$\delta\hat{\underline{u}}^T \, {}^t_t\tilde{\underline{K}}_{NL} \, \hat{\underline{u}} \rightarrow \delta\hat{\underline{u}}^T \underbrace{\underline{T}^T \, {}^t_t\tilde{\underline{K}}_{NL} \, \underline{T}}_{{}^t_t\underline{K}_{NL}} \hat{\underline{u}}$$

$$\delta\hat{\underline{u}}^T \, {}^t_t\tilde{\underline{F}} \rightarrow \delta\hat{\underline{u}}^T \underbrace{\underline{T}^T \, {}^t_t\tilde{\underline{F}}}_{{}^t_t\underline{F}}$$

Transparency 8-18

Performing the indicated matrix multiplications gives

$${}^t_t\underline{K}_L = \frac{EA}{L} \begin{bmatrix} (\cos{}^t\theta)^2 & (\cos{}^t\theta)(\sin{}^t\theta) & -(\cos{}^t\theta)^2 & -(\cos{}^t\theta)(\sin{}^t\theta) \\ & (\sin{}^t\theta)^2 & -(\cos{}^t\theta)(\sin{}^t\theta) & -(\sin{}^t\theta)^2 \\ \text{symmetric} & & (\cos{}^t\theta)^2 & (\cos{}^t\theta)(\sin{}^t\theta) \\ & & & (\sin{}^t\theta)^2 \end{bmatrix}$$

Transparency 8-19

$$ {}^{t}_{t}\underline{K}_{NL} = \frac{{}^{t}P}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ & 1 & 0 & -1 \\ & & 1 & 0 \\ \text{symmetric} & & & 1 \end{bmatrix} $$

and

$$ {}^{t}_{t}\underline{F} = {}^{t}P\begin{bmatrix} -\cos{}^{t}\theta \\ -\sin{}^{t}\theta \\ \cos{}^{t}\theta \\ \sin{}^{t}\theta \end{bmatrix} $$

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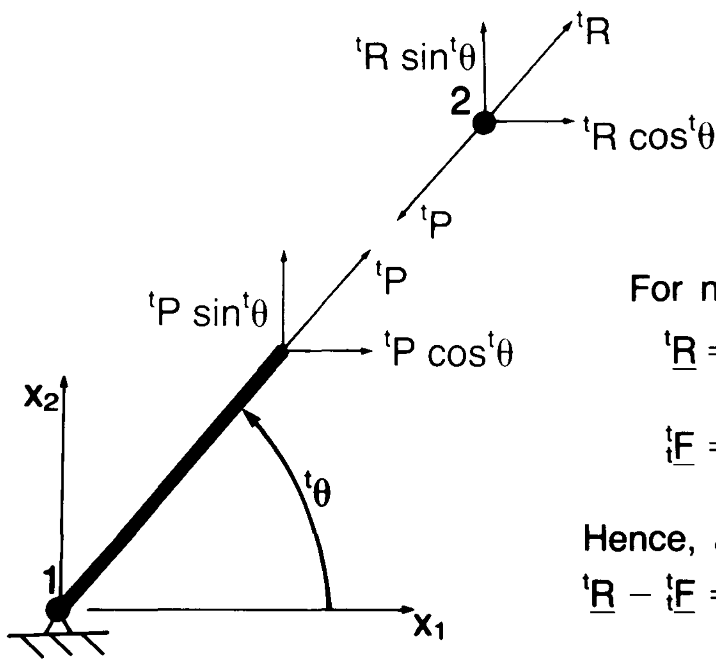
The vector ${}^{t}_{t}\underline{F}$ makes physical sense:



For node 2,

$$ {}^{t}\underline{R} = \begin{bmatrix} {}^{t}R\cos{}^{t}\theta \\ {}^{t}R\sin{}^{t}\theta \end{bmatrix} $$

$$ {}^{t}_{t}\underline{F} = \begin{bmatrix} {}^{t}P\cos{}^{t}\theta \\ {}^{t}P\sin{}^{t}\theta \end{bmatrix} $$

Hence, at equilibrium,

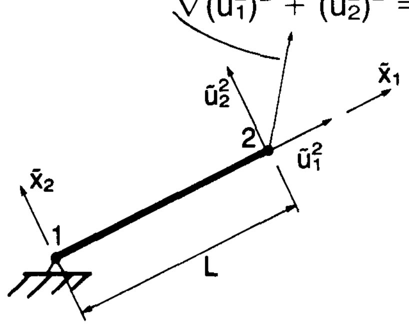
$$ {}^{t}\underline{R} - {}^{t}_{t}\underline{F} = \underline{0} $$

Transparency 8-21

We note that the ${}^t_t\underline{K}_{NL}$ matrix is unchanged by the coordinate transformation.

- The nonlinear strain increment is related only to the vector magnitude of the displacement increment.

$$\sqrt{(\tilde{u}_1^2)^2 + (\tilde{u}_2^2)^2} = \left(\sqrt{\left(\frac{\partial \tilde{u}_1}{\partial\, {}^t\tilde{x}_1}\right)^2 + \left(\frac{\partial \tilde{u}_2}{\partial\, {}^t\tilde{x}_1}\right)^2}\right) L$$



$$= \sqrt{2\, {}_t\eta_{11}}\, L$$

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Physically, ${}^t_t\underline{K}_{NL}$ gives the required change in the externally applied nodal point forces when the truss is rotated.

Consider only $\tilde{u}_2^2$ nonzero.

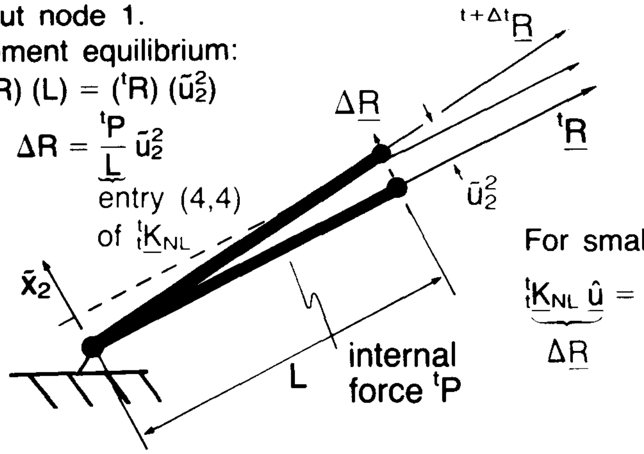
For small $\tilde{u}_2^2$, this gives a rotation about node 1.

Moment equilibrium:

$(\Delta R)\,(L) = ({}^tR)\,(\tilde{u}_2^2)$

or $\Delta R = \frac{{}^tP}{L}\, \tilde{u}_2^2$

entry (4,4) of ${}^t_t\underline{K}_{NL}$



For small $\underline{\hat{u}}$,

$$\underbrace{{}^t_t\underline{K}_{NL}\, \underline{\hat{u}}}_{\Delta \underline{R}} = {}^{t+\Delta t}\underline{R} - {}^t\underline{R}$$

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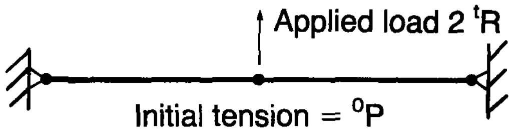
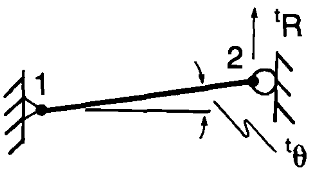
<u>Example</u>: Prestressed cable



Length 2 L
Young's modulus E
Area A

Finite element model (using symmetry):



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Using the U.L. formulation, we obtain

$$\left(\underbrace{\frac{EA}{L}(\sin{}^t\theta)^2}_{{}^t_t\underline{K}_L} + \underbrace{\frac{{}^tP}{L}}_{{}^t_t\underline{K}_{NL}}\right) u_2^2 = {}^{t+\Delta t}R - \underbrace{{}^tP\sin{}^t\theta}_{{}^t_t\underline{F}}$$

Of particular interest is the configuration at time 0, when ${}^t\theta = 0$:

$$\left(\frac{{}^0P}{L}\right) u_2^2 = {}^{\Delta t}R$$

The undeformed cable stiffness is given <u>solely</u> by ${}^t_t\underline{K}_{NL}$.

The cable stiffens as load is applied:

$$ {}^t_tK = \underbrace{\frac{EA}{L}(\sin {}^t\theta)^2}_{{}^t_tK_L} + \underbrace{\frac{{}^tP}{L}}_{{}^t_tK_{NL}} $$

${}^t_tK_L$ increases as ${}^t\theta$ increases (the truss provides axial stiffness as ${}^t\theta$ increases).

As ${}^t\theta \rightarrow 90°$, the stiffness approaches $\frac{EA}{L}$,

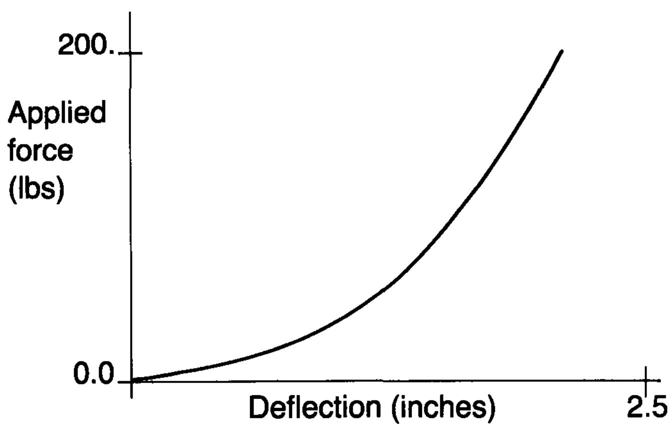
but constant L and A means here that only small values of ${}^t\theta$ are permissible.

Transparency 8-25

Using: $L = 120$ in , $A = 1\ \text{in}^2$,

$E = 30 \times 10^6$ psi , ${}^0P = 1000$ lbs

we obtain



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Transparency 8-27

We also show the stiffness matrix components as functions of the applied load: