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6

POLARIZATION

6.0 INTRODUCTION

The previous chapters postulated surface charge densities that appear and disappear as required by the boundary conditions obeyed by surfaces of conductors. Thus, the idea that the distribution of the charge density may be linked to the field it induces is not new. Thus far, however, no consideration has been given in any detail to the physical laws which determine the occurrence and behavior of charge densities in matter.

To set the stage for this and the next chapter, consider two possible pictures that could be used to explain why an object distorts an initially uniform electric field. In Fig. 6.0.1a, the sphere is composed of a metallic conductor, and therefore composed of atoms having electrons that are free to move from one atomic site to another. Suppose, to begin with, that there are equal numbers of positive sites and negative electrons. In the absence of an applied field and on a scale that is large compared to the distance between atoms (that is, on a *macroscopic* scale), there is therefore no charge density at any point within the material.

When this object is placed in an initially uniform electric field, the electrons are subject to forces that tend to make them concentrate on the south pole of the sphere. This requires only that the electrons migrate downward slightly (on the average, less than an interatomic distance). Because the interior of the sphere must be field free in the final equilibrium (steady) state, the charge density remains zero at each point within the volume of the material. However, to preserve a zero net charge, the positive atomic sites on the north pole of the sphere are uncovered. After a time, the net result is the distribution of surface charge density shown in Fig. 6.0.1b. [In fact, provided the electrodes are well-removed from the sphere, this is the distribution found in Example 5.9.1.]

Now consider an alternative picture of the physics that can lead to a very similar result. As shown in Fig. 6.0.1c, the material is composed of atoms, molecules,
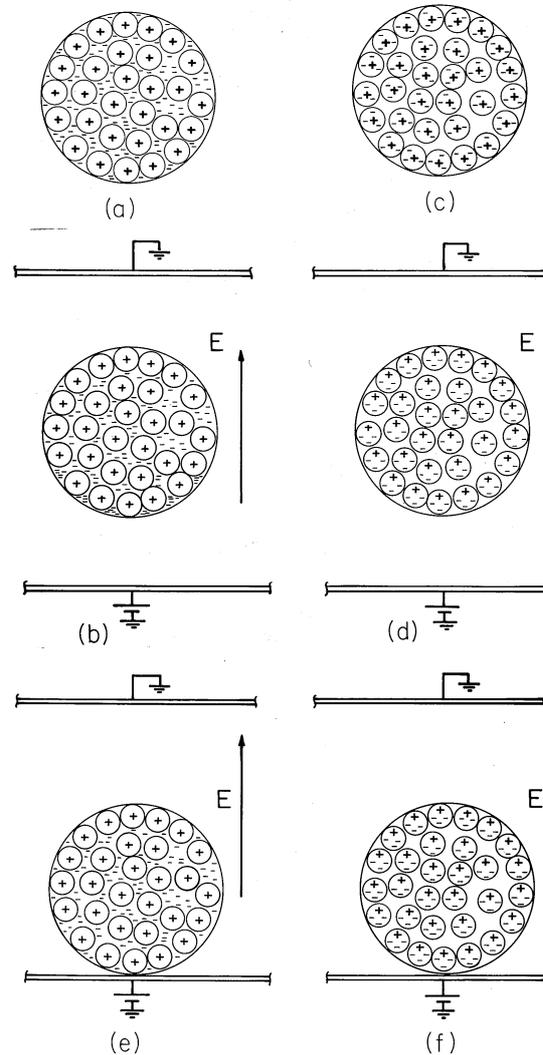
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Fig. 6.0.1   In the left-hand sequence, the sphere is conducting, while on the right, it is polarizable and not conducting.

or groups of molecules (domains) in which the electric field induces dipole moments. For example, suppose that the dipole moments are of an atomic scale and, in the absence of an electric field, do not exist; the moments are induced because atoms contain positively charged nuclei and electrons orbiting around the nuclei. According to quantum theory, electrons orbiting the nuclei are not to be viewed as localized at any particular instant of time. It is more appropriate to think of the electrons as "clouds" of charge surrounding the nuclei. Because the charge of the orbiting electrons is equal and opposite to the charge of the nuclei, a neutral atom has no net charge. An atom with no permanent dipole moment has the further
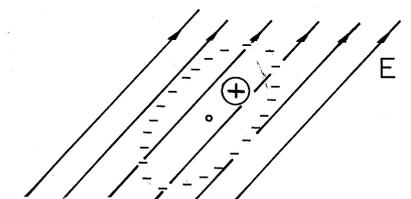
Fig. 6.0.2 Nucleus with surrounding electronic charge cloud displaced by applied electric field.

property that the center of the negative charge of the electron "clouds" coincides with the center of the positive charge of the nuclei. In the presence of an electric field, the center of positive charge is pulled in the direction of the field while the center of negative charge is pushed in the opposite direction. At the atomic level, this relative displacement of charge centers is as sketched in Fig. 6.0.2. Because the two centers of charge no longer coincide, the particle acquires a dipole moment. We can represent each atom by a pair of charges of equal magnitude and opposite sign separated by a distance d.

On the macroscopic scale of the sphere and in an applied field, the dipoles then appear somewhat as shown in Fig. 6.0.1d. In the interior of the sphere, the polarization leaves each positive charge in the vicinity of a negative one, and hence there is no net charge density. However, at the north pole there are no negative charges to neutralize the positive ones, and at the south pole no positive ones to pair up with the negative ones. The result is a distribution of surface charge density that does not differ qualitatively from that for the metal sphere.

How can we distinguish between these two very different situations? Suppose that the two spheres make contact with the lower electrode, as shown in parts (e) and (f) of the figure. By this we mean that in the case of the metal sphere, electrons are now free to pass between the sphere and the electrode. Once again, electrons move slightly downward, leaving positive sites exposed at the top of the sphere. However, some of those at the bottom flow into the lower electrode, thus reducing the amount of negative surface charge on the lower side of the metal sphere.

At the top, the polarized sphere shown by Fig. 6.0.1f has a similar distribution of positive surface charge density. But one very important difference between the two situations is apparent. On an atomic scale in the ideal dielectric, the orbiting electrons are paired with the parent atom, and hence the sphere must remain neutral. Thus, the metallic sphere now has a net charge, while the one made up of dipoles does not.

Experimental evidence that a metallic sphere had indeed acquired a net charge could be gained in a number of different ways. Two are clear from demonstrations in Chap. 1. A pair of spheres, each charged by "induction" in this fashion, would repel each other, and this could be demonstrated by the experiment in Fig. 1.3.10. The charge could also be measured by charge conservation, as in Demonstration 1.5.1. Presumably, the same experiments carried out using insulating spheres would demonstrate the existence of no net charge.

Because charge accumulations occur via displacements of paired charges (polarization) as well as of charges that can move far away from their partners of opposite sign, it is often appropriate to distinguish between these by separating the total charge density ρ into parts ρ_u and ρ_p, respectively, produced by unpaired and

paired charges.

$$\rho = \rho_u + \rho_p \tag{1}$$

In this chapter, we consider insulating materials and therefore focus on the effects of the paired or polarization charge density. Additional effects of unpaired charges are taken up in the next chapter.

Our first step, in Sec. 6.1, is to relate the polarization charge density to the density of dipoles– to the polarization density. We do this because it is the polarization density that can be most easily specified. Sections 6.2 and 6.3 then focus on the first of two general classes of polarization. In these sections, the polarization density is permanent and therefore specified without regard for the electric field. In Sec. 6.4, we discuss simple constitutive laws expressing the action of the field upon the polarization. This field-induced atomic polarization just described is typical of physical situations. The field action on the atom, molecule, or domain is accompanied by a reaction of the dipoles on the field that must be considered simultaneously. That is, within such a polarizable body placed into an electric field, a polarization charge density is produced which, in turn, modifies the electric field. In Secs. 6.5–6.7, we shall study methods by which self-consistent solutions to such problems are obtained.

6.1 POLARIZATION DENSITY

The following development is applicable to polarization phenomena having diverse microscopic origins. Whether representative of atoms, molecules, groups of ordered atoms or molecules (domains), or even macroscopic particles, the dipoles are pictured as opposite charges $\pm q$ separated by a vector distance $\mathbf{d}$ directed from the negative to the positive charge. Thus, the individual dipoles, represented as in Sec. 4.4, have moments $\mathbf{p}$ defined as

$$\mathbf{p} = q\mathbf{d} \tag{1}$$

Because $\mathbf{d}$ is generally smaller in magnitude than the size of the atom, molecule, or other particle, it is small compared with any macroscopic dimension of interest.

Now consider a medium consisting of N such polarized particles per unit volume. What is the net charge q contained within an arbitrary volume V enclosed by a surface S? Clearly, if the particles of the medium within V were unpolarized, the net charge in V would be zero. However, now that they are polarized, some charge centers that were contained in V in their unpolarized state have moved out of the surface S and left behind unneutralized centers of charge. To determine the net unneutralized charge left behind in V, we will assume (without loss of generality) that the negative centers of charge are stationary and that only the positive centers of charge are mobile during the polarization process.

Consider the particles in the neighborhood of an element of area $d\mathbf{a}$ on the surface S, as shown in Fig. 6.1.1. All positive centers of charge now outside S within the volume $dV = \mathbf{d} \cdot d\mathbf{a}$ have left behind negative charge centers. These contribute a net negative charge to V. Because there are $N\mathbf{d} \cdot d\mathbf{a}$ such negative centers of charge in dV, the net charge left behind in V is
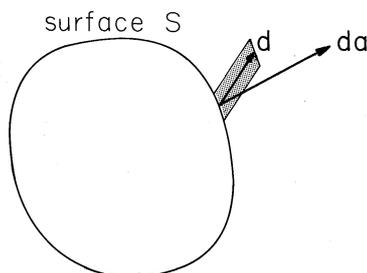
Fig. 6.1.1   Volume element containing positive charges which have left negative charges on the other side of surface S.

$$Q = - \oint_S (qN\mathbf{d}) \cdot d\mathbf{a} \tag{2}$$

Note that the integrand can be either positive or negative depending on whether positive centers of charge are leaving or entering V through the surface element $d\mathbf{a}$. Which of these possibilities occurs is reflected by the relative orientation of $\mathbf{d}$ and $d\mathbf{a}$. If $\mathbf{d}$ has a component parallel (anti-parallel) to $d\mathbf{a}$, then positive centers of charge are leaving (entering) V through $d\mathbf{a}$.

   The integrand of (1) has the dimensions of dipole moment per unit volume and will therefore be defined as the *polarization density*.

$$\mathbf{P} \equiv Nq\mathbf{d} \tag{3}$$

Also by definition, the net charge in V can be determined by integrating the polarization charge density over its volume.

$$Q = \int_V \rho_p dV \tag{4}$$

Thus, we have two ways of calculating the net charge, the first by using the polarization density from (3) in the surface integral of (2).

$$Q = - \oint_S \mathbf{P} \cdot d\mathbf{a} = - \int_V \nabla \cdot \mathbf{P} dV \tag{5}$$

Here Gauss' theorem has been used to convert the surface integral to one over the enclosed volume. The charge found from this volume integral must be the same as given by the second way of calculating the net charge, by (4). Because the volume under consideration is arbitrary, the integrands of the volume integrals in (4) and (5) must be identical.

$$\boxed{\rho_p = -\nabla \cdot \mathbf{P}} \tag{6}$$

In this way, the polarization charge density ρ_p has been related to the polarization density $\mathbf{P}$.
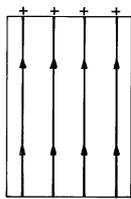
Fig. 6.1.2   Polarization surface charge due to uniform polarization of right cylinder.

It may seem that little has been accomplished in this development because, instead of the unknown ρ_p, the new unknown $\mathbf{P}$ appeared. In some instances, $\mathbf{P}$ is known. But even in the more common cases where the polarization density and hence the polarization charge density is not known a priori but is induced by the field, it is easier to directly link $\mathbf{P}$ with $\mathbf{E}$ than ρ_p with $\mathbf{E}$.

In Fig. 6.0.1, the polarized sphere could acquire no net charge. Our representation of the polarization charge density in terms of the polarization density guarantees that this is true. To see this, suppose V is interpreted as the volume containing the entire polarized body so that the surface S enclosing the volume V falls outside the body. Because $\mathbf{P}$ vanishes on S, the surface integral in (5) must vanish. Any distribution of charge density related to the polarization density by (6) cannot contribute a net charge to an isolated body.

We will often find it necessary to represent the polarization density by a discontinuous function. For example, in a material surrounded by free space, such as the sphere in Fig. 6.0.1, the polarization density can fall from a finite value to zero at the interface. In such regions, there can be a surface polarization charge density. With the objective of determining this density from $\mathbf{P}$, (6) can be integrated over a pillbox enclosing an incremental area of an interface. With the substitution $-\mathbf{P} \rightarrow \epsilon_o \mathbf{E}$ and $\rho_p \rightarrow \rho$, (6) takes the same form as Gauss' law, so the proof is identical to that leading from (1.3.1) to (1.3.17). We conclude that where there is a jump in the normal component of $\mathbf{P}$, there is a *surface polarization charge density*

$$\boxed{\sigma_{sp} = -\mathbf{n} \cdot (\mathbf{P}^a - \mathbf{P}^b)} \tag{7}$$

Just as (6) tells us how to determine the polarization charge density for a given distribution of $\mathbf{P}$ in the volume of a material, this expression serves to evaluate the singularity in polarization charge density (the surface polarization charge density) at an interface.

Note that according to (6), $\mathbf{P}$ originates on negative polarization charge and terminates on positive charge. This contrasts with the relationship between $\mathbf{E}$ and the charge density. For example, according to (6) and (7), the uniformly polarized cylinder of material shown in Fig. 6.1.2 with $\mathbf{P}$ pointing upward has positive σ_{sp} on the top and negative on the bottom.

6.2 LAWS AND CONTINUITY CONDITIONS WITH POLARIZATION

With the unpaired and polarization charge densities distinguished, Gauss' law becomes

$$\nabla \cdot \epsilon_o \mathbf{E} = \rho_u + \rho_p \tag{1}$$

where (6.1.6) relates ρ_p to $\mathbf{P}$.

$$\rho_p = -\nabla \cdot \mathbf{P} \tag{2}$$

Because $\mathbf{P}$ is an "averaged" polarization per unit volume, it is a "smooth" vector function of position on an atomic scale. In this sense, it is a macroscopic variable. The negative of its divergence, the polarization charge density, is also a macroscopic quantity that does not reflect the "graininess" of the microscopic charge distribution. Thus, as it appears in (1), the electric field intensity is also a macroscopic variable.

Integration of (1) over an incremental volume enclosing a section of the interface, as carried out in obtaining (1.3.7), results in

$$\mathbf{n} \cdot \epsilon_o (\mathbf{E}^a - \mathbf{E}^b) = \sigma_{su} + \sigma_{sp} \tag{3}$$

where (6.1.7) relates σ_{sp} to $\mathbf{P}$.

$$\sigma_{sp} = -\mathbf{n} \cdot (\mathbf{P}^a - \mathbf{P}^b) \tag{4}$$

These last two equations, respectively, give expression to the continuity condition of Gauss' law, (1), at a surface of discontinuity.

Polarization Current Density and Ampère's Law.   Gauss' law is not the only one affected by polarization. If the polarization density varies with time, then the flow of charge across the surface S described in Sec. 6.1 comprises an electrical current. Thus, we need to investigate charge conservation, and more generally the effect of a time-varying polarization density on Ampére's law. To this end, the following steps lead to the polarization current density implied by a time-varying polarization density.

According to the definition of $\mathbf{P}$ evolved in Sec. 6.1, the process of polarization transfers an amount of charge dQ

$$dQ = \mathbf{P} \cdot d\mathbf{a} \tag{5}$$

through a surface area element $d\mathbf{a}$. This is perhaps envisioned in terms of the volume $\mathbf{d} \cdot d\mathbf{a}$ shown in Fig. 6.2.1. If the polarization density $\mathbf{P}$ varies with time, then according to this equation, charge is passed through the area element at a finite rate. For a change in $qN\mathbf{d}$, or $\mathbf{P}$, of $\Delta\mathbf{P}$, the amount of charge that has passed through the incremental area element $d\mathbf{a}$ is

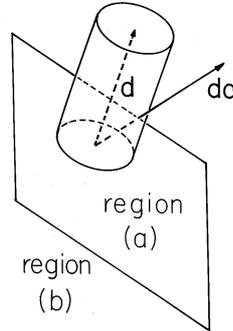$$\Delta(dQ) = \Delta\mathbf{P} \cdot d\mathbf{a} \tag{6}$$

Fig. 6.2.1   Charges passing through area element $d\mathbf{a}$ result in polarization current density.

Note that we have two indicators of differentials in this expression. The d refers to the fact that Q is differential because $d\mathbf{a}$ is a differential. The rate of change with time of dQ, $\Delta(dQ)/\Delta t$, can be identified with a current di_p through $d\mathbf{a}$, from side (b) to side (a).

$$di_p = \frac{\Delta(dQ)}{\Delta t} = \frac{\partial \mathbf{P}}{\partial t} \cdot d\mathbf{a} \qquad (7)$$

The partial differentiation symbol is used to distinguish the differentiation with respect to t from the space dependence of $\mathbf{P}$.

A current di_p through an area element $d\mathbf{a}$ is usually written as a current density dot-multiplied by $d\mathbf{a}$

$$di_p = \mathbf{J}_p \cdot d\mathbf{a} \qquad (8)$$

Hence, we compare these last two equations and deduce that the polarization current density is

$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t} \qquad (9)$$

Note that $\mathbf{J}_p$ and ρ_p, via (2) and (9), automatically obey a continuity law having the same form as the charge conservation equation, (2.3.3).

$$\nabla \cdot \mathbf{J}_p + \frac{\partial \rho_p}{\partial t} = 0 \qquad (10)$$

Hence, we can think of a rate of charge transport in a material medium as consisting of a current density of unpaired charges $\mathbf{J}_u$ and a polarization current density $\mathbf{J}_p$, each obeying its own conservation law. This is also implied by Ampère's law, as now generalized to include the effects of polarization.

In the EQS approximation, the magnetic field intensity is not usually of interest, and so Ampère's law is of secondary importance. But if $\mathbf{H}$ were to be determined, $\mathbf{J}_p$ would make a contribution. That is, Ampère's law as given by (2.6.2) is now written with the current density divided into paired and unpaired parts. With the latter given by (9), Ampère's differential law, generalized to include polarization, is

$$\nabla \times \mathbf{H} = \mathbf{J}_u + \frac{\partial}{\partial t}(\epsilon_o \mathbf{E} + \mathbf{P}) \qquad (11)$$

This law is valid whether quasistatic approximations are to be made or not. However, it is its implication for charge conservation that is usually of interest in the EQS approximation. Thus, the divergence of (11) gives zero on the left and, in view of (1), (2), and (9), the expression becomes

$$\nabla \cdot \mathbf{J}_u + \frac{\partial \rho_u}{\partial t} + \nabla \cdot \mathbf{J}_p + \frac{\partial \rho_p}{\partial t} = 0 \tag{12}$$

Thus, with the addition of the polarization current density to (11), the divergence of Ampère's law gives the sum of the conservation equations for polarization charges, (10), and unpaired charges

$$\nabla \cdot \mathbf{J}_u + \frac{\partial \rho_u}{\partial t} = 0 \tag{13}$$

In the remainder of this chapter, it will be assumed that in the polarized material, ρ_u is usually zero. Thus, (13) will not come into play until Chap. 7.

Displacement Flux Density.   Primarily in dealing with field-dependent polarization phenomena, it is customary to define a combination of quantities appearing in Gauss' law and Ampère's law as the *displacement flux density* $\mathbf{D}$.

$$\boxed{\mathbf{D} \equiv \epsilon_o \mathbf{E} + \mathbf{P}} \tag{14}$$

We regard $\mathbf{P}$ as representing the material and $\mathbf{E}$ as a field quantity induced by the external sources and the sources within the material. This suggests that $\mathbf{D}$ be considered a "hybrid" quantity. Not all texts on electromagnetism take this point of view. Our separation of all quantities appearing in Maxwell's equations into field and material quantities aids in the construction of models for the interaction of fields with matter.

With ρ_p replaced by (2), Gauss' law (1) can be written in terms of $\mathbf{D}$ defined by (14),

$$\boxed{\nabla \cdot \mathbf{D} = \rho_u} \tag{15}$$

while the associated continuity condition, (3) with σ_{sp} replaced by (4), becomes

$$\boxed{n \cdot (\mathbf{D}^a - \mathbf{D}^b) = \sigma_{su}} \tag{16}$$

The divergence of $\mathbf{D}$ and the jump in normal $\mathbf{D}$ determine the unpaired charge densities. Equations (15) and (16) hold, unchanged in form, both in free space and matter. To adapt the laws to free space, simply set $\mathbf{D} = \epsilon_o \mathbf{E}$.

Ampère's law is also conveniently written in terms of $\mathbf{D}$. Substitution of (14) into (11) gives

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}_u + \frac{\partial \mathbf{D}}{\partial t}} \tag{17}$$

Now the *displacement current density* $\partial \mathbf{D}/\partial t$ includes the polarization current density.

6.3 PERMANENT POLARIZATION

Usually, the polarization depends on the electric field intensity. However, in some materials a permanent polarization is "frozen" into the material. Ideally, this means that $\mathbf{P}(\mathbf{r}, t)$ is prescribed, independent of $\mathbf{E}$. Electrets, used to make microphones and telephone speakers, are often modeled in this way.

With $\mathbf{P}$ a given function of space, and perhaps of time, the polarization charge density and surface charge density follow from (6.2.2) and (6.2.4) respectively. If the unpaired charge density is also given throughout the material, the total charge density in Gauss' law and surface charge density in the continuity condition for Gauss' law are known. [The right-hand sides of (6.2.1) and (6.2.3) are known.] Thus, a description of permanent polarization problems follows the same format as used in Chaps. 4 and 5.

Examples in this section are intended to develop an appreciation for the relationship between the polarization density $\mathbf{P}$, the polarization charge density ρ_p, and the electric field intensity $\mathbf{E}$. It should be recognized that once ρ_p is determined from the given $\mathbf{P}$, the methods of Chaps. 4 and 5 are directly applicable.

The distinction between paired and unpaired charges is sometimes academic. By subjecting an insulating material to an extremely large field, especially at an elevated temperature, it is possible to coerce molecules or domains of molecules into a polarization state that is retained for some period of time at lower fields and temperatures. It is natural to take this as a state of permanent polarization. But, if ions are made to impact the surface of the material, they can form sites of permanent charge. Certainly, the origin of these ions suggests that they be regarded as unpaired. Yet if the material attracts other charges to become neutral, as it tends to do, these permanent charges could also be regarded as due to polarization and represented by a permanent polarization charge density.

In this section, the EQS laws prevail. Thus, with the understanding that throughout the region of interest (exclusive of enclosing boundaries) the charge densities are given,

$$\mathbf{E} = -\nabla \Phi \tag{1}$$

$$\nabla^2 \Phi = -\frac{1}{\epsilon_o}(\rho_u + \rho_p) \tag{2}$$

The example now considered is akin to that pictured qualitatively in Fig. 6.1.2. By making the uniformly polarized material spherical, it is possible to obtain a simple solution for the field distribution.

Example 6.3.1.   A Permanently Polarized Sphere

A sphere of material having radius R is uniformly polarized along the z axis,

$$\mathbf{P} = P_o \mathbf{i_z} \tag{3}$$

Given that the surrounding region is free space with no additional field sources, what is the electric field intensity **E** produced by this permanent polarization?

The first step is to establish the distribution of ρ_p, in the material volume and on its surfaces. In the volume, the negative divergence of **P** is zero, so there is no volumetric polarization charge density (6.2.2). This is obvious with **P** written in Cartesian coordinates. It is less obvious when **P** is expressed in its spherical coordinate components.

$$\mathbf{P} = P_o \cos \theta \mathbf{i_r} - P_o \sin \theta \mathbf{i_\theta} \tag{4}$$

Abrupt changes of the normal component of **P** entail polarization surface charge densities. These follow from using (4) to evaluate the continuity condition of (6.2.4) applied at $r = R$, where the normal component is $\mathbf{i_r}$ and region (a) is outside the sphere.

$$\sigma_{sp} = P_o \cos \theta \tag{5}$$

This surface charge density gives rise to **E**.

Now that the field sources have been identified, the situation reverts to one much like that illustrated by Problem 5.9.2. Both within the sphere and in the surrounding free space, the potential must satisfy Laplace's equation, (2), with $\rho_u + \rho_p = 0$. In terms of Φ the continuity conditions at $r = R$ implied by (1) and (2) [(5.3.3) and (6.2.3)] with the latter evaluated using (5) are

$$\Phi^o - \Phi^i = 0 \tag{6}$$

$$-\epsilon_o \frac{\partial \Phi^o}{\partial r} + \epsilon_o \frac{\partial \Phi^i}{\partial r} = P_o \cos \theta \tag{7}$$

where (o) and (i) denote the regions outside and inside the sphere.

The source of the **E** field represented by this potential is a surface polarization charge density that varies cosinusoidally with θ. It is possible to fulfill the boundary conditions, (6) and (7), with the two spherical coordinate solutions to Laplace's equation (from Sec. 5.9) having the θ dependence $\cos \theta$. Because there are no sources in the region outside the sphere, the potential must go to zero as $r \to \infty$. Of the two possible solutions having the $\cos \theta$ dependence, the dipole field is used outside the sphere.

$$\Phi^o = A \frac{\cos \theta}{r^2} \tag{8}$$

Inside the sphere, the potential must be finite, so this solution is excluded. The solution is

$$\Phi^i = Br \cos \theta \tag{9}$$

which is that of a uniform electric field intensity. Substitution of these expressions into the continuity conditions, (6) and (7), gives expressions from which $\cos \theta$ can be factored. Thus, the boundary conditions are satisfied at every point on the surface if

$$\frac{A}{R^2} - BR = 0 \tag{10}$$

$$2\epsilon_o \frac{A}{R^3} + \epsilon_o B = P_o \tag{11}$$

These expressions can be solved for A and B, which are introduced into (8) and (9) to give the potential distribution

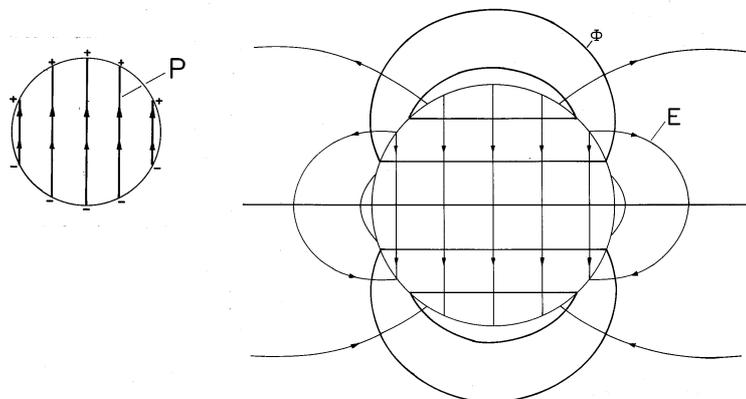$$\Phi^o = \frac{P_o R^3}{3\epsilon_o} \frac{\cos \theta}{r^2} \tag{12}$$

Fig. 6.3.1 Equipotentials and lines of electric field intensity of perma-
nently polarized sphere having uniform polarization density. Inset shows
polarization density and associated surface polarization charge density.

$$\Phi^i = \frac{P_o}{3\epsilon_o} r \cos\theta \tag{13}$$

Finally, the desired distribution of electric field is obtained by taking the negative
gradient of this potential.

$$\mathbf{E}^o = \frac{P_o R^3}{3\epsilon_o r^3}(2\cos\theta\mathbf{i_r} + \sin\theta\mathbf{i_\theta}) \tag{14}$$

$$\mathbf{E}^i = \frac{P_o}{3\epsilon_o}(-\cos\theta\mathbf{i_r} + \sin\theta\mathbf{i_\theta}) \tag{15}$$

With the distribution of polarization density shown in the inset, Fig. 6.3.1 shows
this electric field intensity. It comes as no surprise that the **E** lines originate on the
positive charge and terminate on the negative. The polarization density originates
on negative polarization charge and terminates on positive polarization charge. The
resulting electric field is classic because outside it is exactly that of a dipole at the
origin, while inside it is uniform.

What would be the moment of the dipole at the origin giving rise to the same
external field as the uniformly polarized sphere? This can be seen from a comparison
of (12) and (4.4.10).

$$|P| = \frac{4}{3}\pi R^3 P_o \tag{16}$$

The moment is simply the volume multiplied by the uniform polarization density.

There are two new ingredients in the next example. First, the region of interest
has boundaries upon which the potential is constrained. Second, the given polar-
ization density represents a volumetric distribution of polarization charge density
rather than a surface distribution.

Example 6.3.2. Fields Due to Volume Polarization Charge
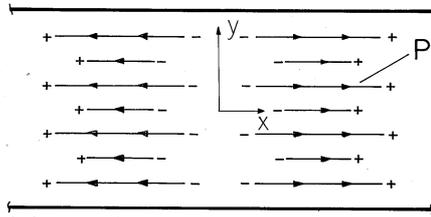with Boundary Conditions

Fig. 6.3.2 Periodic distribution of polarization density and associated polarization charge density ($\rho_o < 0$) gives rise to potential and field shown in Fig. 5.6.2.
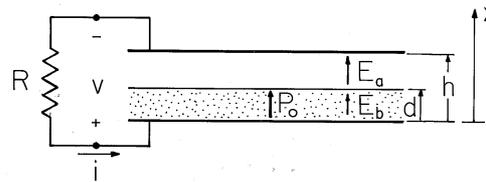

Fig. 6.3.3 Cross-section of electret microphone.

Plane parallel electrodes, in the planes $y = \pm a$, are constrained to zero potential. In the planar region between, the polarization density is the spatially periodic function

$$\mathbf{P} = -\mathbf{i_x}\frac{\rho_o}{\beta}\sin\beta x \tag{17}$$

We wish to determine the field distribution.

First, the distribution of polarization charge density is determined by taking the negative divergence of (17) [(17) is substituted into (6.1.6)].

$$\rho_p = \rho_o \cos\beta x \tag{18}$$

The distribution of polarization density and polarization charge density which has been found is shown in Fig. 6.3.2 ($\rho_o < 0$).

Now the situation reverts to solving Poisson's equation, given this source distribution and subject to the zero potential conditions on the boundaries at $y = \pm a$. The problem is identical to that considered in Example 5.6.1. The potential and field are the superposition of particular and homogeneous parts depicted in Fig. 5.6.2.

The next example illustrates how a permanent polarization can conspire with a mechanical deformation to produce a useful electrical signal.

Example 6.3.3. An Electret Microphone

Shown in cross-section in Fig. 6.3.3 is a thin sheet of permanently polarized material having thickness d. It is bounded from below by a fixed electrode having the potential v and from above by an air gap. On the other side of this gap is a conducting grounded diaphragm which serves as the movable element of a microphone. It is mounted so that it can undergo displacements. Thus, the spacing $h = h(t)$. Given $h(t)$, what is the voltage developed across a load resistance R?

In the sheet, the polarization density is uniform, with magnitude P_o, and directed from the lower electrode toward the upper one. This vector has no divergence,
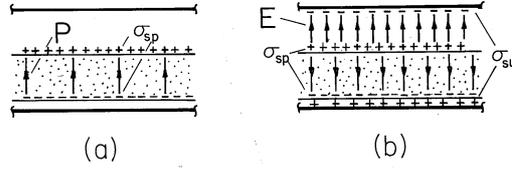
Fig. 6.3.4   (a) Distribution of polarization density and surface charge density in electret microphone. (b) Electric field intensity and surface polarization and unpaired charges.

and so evaluation of (6.1.6) shows that the polarization charge density is zero in the volume of the sheet. The polarization surface charge density on the electret air gap interface follows from (6.1.7) as

$$\sigma_{sp} = -\mathbf{n} \cdot (\mathbf{P}^a - \mathbf{P}^b) = P_o \tag{19}$$

Because σ_{sp} is uniform and the equipotential boundaries are plane and parallel, the electric field in the air gap [region (a)] and in the electret [region (b)] are taken as uniform.

$$\mathbf{E} = \mathbf{i_x} \begin{cases} E_a; & d < x < h \\ E_b; & 0 < x < d \end{cases} \tag{20}$$

Formally, we have just solved Laplace's equation in each of the bulk regions. The fields E_a and E_b must satisfy two conditions. First, the potential difference between the electrodes is v, so

$$v = \int_0^h E_x dx = dE_b + (h-d)E_a \tag{21}$$

Second, Gauss' jump condition at the electret air gap interface, (6.2.3), requires that

$$\epsilon_o E_a - \epsilon_o E_b = P_o \tag{22}$$

Simultaneous solution of these last two expressions evaluates the electric fields in terms of v and h.

$$E_a = \frac{v}{h} + \frac{d}{h}\frac{P_o}{\epsilon_o} \tag{24a}$$

$$E_b = \frac{v}{h} - \frac{(h-d)}{h}\frac{P_o}{\epsilon_o} \tag{24b}$$

What has been found is illustrated in Fig. 6.3.4. The uniform $\mathbf{P}$ and associated σ_{sp} shown in part (a) combine with the unpaired charges on the lower electrode and upper diaphragm to produce the fields shown in part (b). In this picture, it is assumed that v is positive and $(h-d)P_o/\epsilon_o > v$. In the air gap, the field due to the unpaired charges on the electrodes reinforces that due to σ_{sp}, while in the electret, it opposes the downward-directed field due to σ_{sp}.

To compute the current i, defined in Fig. 6.3.3, the lower electrode and the electret are enclosed by a surface S, and Gauss' law is used to evaluate the enclosed *unpaired* charge.

$$\nabla \cdot (\epsilon_o \mathbf{E} + \mathbf{P}) = \rho_u \Rightarrow q = \oint_S (\epsilon_o \mathbf{E} + \mathbf{P}) \cdot \mathbf{n} da \tag{25}$$

Just how the surface S cuts through the system does not matter. Here we take the surface as enclosing the lower electrode by passing through the air gap. It follows from (24) that the unpaired charge is

$$q = A\epsilon_o E_a = \frac{A\epsilon_o}{h}\left(v + \frac{dP_o}{\epsilon_o}\right) \tag{26}$$

where A is the area of the electrode.

Conservation of unpaired charge requires that the current be the rate of change of the total unpaired charge on the lower electrode.

$$i = \frac{dq}{dt} \tag{27}$$

With the resistor attached to the terminals (the input resistance of an amplifier driven by the microphone), the voltage and current must also satisfy Ohm's law.

$$v = -iR \tag{28}$$

These last three relations combine to give an expression for $v(t)$, given $h(t)$.

$$-\frac{v}{R} = -\frac{A\epsilon_o}{h^2}\left(v + \frac{dP_o}{\epsilon_o}\right)\frac{dh}{dt} + \frac{A\epsilon_o}{h}\frac{dv}{dt} \tag{29}$$

This differential equation has time-varying coefficients. Not only is this equation difficult to solve, but also the predicted voltage response cannot be a good replica of $h(t)$, as required for a good microphone, if all terms are of equal importance. That situation can be remedied if the deflections h_1 are kept small compared with the equilibrium position, $h_o \gg h_1$. In the absence of a time variation of h_1, it is clear from (29) that v is zero. By making h_1 small, we can make v small.

Expanding the right-hand side of (29) to first order in h_1, dh_1/dt, v, and dv/dt, we obtain

$$C_o\frac{dv}{dt} + \frac{v}{R} = \frac{C_o}{h_o}\left(\frac{dP_o}{\epsilon_o}\right)\frac{dh_1}{dt} \tag{30}$$

where $C_o = A\epsilon_o/h_o$.

We could solve this equation for its response to a sinusoidal drive. Alternatively, the resulting frequency response can be determined, with more physical insight, by considering two limits. First, suppose that time rate of change is so slow (frequencies so low) that the first term on the left is negligible compared to the second. Then the output voltage is

$$v = \frac{C_o R}{h_o}\left(\frac{dP_o}{\epsilon_o}\right)\frac{dh_1}{dt}; \qquad \omega R C_o \ll 1 \tag{31}$$

In this limit, the resistor acts as a short. The charge can be determined by the diaphragm displacement with the contribution of v ignored (i.e., the charge required to produce v by charging the capacitance C_o is ignored). The small but finite voltage is then obtained as the time rate of change of the charge multiplied by $-R$.

Second, suppose that time rates of change are so rapid that the second term is negligible compared to the first. Within an integration constant,

$$v = \frac{dP_o}{\epsilon_o}\frac{h_1}{h_o}; \qquad \omega R C_o \gg 1 \tag{32}$$
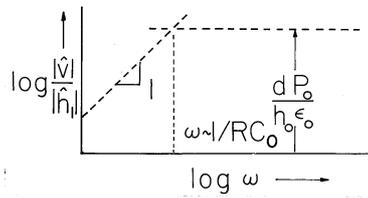
Fig. 6.3.5   Frequency response of electret microphone for imposed diaphragm displacement.

In this limit, the electrode charge is essentially constant. The voltage is obtained from (26) with q set equal to its equilibrium value, $(A\epsilon_o/h_o)(dP_o/\epsilon_o)$.

The frequency response gleaned from these asymptotic responses is in Fig. 6.3.5.

Because its displacement was taken as known, we have been able to ignore the dynamical equations of the diaphragm. If the mass and damping of the diaphragm are ignored, the displacement indeed reflects the pressure of a sound wave. In this limit, a linear distortion-free response of the microphone to pressure is assured at frequencies $\omega > 1/RC$. However, in predicting the response to a sound wave, it is usually necessary to include the detailed dynamics of the diaphragm.

In a practical microphone, subjecting the electret sheet to an electric field would induce some polarization over and beyond the permanent component P_o. Thus, a more realistic model would incorporate features of the linear dielectrics introduced in Sec. 6.4.

6.4 CONSTITUTIVE LAWS OF POLARIZATION

Dipole formation, or orientation of dipolar particles, usually depends on the local field in which the particles are situated. This local microscopic field is not necessarily equal to the macroscopic E field. Yet certain relationships between the macroscopic quantities $\mathbf{E}$ and $\mathbf{P}$ can be established without a knowledge of the relations between the local microscopic fields and the macroscopic E fields. Usually, these relations, called constitutive laws, originate in experimental observations characteristic of the material being investigated.

First, the permanent polarization model developed in the previous section is one constitutive law. In such a medium, $\mathbf{P}(\mathbf{r})$ is prescribed independent of $\mathbf{E}$.

There are media, and these are much more common, in which the polarization depends on $\mathbf{E}$. Consider an isotropic medium, which, in the absence of an electric field has no preferred orientation. Amorphous media such as glass are isotropic. Crystalline media, made up of randomly oriented microscopic crystals, also behave as isotropic media on a macroscopic scale. If we assume that the polarization $\mathbf{P}$ in an isotropic medium depends on the instantaneous field and not on its past history, then $\mathbf{P}$ is a function of $\mathbf{E}$

$$\mathbf{P} = \mathbf{P}(\mathbf{E}) \tag{1}$$

where $\mathbf{P}$ and $\mathbf{E}$ are parallel to each other. Indeed, if $\mathbf{P}$ were not parallel to $\mathbf{E}$, then a preferred direction different from the direction of $\mathbf{E}$ would need to exist in the medium, which contradicts the assumption of isotropy. A possible relation between
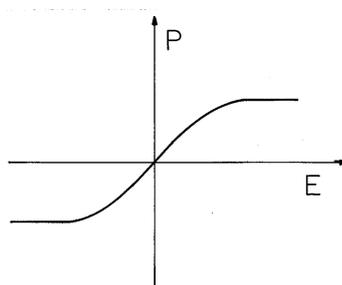
Fig. 6.4.1   Polarization characteristic for nonlinear isotropic material.

the magnitudes of $\mathbf{E}$ and $\mathbf{P}$ is shown in Fig. 6.4.1 and represents an "electrically nonlinear" medium for which $\mathbf{P}$ "saturates" for large values of $\mathbf{E}$.

If the medium is electrically linear, in addition to being isotropic, then a linear relationship exists between $\mathbf{E}$ and $\mathbf{P}$

$$\mathbf{P} = \epsilon_o \chi_e \mathbf{E} \tag{2}$$

where χ_e is the *dielectric susceptibility*. Typical values are given in Table 6.4.1. All isotropic media behave as linear media and obey (2) if the applied $\mathbf{E}$ field is sufficiently small. As long as $\mathbf{E}$ is small enough, any continuous function $\mathbf{P}(\mathbf{E})$ can be expanded in a Taylor series of $\mathbf{E}$ and broken off with the first term in $\mathbf{E}$. (An isotropic medium cannot have a term in the Taylor expansion independent of $\mathbf{E}$.)

For a linear isotropic material, where (2) is obeyed, it follows that $\mathbf{D}$ and $\mathbf{E}$ are related by

$$\mathbf{D} = \epsilon \mathbf{E} \tag{3}$$

where

$$\epsilon \equiv \epsilon_o (1 + \chi_e) \tag{4}$$

is the *permittivity* or *dielectric constant*. The permittivity normalized to ϵ_o, $(1+\chi_e)$, is the *relative dielectric constant*.

In our discussion, it has been assumed that the state of polarization depends only on the instantaneous electric field intensity. There are materials in which the polarization depends not only on the current electric field intensity but on the sequence of preceding states as well (hysteresis). Because we will find magnetization phenomena analogous in many ways to polarization phenomena, we will defer consideration of hysteretic phenomena to Chap. 9.

Many types of transducers exploit the dependence of polarization on variables other than the electric field. In pyroelectric materials, polarization is a function of temperature. Pyroelectrics are used for optical detectors of high-power infrared radiation. Piezoelectric materials have a polarization which is a function of strain (deformation). Such media are suited to low-power electromechanical energy conversion.

TABLE 6.4.1
MATERIAL DIELECTRIC SUSCEPTIBILITIES

Gases	χ_e
Air,	
$0°$C....................................	0.00059
40 atmospheres	0.0218
80 atmospheres	0.0439
Carbon dioxide, $0°$C........................	0.000985
Hydrogen, $0°$C	0.000264
Water vapor, $145°$C	0.00705

Liquids	χ_e
Acetone, $0°$C...............................	25.6
Air, -191$°$C.................................	0.43
Alcohol	
amyl...................................	16.0
ethyl..................................	24.8
methyl.................................	30.2
Benzene....................................	1.29
Glycerine, $15°$C	55.2
Oils,	
castor..................................	3.67
linseed.................................	2.35
corn	2.1
Water, distilled.............................	79.1

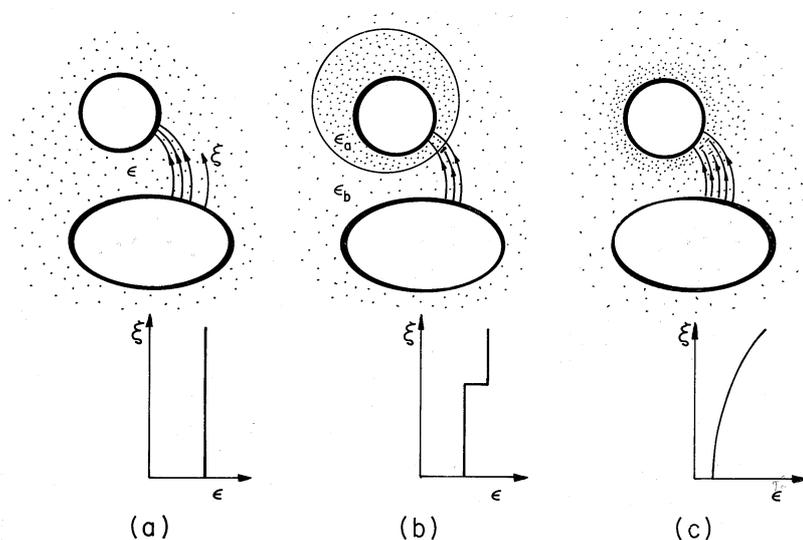| Solids	χ_e
Diamond...................................	15.5
Glass,	
flint, density 4.5	8.90
flint, density 2.87	5.61
lead, density 3.0-3.5.....................	4.4-7.0
Mica......................................	4.6-5.0
Paper (cable insulation)	1.0-1.5
Paraffin...................................	1.1
Porcelain..................................	4.7
Quartz,	
1 to axis...............................	3.69
11 to axis..............................	4.06
Rubber....................................	1.3-3.0
Shellac....................................	2.1

Fig. 6.5.1   Field region filled by (a) uniform dielectric, (b) piece-wise uniform dielectric and (c) smoothly varying dielectric.

6.5 FIELDS IN THE PRESENCE OF ELECTRICALLY LINEAR DIELECTRICS

In Secs. 6.2 and 6.3, the polarization density was given independently of the electric field intensity. In this and the next two sections, the polarization is induced by the electric field. Not only does the electric field give rise to the polarization, but in return, the polarization modifies the field. The polarization feeds back on the electric field intensity.

This "feedback" is described by the constitutive law for a linear dielectric. Thus, (6.4.3) and Gauss' law, (6.2.15), combine to give

$$\nabla \cdot \epsilon \mathbf{E} = \rho_u \tag{1}$$

and the electroquasistatic form of Faraday's law requires that

$$\nabla \times \mathbf{E} = 0 \Rightarrow \mathbf{E} = -\nabla \Phi \tag{2}$$

The continuity conditions implied by these two laws across an interface separating media having different permittivities are (6.2.16) expressed in terms of the constitutive law and either (5.3.1) or (5.3.4). These are

$$\mathbf{n} \cdot (\epsilon_a \mathbf{E}^a - \epsilon_b \mathbf{E}^b) = \sigma_{su} \tag{3}$$

$$\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b) = 0 \Rightarrow \Phi^a - \Phi^b = 0 \tag{4}$$

Figure 6.5.1 illustrates three classes of situations involving linear dielectrics. In the first, the entire region of interest is filled with a *uniform* dielectric. In the second, the region of interest can be broken into uniform subregions within which

the permittivity is constant. The continuity conditions are needed to insure that the basic laws are satisfied through the interfaces between these regions. Systems of this type are said to be composed of *piece-wise uniform* dielectrics. Finally, the dielectric material may vary in its permittivity over dimensions that are on the same order as those of interest. Such a *smoothly inhomogeneous* dielectric is illustrated in Fig. 6.5.1c.

The remainder of this section makes some observations that are generally applicable provided that $\rho_u = 0$ throughout the volume of the region of interest. Section 6.6 is devoted to systems having uniform and piece-wise uniform dielectrics, while Sec. 6.7 illustrates fields in smoothly inhomogeneous dielectrics.

Capacitance. How does the presence of a dielectric alter the capacitance? To answer this question, recognize that conservation of unpaired charge, as expressed by (6.2.13), still requires that the current i measured at terminals connected to a pair of electrodes is the time rate of change of the *unpaired* charge on the electrode. In view of Gauss' law, with the effects of polarization included, (6.2.15), the net unpaired charge on an electrode enclosed by a surface S is

$$q = \int_V \rho_u dV = \int_V \nabla \cdot \mathbf{D}\, dV = \oint_S \mathbf{D} \cdot \mathbf{n}\, da \qquad (5)$$

Here, Gauss' theorem has been used to convert the volume integral to a surface integral.

We conclude that the capacitance of an electrode (a) relative to a reference electrode (b) is

$$C = \frac{\oint_S \mathbf{D} \cdot \mathbf{n}\, da}{\int_a^b {}_{C'}\mathbf{E} \cdot d\mathbf{s}} = \frac{\oint_S \mathbf{D} \cdot \mathbf{n}\, da}{v} \qquad (6)$$

Note that this is the same as for electrodes in free space except that $\epsilon_o \mathbf{E} \to \mathbf{D}$. Because there is no unpaired charge density in the region between the electrodes, S is any surface that encloses the electrode (a). As before, with no polarization, $\mathbf{E}$ is irrotational, and therefore C' is any contour connecting the electrode (a) to the reference (b).

In an electrically linear dielectric, where $\mathbf{D} = \epsilon \mathbf{E}$, both the numerator and denominator of (6) are proportional to the voltage, and as a result, the capacitance C is independent of the voltage. However, with the introduction of an electrically nonlinear material, perhaps having the polarization constitutive law of Fig. 6.4.1, the numerator of (6) is not a linear function of the voltage. As defined by (6), the capacitance is then a function of the applied voltage.

Induced Polarization Charge. Stated as (1)–(4), the laws and continuity conditions for fields in a linear dielectric put the polarization charge out of view. Yet it is this charge that contains the effect of the dielectric on the field. Where does the polarization charge accumulate?

Again, assuming that ρ_u is zero, a vector identity casts Gauss' law as given by (1) into the form

$$\epsilon \nabla \cdot \mathbf{E} + \mathbf{E} \cdot \nabla \epsilon = 0 \qquad (7)$$

Multiplied by ϵ_o and divided by ϵ, this expression can be written as

$$\nabla \cdot \epsilon_o \mathbf{E} = \frac{-\epsilon_o}{\epsilon} \mathbf{E} \cdot \nabla \epsilon \tag{8}$$

Comparison of this expression to Gauss' law written in terms of ρ_p, (6.2.1), shows that the polarization charge density is

$$\rho_p = -\frac{\epsilon_o}{\epsilon} \mathbf{E} \cdot \nabla \epsilon \tag{9}$$

This equation makes it clear that polarization charge will be induced only where there are gradients in ϵ. A special case is where there is an abrupt discontinuity in ϵ. Then the gradient in (9) is singular and represents a polarization surface charge density (the gradient represents the spatial derivative of a step function, which is an impulse). This surface charge density can best be determined by making use of the polarization charge density continuity condition, (6.1.7). Substitution of the constitutive law $\mathbf{P} = (\epsilon - \epsilon_o)\mathbf{E}$ then gives

$$\sigma_{sp} = -\mathbf{n} \cdot [(\epsilon_a - \epsilon_o)\mathbf{E}^a - (\epsilon_b - \epsilon_o)\mathbf{E}^b] \tag{10}$$

Because $\sigma_{su} = 0$, it follows from the jump condition for $\mathbf{n} \cdot \mathbf{D}$, (3), that

$$\sigma_{sp} = \mathbf{n} \cdot \epsilon_o \mathbf{E}^a \left(1 - \frac{\epsilon_a}{\epsilon_b}\right) \tag{11}$$

Remember that $\mathbf{n}$ is directed from region (b) to region (a).

Because $\mathbf{D}$ is solenoidal, we can construct tubes of $\mathbf{D}$ containing constant flux. Lines of $\mathbf{D}$ must therefore begin and terminate on the boundaries. The constitutive law, $\mathbf{D} = \epsilon \mathbf{E}$, requires that $\mathbf{D}$ is proportional to $\mathbf{E}$. Thus, although $\mathbf{E}$ can intensify or rarify as it passes through a flux tube, it can not reverse direction. Therefore, if we follow a bundle of electric field lines from the boundary point of high potential to the one of low potential, the polarization charge encountered [in accordance with (9) and (11)] is positive at points where ϵ is decreasing, negative where it is increasing.

Consider the examples in Fig. 6.5.1. In the case of the uniform dielectric, Fig. 6.5.1a, the typical flux tube shown passes through no variations in ϵ, and it follows from (8) that there is no volume polarization charge density. Thus, it will come as no surprise that the field distribution in this case is predicted by Laplace's equation.

In the piece-wise uniform dielectrics, there is no polarization charge density in a flux tube except where it passes through an interface. For the flux tube shown, (11) shows that if the upper region has the greater permittivity ($\epsilon_a > \epsilon_b$), then there is an accumulation of negative surface charge density at the interface. Thus, the field originating on positive charges at the lower electrode is in part terminated by negative polarization surface charge at the interface, and the field in the upper region tends to be weakened relative to that below.

In the smoothly inhomogeneous dielectric of Fig. 6.5.1c, the typical flux tube shown passes through a region where ϵ increases with ξ. It follows from (8) that negative polarization charge density is induced in the volume of the material. Here

again, the electric field associated with positive charge on the lower electrode is in part terminated on the polarization charge density induced in the volume. As a result, the dielectric tends to make the electric field weaken with increasing ξ.

The next two sections give the opportunity to solve for the fields in simple configurations and then see that the results are consistent with the physical picture that has been found here.

6.6 PIECE-WISE UNIFORM ELECTRICALLY LINEAR DIELECTRICS

In a region where the permittivity is uniform and where there is no unpaired charge, the electric potential obeys Laplace's equation.

$$\nabla^2 \Phi = 0 \tag{1}$$

This follows from (6.5.1) and (6.5.2).

Uniform Dielectrics.   If all of the region of interest is filled by a uniform dielectric, it is clear from the foregoing that all equations developed for fields in free space are now valid in the presence of the uniform dielectric. The only alteration is the replacement of the permittivity of free space ϵ_o by that of the uniform dielectric. In every problem from Chaps. 4 and 5 where Φ and $\mathbf{E}$ were determined in a region of free space bounded by equipotentials, that region could just as well be filled with a uniform dielectric, and for the same potentials the electric field intensity would be unaltered. However, the surface charge density σ_{su} on the boundaries would then be increased by the ratio ϵ/ϵ_o.

Illustration.   Capacitance of a Sphere

A sphere having radius R has a potential v relative to infinity. Formally, the potential, and hence the electric field, follow from (1).

$$\Phi = v\frac{R}{r} \Rightarrow \mathbf{E} = v\frac{R}{r^2} \tag{2}$$

Evaluation of the capacitance, (6.5.6), then gives

$$C \equiv \frac{q}{v} = \frac{4\pi R^2}{v}\epsilon E_r|_{r=R} = 4\pi R\epsilon \tag{3}$$

The dielectric has increased the capacitance in the ratio of the dielectric constant of the material to the dielectric constant of free space.

The susceptibilities listed in Table 6.4.1 illustrate the increase in capacitance that would be observed if vacuum were replaced by one of the materials. In gases, atoms or molecules are so dilute that the increase in capacitance is usually negligible. With solids and liquids, the increase is of practical importance. Some, having
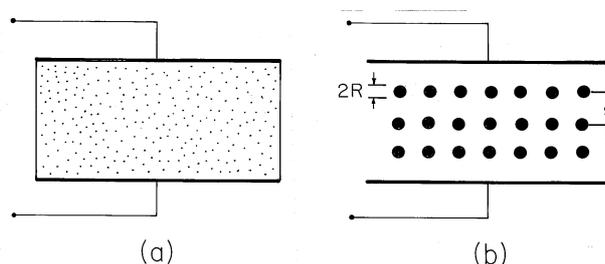
Fig. 6.6.1   (a) Plane parallel capacitor with region between electrodes
occupied by a dielectric. (b) Artificial dielectric composed of cubic array
of perfectly conducting spheres having radius R and spacing s.

molecules of large permanent dipole moments that are aligned by the field, increase
the capacitance dramatically.

The following example is intended to provide an appreciation for why the
polarized dielectric increases the capacitance.

Example 6.6.1.   An Artificial Dielectric

In the plane parallel capacitor of Fig. 6.6.1, the electric field intensity is $(v/d)\mathbf{i_z}$.
Thus, the unpaired charge density on the lower electrode is $D_z = \epsilon v/d$, and if the
electrode area is A, the capacitance is

$$C \equiv \frac{q}{v} = \frac{A}{v} D_z|_{z=0} = \frac{A\epsilon}{d} \tag{4}$$

Here we assume that d is much less than either of the electrode dimensions, so the
fringing fields can be ignored.

Now consider the plane parallel capacitor of Fig. 6.6.1b. The dielectric is com-
posed of "molecules" that are actually perfectly conducting spheres. These have
radius R and are in a cubic array with spacing $s >> R$. With the application of
a voltage, the spheres acquire the positive and negative surface charges on their
northern and southern poles required to make their surfaces equipotentials. In so
far as the field outside the spheres is concerned, the system is modeled as an array
of dipoles, each induced by the applied field.

If there are many of the spheres, the change in capacitance caused by inserting
the array between the plates can be determined by treating it as a continuum. This
we will do under the assumption that $s >> R$. In that case, the field in regions
removed several radii from the sphere centers is essentially uniform, and taken as
$E_z = v/d$. The resulting field in the vicinity of a sphere is then as determined in
Example 5.9.1. The dipole moment of each sphere follows from a comparison of the
potential for the perfectly conducting sphere in a uniform electric field, (5.9.7), with
that of a dipole, (4.4.10).

$$p = 4\pi\epsilon_o R^3 E_a \tag{5}$$

The polarization density is the moment/dipole multiplied by the number of
dipoles per unit volume, the number density N.

$$P_z = \epsilon_o(4\pi R^3 N)E_a \tag{6}$$

For the cubic array, a unit volume contains $1/s^3$ spheres, and so
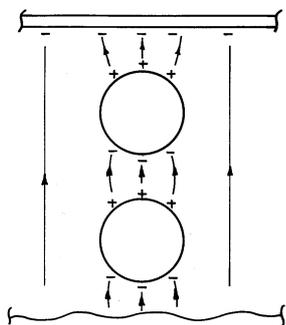
$$N = \frac{1}{s^3} \tag{7}$$

Fig. 6.6.2   From the microscopic point of view, the increase in capacitance results because the dipoles adjacent to the electrode induce image charges on the electrode in addition to those from the unpaired charges on the opposite electrode.

From (6) and (7) it follows that

$$\mathbf{P} = \epsilon_o \Big[4\pi \Big(\frac{R}{s}\Big)^3\Big]\mathbf{E} \tag{8}$$

Thus, the polarization density is a linear function of $\mathbf{E}$. The susceptibility follows from a comparison of (8) with (6.4.2) and, in turn, the permittivity is given by (6.4.4).

$$\chi_e = 4\pi \Big(\frac{R}{s}\Big)^3 \Rightarrow \epsilon = \Big[1 + 4\pi \Big(\frac{R}{s}\Big)^3\Big]\epsilon_o \tag{9}$$

Of course, this expression is accurate only if the interaction between spheres is negligible.

As the array of spheres is inserted between the electrodes, surface charges are induced, as shown in Fig. 6.6.2. Within the array, each cap of positive surface charge on the north pole of a sphere is compensated by an opposite charge on the south pole of a neighboring sphere. Thus, on a scale large compared to the spacing s, there is no charge density in the volume of the array. Nevertheless, the average field at the electrode is larger than the applied field $\mathbf{E}_a$. This is caused by surface charges on the last layers of spheres which have their images in unpaired charges on the electrodes. For a given applied voltage, the field between the top and bottom layers of spheres and the adjacent electrodes is increased, with an attendant increase in observed capacitance.

Demonstration 6.6.1.   Artificial Dielectric

In Fig. 6.6.3, the artificial dielectric is composed of an array of ping-pong balls with conducting coatings. The parallel plate capacitor is in one leg of a bridge, as shown in the circuit pictured in Fig. 6.6.4. The resistors shunt the input terminals of balanced amplifiers so that the oscilloscope displays v_o. With the array removed, capacitor C_2 is adjusted to null the output voltage v_o. The output voltage resulting from the the insertion of the array is a measure of the change in capacitance. To simplify the interpretation of this voltage, the resistances R_s are made small compared to the impedance of the parallel plate capacitor. Thus, almost all of the applied voltage V appears across the lower legs of the bridge. With the introduction of the array, the change in current through the parallel plate capacitor is
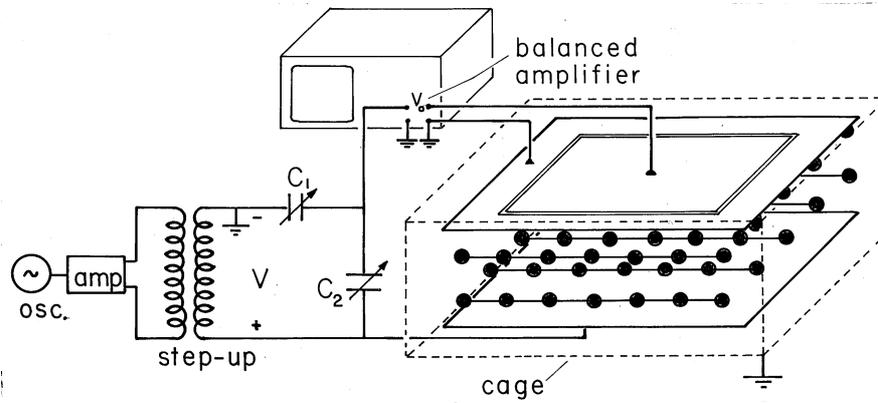
Fig. 6.6.3 Demonstration in which change in capacitance is used to measure the equivalent dielectric constant of an artificial dielectric.
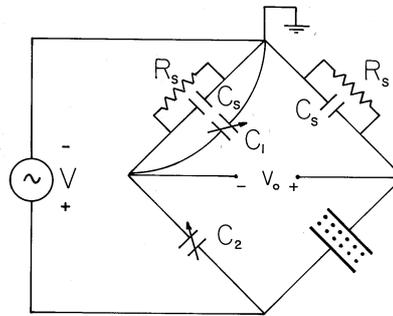

Fig. 6.6.4 Balanced amplifiers of oscilloscope, balancing capacitors, and demonstration capacitor shown in Fig. 6.6.4 comprise the elements in the bridge circuit. The driving voltage comes from the transformer, while v_o is the oscilloscope voltage.

$$|\Delta i| = \omega(\Delta C)|V| \tag{10}$$

Thus, there is a change of current through the resistance in the right leg and hence a change of voltage across that resistance given by

$$v_o = R_s\omega(\Delta C)V \tag{11}$$

Because the current through the left leg has remained the same, this change in voltage is the measured output voltage.

Typical experimental values are $R = 1.87$ cm, $s = 8$ cm, $A = (0.40)^2$ m^2, $d = 0.15$ m, $\omega = 2\pi$ (250 Hz), $R_s = 100$ kΩ and $V = 566$ v peak with a measured voltage of $v_o = 0.15$ V peak. From (4), (9), and (11), the output voltage is predicted to be 0.135 V peak.

Piece-Wise Uniform Dielectrics.   So far we have only considered systems filled with uniform dielectrics, as in Fig. 6.5.1a. We turn now to the description of fields in piece-wise uniform dielectrics, as exemplified by Fig. 6.5.1b.
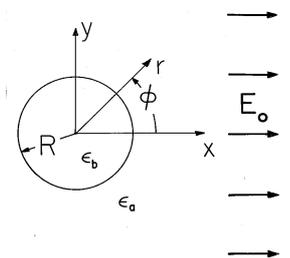
Fig. 6.6.5   Insulating rod having uniform permittivity ϵ_b surrounded
by material of uniform permittivity ϵ_a. Uniform electric field is imposed
by electrodes that are at "infinity."

In each of the regions of constant permittivity, the field distribution is de-
scribed by Laplace's equation, (1). The field problem is attacked by solving this
equation in each of the regions and then using the jump conditions to match these
solutions at the surfaces of discontinuity between the dielectrics. The following ex-
ample has a relatively simple solution that helps form further insights.

Example 6.6.2.   Dielectric Rod in Uniform Transverse Field

A uniform electric field $E_o\mathbf{i_x}$, perhaps produced by means of a parallel plate ca-
pacitor, exists in a dielectric having permittivity ϵ_a. With its axis perpendicular to
this field, a circular cylindrical dielectric rod having permittivity ϵ_b and radius R is
introduced, as shown in Fig. 6.6.5. With the understanding that the electrodes are
sufficiently far from the rod so that the field at "infinity" is essentially uniform, our
objective is to determine and then interpret the electric field inside and outside the
rod.

The shape of the circular cylindrical boundary suggests that we use polar
coordinates. In these coordinates, $x = r\cos\phi$, and so the potential far from the
cylinder is

$$\Phi(r \to \infty) \to -E_o r \cos\phi \tag{12}$$

Because this potential varies like the cosine of the angle, it is reasonable to attempt
satisfying the jump conditions with solutions of Laplace's equation having the same
ϕ dependence. Thus, outside the cylinder, the potential is assumed to take the form

$$\Phi^a = -E_o r \cos\phi + A\frac{R}{r}\cos\phi \tag{13}$$

Here the dipole field is multiplied by an adjustable coefficient A, but the uniform
field has a magnitude set to match the potential at large r, (12).

Inside the cylinder, the solution with a $1/r$ dependence cannot be accepted
because it becomes singular at the origin. Thus, the only solution having the cosine
dependence on ϕ is a uniform field, with the potential

$$\Phi_b = B\frac{r}{R}\cos\phi \tag{14}$$

Can the coefficients A and B be adjusted to satisfy the two jump conditions implied
by the laws of Gauss and Faraday, (6.5.3) and (6.5.4), at $r = R$?

$$\epsilon_a E_r^a - \epsilon_b E_r^b = 0 \tag{15}$$

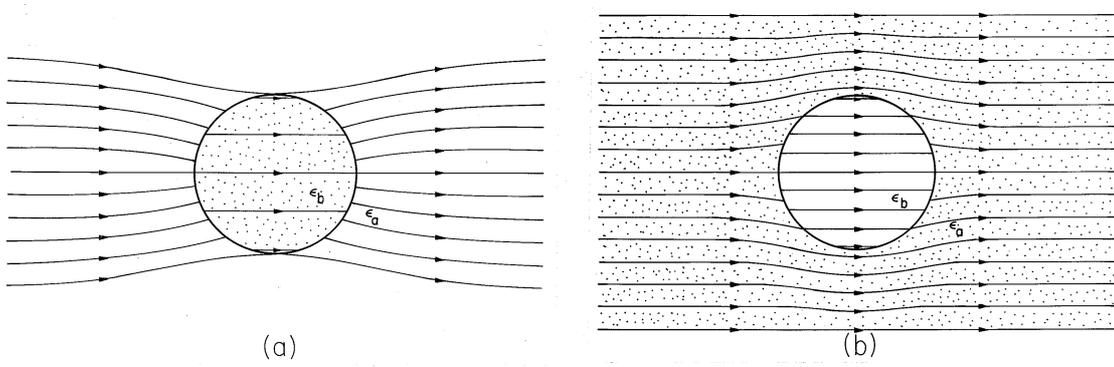(a)   (b)

Fig. 6.6.6   Electric field intensity in and around dielectric rod of Fig.
6.6.5 for (a) $\epsilon_b > \epsilon_a$ and (b) $\epsilon_b \leq \epsilon_a$.

$$\Phi^a - \Phi^b = 0 \qquad (16)$$

Substitution of (13) and (14) into these conditions shows that the answer is yes.
Continuity of potential, (16), requires that

$$(-E_oR + A)\cos\phi = B\cos\phi \qquad (17)$$

while continuity of normal **D**, (15), is satisfied if

$$\left(-\epsilon_a E_o - \epsilon_a \frac{A}{R}\right)\cos\phi = \frac{\epsilon_b B}{R}\cos\phi \qquad (18)$$

Note that these conditions contain the $\cos\phi$ dependence on both sides, and so can
be satisfied at each angle ϕ. This confirms the correctness of the originally assumed
ϕ dependence of our solutions. Simultaneous solution of (17) and (18) for A and B
gives

$$A = \frac{\epsilon_b - \epsilon_a}{\epsilon_b + \epsilon_a}E_oR \qquad (19)$$

$$B = \frac{-2\epsilon_a}{\epsilon_b + \epsilon_a}E_oR \qquad (20)$$

Introducing these values of the coefficients into the potentials, (13) and (14), gives

$$\Phi_a = -RE_o\cos\phi\left[\left(\frac{r}{R}\right) - \left(\frac{R}{r}\right)\frac{(\epsilon_b - \epsilon_a)}{(\epsilon_b + \epsilon_a)}\right] \qquad (21)$$

$$\Phi_b = \frac{-2\epsilon_a}{\epsilon_b + \epsilon_a}E_or\cos\phi \qquad (22)$$

The electric field is obtained as the gradient of this potential.

$$\mathbf{E}^a = E_o\left\{\mathbf{i_r}\cos\phi\left[1 + \left(\frac{R}{r}\right)^2\frac{(\epsilon_b - \epsilon_a)}{(\epsilon_b + \epsilon_a)}\right] - \mathbf{i}_\phi\sin\phi\left[1 - \left(\frac{R}{r}\right)^2\frac{(\epsilon_b - \epsilon_a)}{\epsilon_b + \epsilon_a}\right]\right\} \qquad (23)$$

$$\mathbf{E}^b = \frac{2\epsilon_a}{\epsilon_b + \epsilon_a}E_o(\mathbf{i_r}\cos\phi - \mathbf{i}_\phi\sin\phi) \qquad (24)$$
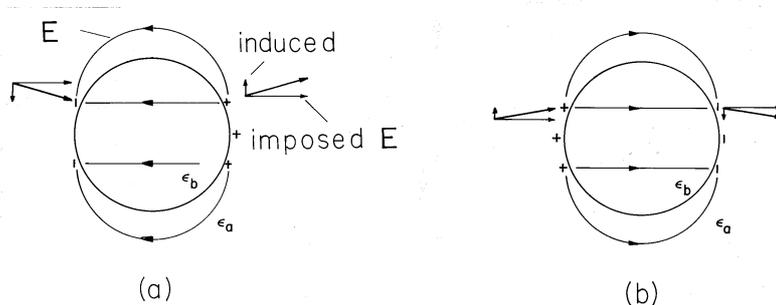
Fig. 6.6.7 Surface polarization charge density responsible for distortion of fields as shown in Fig. 6.6.6. (a) $\epsilon_b > \epsilon_a$, (b) $\epsilon_a > \epsilon_b$.

The electric field intensity given by these expressions is shown in Fig. 6.6.6. If the cylinder has the higher dielectric constant, as would be the case for a dielectric rod in air, the lines of electric field intensity tend to concentrate in the rod. In the opposite case– for example, representing a cylindrical void in a dielectric– the field lines tend to skirt the cylinder.

With an understanding of the relationship between the electric field intensity and the induced polarization charge comes the ability to see in advance how dielectrics distort the electric field. The circular cylindrical dielectric rod introduced into a uniform tranverse electric field in Example 6.6.2 serves as an illustration. Without carrying out the detailed analysis which led to (23) and (24), could we see in advance that the electric field has the distribution illustrated in Fig. 6.6.6?

The induced polarization charge provides the sources for the field induced by polarized material. For piece-wise uniform dielectrics, this is a polarization surface charge, given by (6.5.11).

$$\sigma_{sp} = \mathbf{n} \cdot \epsilon_o \mathbf{E}^a \big(1 - \frac{\epsilon_a}{\epsilon_b} \big) \tag{25}$$

The electric field intensity in the cylindrical rod example is generally directed to the right. It follows from (25) that the distribution of surface polarization charge at the cylindrical interface is as illustrated in Fig. 6.6.7. With the rod having the higher permittivity, Fig. 6.6.7a, the induced positive polarization surface charge density is at the right and the negative surface charge is at the left. These charges give rise to fields that generally originate at the positive charge and terminate at the negative. Thus, it is clear without any analysis that if $\epsilon_b > \epsilon_a$, the induced field inside tends to cancel the imposed field. In this case, the interior field is decreased or "depolarized." In the exterior region, vector addition of the induced field to the right-directed imposed field shows that incoming field lines at the left must be deflected inward, while outgoing ones at the right are deflected outward.

These same ideas, applied to the case where $\epsilon_a > \epsilon_b$, show that the interior field is increased while the exterior one tends to be ducted around the cylinder.

The circular cylinder is one of a series of examples having exact solutions. These give the opportunity to highlight the physical phenomena without encumbering mathematics. If it is actually necessary to account for detailed geometry,
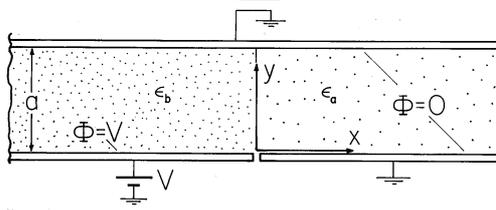
Fig. 6.6.8 Grounded upper electrode and lower electrode extending from $x = 0$ to $x \to \infty$ form plane parallel capacitor with fringing field that extends into the region $0 < x$ between grounded electrodes.

then some of the approaches introduced in Chaps. 4 and 5 can be used. The following example illustrates the use of the orthogonal modes approach introduced in Sec. 5.5.

Example 6.6.3.   Fringing Field of Dielectric Filled Parallel
                                  Plate Capacitor

Fields are to be determined in the planar region between a grounded conductor in the plane $y = a$ and a pair of conductors in the plane $y = 0$, shown in Fig. 6.6.8. To the right of $x = 0$ in the $y = 0$ plane is a second grounded conductor. To the left of $x = 0$ in this same plane is an electrode at the potential V. The regions to the right and left of the plane $x = 0$ are, respectively, filled with uniform dielectrics having permittivities ϵ_a and ϵ_b. Under the assumption that the system extends to infinity in the $\pm x$ and $\pm z$ directions, we now determine the fringing fields in the vicinity of the interface between dielectrics.

Our approach is to write solutions to Laplace's equation in the respective regions that satisfy the boundary conditions in the planes $y = 0$ and $y = a$ and as $x \to \pm\infty$. These are then matched up by the jump conditions at the interface between dielectrics.

Consider first the region to the right, where $\Phi = 0$ in the planes $y = 0$ and $y = a$ and goes to zero as $x \to \infty$. From Table 5.4.1, we select the infinite set of solutions

$$\Phi^a = \sum_{n=1}^{\infty} A_n e^{-\frac{n\pi}{a}x} \sin \frac{n\pi}{a} y \tag{26}$$

Here we have set $k = n\pi/a$ so that the sine functions are zero at each of the boundaries.

In the region to the left, the field is uniform in the limit $x \to -\infty$. This suggests writing the solution as the sum of a "particular" part meeting the "inhomogeneous part" of the boundary condition and a homogeneous part that is zero on each of the boundaries.

$$\Phi^b = -V\left(\frac{y}{a} - 1\right) + \sum_{n=1}^{\infty} B_n e^{\frac{n\pi}{a}x} \sin \frac{n\pi}{a} y \tag{27}$$

The coefficients A_n and B_n must now be adjusted so that the jump conditions are met at the interface between the dielectrics, where $x = 0$. First, consider the jump condition on the potential, (6.5.4). Evaluated at $x = 0$, (26) and (27) must give the same potential regardless of y.

$$\Phi^a\big|_{x=0} = \Phi^b\big|_{x=0} \Rightarrow \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} y = -V\left(\frac{y}{a} - 1\right) + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{a} y \tag{28}$$

To satisfy this relation at each value of y, expand the linear potential distribution on the right in a series of the same form as the other two terms.

$$-V\left(\frac{y}{a} - 1\right) = \sum_{n=1}^{\infty} V_n \sin\frac{n\pi}{a}y \tag{29}$$

Multiplication of both sides by $\sin(m\pi y/a)$ and integration from $y = 0$ to $y = a$ gives only one term on the right and an integral that can be carried out on the left. Hence, we can solve for the coefficients V_n in (29).

$$\int_0^a -V\left(\frac{y}{a} - 1\right)\sin\frac{m\pi}{a}y\,dy = \frac{aV_m}{2} \Rightarrow V_n = \frac{2V}{n\pi} \tag{30}$$

Thus, the series provided by (29) and (30) can be substituted into (28) to obtain an expression with each term a sum over the same type of series.

$$\sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{a}y = \sum_{n=1}^{\infty} \frac{2V}{n\pi}\sin\frac{n\pi}{a}y + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{a}y \tag{31}$$

This expression is satisfied if the coefficients of the like terms are equal. Thus, we have

$$A_n = \frac{2V}{n\pi} + B_n \tag{32}$$

To make the normal component of $\mathbf{D}$ continuous at the interface,

$$-\epsilon_a \frac{\partial \Phi^a}{\partial x}\Big|_{x=0} = -\epsilon_b \frac{\partial \Phi^b}{\partial x}\Big|_{x=0} \Rightarrow \sum_{n=1}^{\infty} \epsilon_a \frac{n\pi}{a} A_n \sin\frac{n\pi}{a}y$$

$$= -\sum_{n=1}^{\infty} \epsilon_b \frac{n\pi}{a} B_n \sin\frac{n\pi}{a}y \tag{33}$$

and a second relation between the coefficients results.

$$\epsilon_a A_n = -\epsilon_b B_n \tag{34}$$

The coefficients A_n and B_n are now determined by simultaneously solving (32) and (34). These are substituted into the original expressions for the potential, (26) and (27), to give the desired potential distribution.

$$\Phi^a = \sum_{n=1}^{\infty} \frac{2V}{n\pi\left(1 + \frac{\epsilon_a}{\epsilon_b}\right)} e^{-\frac{n\pi}{a}x} \sin\frac{n\pi}{a}y \tag{35}$$

$$\Phi^b = -V\left(\frac{y}{a} - 1\right) - \sum_{n=1}^{\infty} \frac{2}{n\pi}\frac{\epsilon_a}{\epsilon_b}\frac{V}{\left(1 + \frac{\epsilon_a}{\epsilon_b}\right)} e^{\frac{n\pi}{a}x} \sin\frac{n\pi}{a}y \tag{36}$$

These potential distributions, and sketches of the associated fields, are illustrated in Fig. 6.6.9. Shown first is the uniform dielectric. Laplace's equation prevails throughout, even at the "interface." Far to the left, we know that the potential is
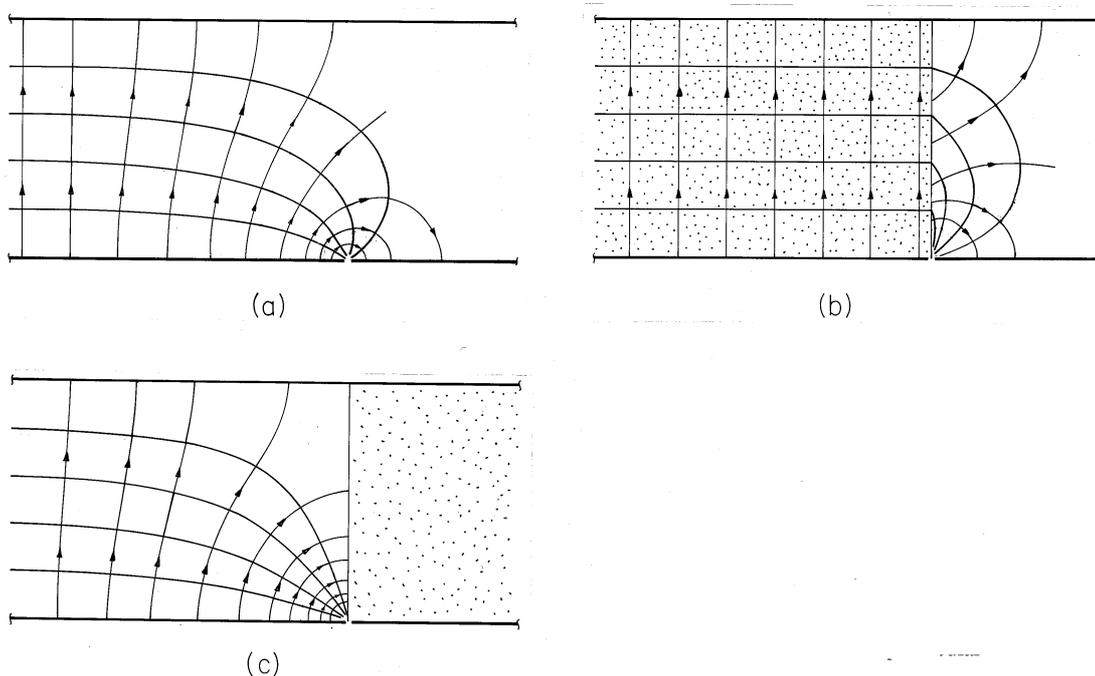
(a)

(b)

(c)

Fig. 6.6.9   Equipotentials and field lines for configuration of Fig. 6.6.8.
(a) Fringing for uniform dielectric. (b) With high permittivity material
between capacitor plates, field inside tends to become tangential to the
interface and uniform throughout the region to the left. (c) With high
permittivity material outside the region between the capacitor plates,
the field inside tends to be perpendicular to the interface.

linear in y, and hence represented by the equally spaced parallel straight lines. These
lines must end at other points on the bounding surface having the same potential.
The only place where this is possible is in the singular region at the origin where
the potential makes an abrupt change from V to 0. These observations provide a
starting point in sketching the field lines.

Shown next is the field distribution in the limit where the permittivity between
the capacitor plates (to the left) is very large compared to that outside. As is clear
by taking the limit $\epsilon_a/\epsilon_b \rightarrow 0$ in (36), the field inside the capacitor tends to be
uniform right up to the edge of the capacitor. The dielectric effectively ducts the
electric field. As far as the field inside the capacitor is concerned, there tends to be
no normal component of **E**.

In the opposite extreme, where the region to the right has a high permittivity
compared to that between the capacitor plates, the electric field inside the capaci-
tor tends to approach the interface normally. As far as the potential to the left is
concerned, the interface is an equipotential.

In Chap. 9, we find that magnetization and polarization phenomena are analo-
gous. There we delve further into approximations on magnetic field distributions in
the presence of magnetizable materials that can just as well be used to understand
systems of piece-wise uniform dielectrics.

6.7 SMOOTHLY INHOMOGENEOUS ELECTRICALLY LINEAR DIELECTRICS

The potential distribution in a dielectric that is free of unpaired charge and which has a space-varying permittivity is governed by

$$\nabla \cdot \epsilon \nabla \Phi = 0 \tag{1}$$

This is (6.5.1) combined with (6.5.2) and with $\rho_u = 0$. The contribution of the spatially varying permittivity is emphasized by using the vector identity for the divergence of a scalar (ϵ) times a vector ($\nabla \Phi$).

$$\nabla^2 \Phi + \nabla \Phi \cdot \frac{\nabla \epsilon}{\epsilon} = 0 \tag{2}$$

With a spatially varying permittivity, polarization charge is induced in proportion to the component of **E** that is in the direction of the gradient in ϵ. Thus, in general, the potential is not a solution to Laplace's equation.

Equation (2) gives a different perspective to the approach taken in dealing with piece-wise uniform systems. In Sec. 6.6, the polarization charge density represented by the $\nabla \epsilon$ term in (2) is confined to interfaces and accounted for by jump conditions. Thus, the section was a variation on the theme of Laplace's equation. The theme of this section broadens the developments of Sec. 6.6.

It is the objective in this section to demonstrate how familiar methods are adapted to dealing with unfamiliar laws. In general, (2) has spatially varying coefficients. Thus, even though it is linear, we are not guaranteed simple closed-form solutions. However, if the spatial dependence of ϵ is exponential, the equation does have constant coefficients and simple solutions. Our example exploits this fact.

Example 6.7.1.   Fields in an Exponentially Varying Dielectric

A dielectric has a permittivity that varies exponentially in the y direction, as illustrated in Fig. 6.7.1a.

$$\epsilon = \epsilon(y) = \epsilon_p e^{-\beta y} \tag{3}$$

Here ϵ_p and β are given constants.

In this example, the dielectric fills the rectangular region shown in Fig. 6.7.1b. This configuration is familiar from Sec. 5.5. The fields are two dimensional, $\Phi = 0$ at $x = 0$ and $x = a$ and $y = 0$. The potential on the "last" surface, where $y = b$, is $v(t)$.

It follows from (3) that

$$\nabla \Phi \cdot \frac{\nabla \epsilon}{\epsilon} = -\beta \frac{\partial \Phi}{\partial y} \tag{4}$$

and (2) becomes

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} - \beta \frac{\partial \Phi}{\partial y} = 0 \tag{5}$$
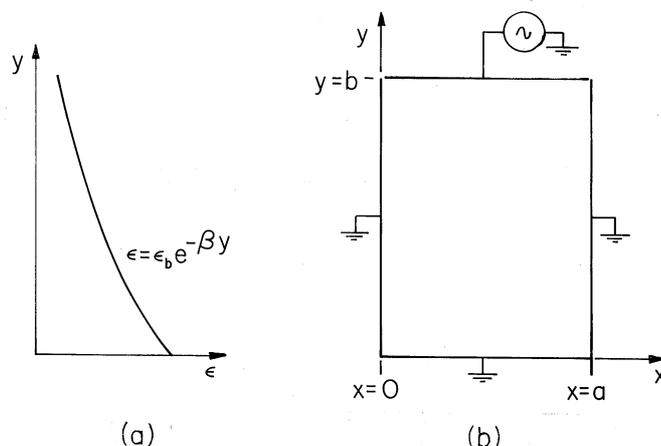
Fig. 6.7.1   (a) Smooth permittivity distribution of material enclosed by (b) zero potential boundaries at $x = 0$, $x = a$, and $y = 0$, and electrode at potential v at $y = b$.

The dielectric fills a region having boundaries that are natural in Cartesian coordinates. Thus, we look for product solutions having the form $\Phi = X(x)Y(y)$. Substitution into (5) gives

$$\frac{1}{Y}\left(\frac{d^2Y}{dy^2} - \frac{1}{\beta}\frac{dY}{dy}\right) + \frac{1}{X}\frac{d^2X}{dx^2} = 0 \tag{6}$$

The first term, a function of y alone, must sum with the function of x alone to give zero. Thus, the first is set equal to the separation coefficient k^2 and the second equal to $-k^2$.

$$\frac{d^2X}{dx^2} + k^2X = 0 \tag{7}$$

$$\frac{d^2Y}{dy^2} - \beta\frac{dY}{dy} - k^2Y = 0 \tag{8}$$

This assignment of sign for the separation coefficient is motivated by the requirement that $\Phi = 0$ at two locations. This results in periodic solutions for (7).

$$X = \begin{cases} \sin kx \\ \cos kx \end{cases} \tag{9}$$

Because it also has constant coefficients, the solutions to (8) are exponentials. Substitution of $\exp(py)$ shows that

$$p = \frac{\beta}{2} \pm \sqrt{\left(\frac{\beta}{2}\right)^2 + k^2} \tag{10}$$

and it follows that solutions are linear combinations of two exponentials.

$$Y = e^{\frac{\beta}{2}y}\left[\begin{array}{c} \cosh\sqrt{\left(\frac{\beta}{2}\right)^2 + k^2}\,y \\ \sinh\sqrt{\left(\frac{\beta}{2}\right)^2 + k^2}\,y \end{array}\right] \tag{11}$$

For the specific problem at hand, we look for the products of these sets of solutions that satisfy the homogeneous boundary conditions. Those at $x = 0$ and $x = a$ are met by making $k = n\pi/a$, with n an integer. The origin of the y axis was made to coincide with the third zero potential boundary so that the hyperbolic sine function could be used. Thus, we arrive at an infinite series of solutions, each satisfying the homogeneous boundary conditions.

$$\Phi = \sum_{n=1}^{\infty} A_n e^{\frac{\beta}{2} y} \sinh \sqrt{\left(\frac{\beta}{2}\right)^2 + \left(\frac{n\pi}{a}\right)^2}\, y\, \sin\left(\frac{n\pi}{a} x\right) \tag{12}$$

The assignment of the coefficients so that the potential constraint at $y = b$ is met follows the procedure familiar from Sec. 5.5.

$$\Phi = \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{4v}{n\pi} e^{\frac{\beta}{2}(y-b)} \frac{\sinh \sqrt{\left(\frac{\beta}{2}\right)^2 + \left(\frac{n\pi}{a}\right)^2}\, y}{\sinh \sqrt{\left(\frac{\beta}{2}\right)^2 + \left(\frac{n\pi}{a}\right)^2}\, b} \sin\left(\frac{n\pi}{a} x\right) \tag{13}$$

For interpretation of (13), suppose that β is positive so that ϵ decreases with y, as illustrated in Fig. 6.7.1a. Without the analysis, we know that the lines of **D** originate on the electrode at $y = b$ and terminate on the zero potential walls. This means that E lines either terminate on the grounded walls or on polarization charges induced in the volume. If $v > 0$, we can see from (6.5.9) that because $\mathbf{E} \cdot \nabla \epsilon$ is positive, the induced polarization charge density must be negative. Thus, some of the E lines terminate on this negative charge density and it comes as no surprise that we have found a potential that decays away from the excitation electrode at $y = b$ at a rate that is faster than if the potential were governed by Laplace's equation. The electric field is effectively shielded out of the lower region of higher permittivity by the induced polarization charge.

One approach to determining fields in spatially varying dielectrics is suggested in Fig. 6.7.2. The smooth distribution has been approximated by "stair steps." Physically, the equivalent system consists of uniform layers. Thus, the fields revert to the solutions of Laplace's equation matched to each other at the interfaces by the jump conditions. According to (6.5.11), E lines originating at $y = b$ and passing downward through these interfaces will induce positive surface polarization charge. Thus, replacing the smoothly varying dielectric with the layers of uniform dielectric is equivalent to representing the volume polarization charge density by a distribution of surface polarization charges.

6.8 SUMMARY

Table 6.8.1 is useful both as an outline of this chapter and as a reference. Gauss' theorem is the basis for deriving the surface relations in the right-hand column from the respective volume relations in the left-hand column. By remembering the volume relations, one is able to recall the surface relations.

Our first task, in Sec. 6.1, was to introduce the polarization density as a way of representing the polarization charge density. The first volume and surface
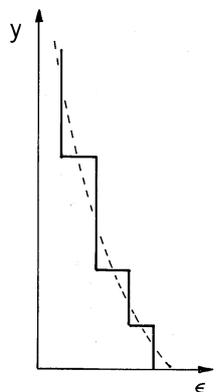
Fig. 6.7.2 Stair-step distribution of permittivity approximating smooth distribution.

relations resulted. These are deceptively similar in appearance to Gauss' law and the associated jump condition. However, they are not electric field laws. Rather, they simply relate the volume and surface sources representing the material to the polarization density.

Next we considered the fields due to permanently polarized materials. The polarization density was given. For this purpose, Gauss' law and the associated jump condition were conveniently written as (6.2.2) and (6.2.3), respectively.

With the polarization induced by the field itself, it was convenient to introduce the displacement flux density $\mathbf{D}$ and write Gauss' law and the jump condition as (6.2.15) and (6.2.16). In particular, for linear polarization, the equivalent constitutive laws of (6.4.2) and (6.4.3) were introduced.

The theme of this chapter has been the determination of EQS fields when the polarization charge density makes a contribution. In cases where the polarization density is given, this is easy to keep in mind, because the first step in formulating a problem is to evaluate ρ_p from the given $\mathbf{P}$. However, when ρ_p is induced, variables such as $\mathbf{D}$ are used and we must be reminded that when all is said and done, ρ_p (or its surface counterpart, σ_{sp}) is still responsible for the effect of the material on the field. The expressions for ρ_p and σ_{sp} given by the last two relations in the table are useful not only for interpreting the distributions of fields after they have been found but for forming an impression of the fields in complex systems where it would not be worthwhile to find an analytic solution. Remember that these relations hold only in regions where there is no unpaired charge density.

In Chap. 9, we will find that most of this chapter is directly applicable to the description of magnetization. There we will continue to develop insights that will be equally applicable to the polarization phenomena of this chapter.

<table>
<tr><td colspan="4" align="center">**TABLE 6.8.1**

SUMMARY OF POLARIZATION RELATIONS AND LAWS</td></tr>
<tr><td colspan="4" align="center">Polarization Charge Density and Polarization Density</td></tr>
<tr><td align="center">$\rho_p = -\nabla \cdot \mathbf{P}$</td><td align="center">(6.1.6)</td><td align="center">$\sigma_{sp} = -\mathbf{n} \cdot (\mathbf{P}^a - \mathbf{P}^b)$</td><td align="center">6.1.7)</td></tr>
<tr><td colspan="4" align="center">Gauss' Law with Polarization</td></tr>
<tr><td align="center">$\nabla \cdot \epsilon_o \mathbf{E} = \rho_p + \rho_u$</td><td align="center">(6.2.1)</td><td align="center">$n \cdot \epsilon_o(\mathbf{E}^a - \mathbf{E}^b) = \sigma_{sp} + \sigma_{su}$</td><td align="center">(6.2.3)</td></tr>
<tr><td align="center">$\nabla \cdot \mathbf{D} = \rho_u$</td><td align="center">(6.2.15)</td><td align="center">$n \cdot (\mathbf{D}^a - \mathbf{D}^b) = \sigma_{su}$</td><td align="center">(6.2.16)</td></tr>
<tr><td align="center">where

$\mathbf{D} \equiv \epsilon_o \mathbf{E} + \mathbf{P}$</td><td align="center">(6.2.14)</td><td></td><td></td></tr>
<tr><td colspan="4" align="center">Electrically Linear Polarization</td></tr>
<tr><td align="center">Constitutive Law

$\mathbf{P} = \epsilon_o \chi_e \mathbf{E} = (\epsilon - \epsilon_o)\mathbf{E}$

$\mathbf{D} = \epsilon \mathbf{E}$</td><td align="center">(6.4.2)

(6.4.3)</td><td></td><td></td></tr>
<tr><td colspan="4" align="center">Source Distribution, $\rho_u = 0$</td></tr>
<tr><td align="center">$\rho_p = -\frac{\epsilon_o}{\epsilon} \mathbf{E} \cdot \nabla \epsilon$</td><td align="center">(6.5.9)</td><td align="center">$\sigma_{sp} = \mathbf{n} \cdot \epsilon_o \mathbf{E}^a \left(1 - \frac{\epsilon_a}{\epsilon_b}\right)$</td><td align="center">(6.5.11)</td></tr>
</table>

PROBLEMS

6.1 Polarization Density

6.1.1 The layer of polarized material shown in cross-section in Fig. P6.1.1, having thickness d and surfaces in the planes $y = d$ and $y = 0$, has the polarization density $\mathbf{P} = P_o \cos \beta x (\mathbf{i_x} + \mathbf{i_y})$.

(a) Determine the polarization charge density throughout the slab.

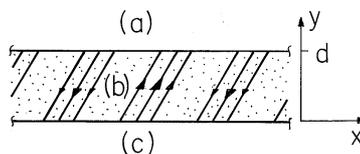(b) What is the surface polarization charge density on the layer surfaces?

Fig. P6.1.1

6.2 Laws and Continuity Conditions with Polarization

6.2.1 For the polarization density given in Prob. 6.1.1, with $P_o(t) = P_o \cos \omega t$:

  (a) Determine the polarization current density and polarization charge density.
  (b) Using $\mathbf{J}_p$ and ρ_p, show that the differential charge conservation law, (10), is indeed satisfied.

6.3 Permanent Polarization

6.3.1* A layer of permanently polarized material is sandwiched between plane parallel perfectly conducting electrodes in the planes $x = 0$ and $x = a$, respectively, having potentials $\Phi = 0$ and $\Phi = -V$. The system extends to infinity in the $\pm y$ and $\pm z$ directions.

  (a) Given that $\mathbf{P} = P_o \cos \beta x \mathbf{i_x}$, show that the potential between the electrodes is

$$\Phi = \frac{P_o}{\beta \epsilon_o} \left(\sin \beta x - \frac{x}{a} \sin \beta a \right) - \frac{Vx}{a} \tag{a}$$

  (b) Given that $\mathbf{P} = P_o \cos \beta y \mathbf{i_y}$, show that the potential between the electrodes is

$$\Phi = \frac{P_o}{\beta \epsilon_o} \sin \beta y \left[1 - \frac{\cosh \beta(x - a/2)}{\cosh(\beta a/2)} \right] - \frac{Vx}{a} \tag{b}$$

6.3.2 The cross-section of a configuration that extends to infinity in the $\pm z$ directions is shown in Fig. P6.3.2. What is the potential distribution inside the cylinder of rectangular cross-section?

6.3.3* A polarization density is given in the semi-infinite half-space $y < 0$ to be $\mathbf{P} = P_o \cos[(2\pi/\Lambda)x]\mathbf{i_y}$. There are no other field sources in the system and P_o and Λ are given constants.

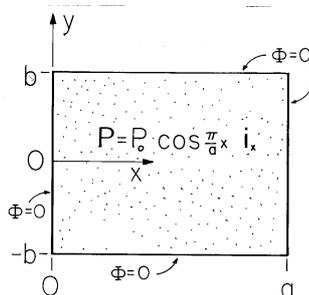 (a) Show that $\rho_p = 0$ and $\sigma_{sp} = P_o \cos(2\pi x/\Lambda)$.
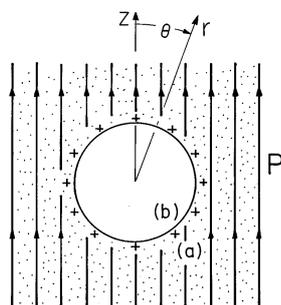
Fig. P6.3.2

Fig. P6.3.5

(b) Show that

$$\Phi = \frac{P_o \Lambda}{4\epsilon_o \pi} \cos(2\pi x/\Lambda) \exp(\mp 2\pi y/\Lambda); \qquad y \gtrless 0 \qquad (a)$$

6.3.4 A layer in the region $-a < y < 0$ has the polarization density $\mathbf{P} = P_o \mathbf{i_y} \sin \beta(x - x_o)$. In the planes $y = \pm a$, the potential is constrained to be $\Phi = V \cos \beta x$, where P_o, β and V are given constants. The region $0 < y < a$ is free space and the system extends to infinity in the $\pm x$ and $\pm z$ directions. Find the potential in regions (a) and (b) in the free space and polarized regions, respectively. (If you have already solved Prob. 5.6.12, you can solve this problem by inspection.)

6.3.5* Figure P6.3.5 shows a material having the uniform polarization density $\mathbf{P} = P_o \mathbf{i_z}$, with a spherical cavity having radius R. On the surface of the cavity is a uniform distribution of unpaired charge having density $\sigma_{su} = \sigma_o$. The interior of the cavity is free space, and P_o and σ_o are given constants. The potential far from the cavity is zero. Show that the electric potential is

$$\Phi = \begin{cases} -\frac{P_o}{3\epsilon_o} r \cos\theta + \frac{\sigma_o R}{\epsilon_o}; & r \leq R \\ -\frac{P_o R^3}{3\epsilon_o r^2} \cos\theta + \frac{\sigma_o R^2}{\epsilon_o r}; & r \geq R \end{cases} \qquad (a)$$

6.3.6 The cross-section of a groove (shaped like a half-cylinder having radius R) cut from a uniformly polarized material is shown in Fig. P6.3.6. The
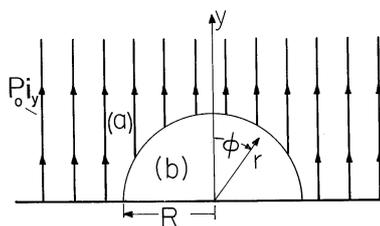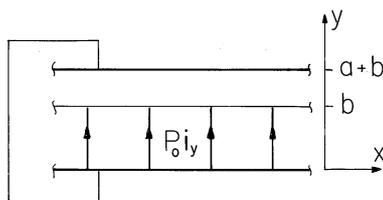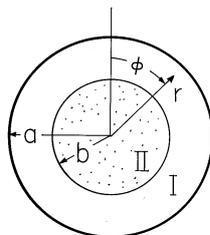
Fig. P6.3.6

Fig. P6.3.7

Fig. P6.3.8

material rests on a grounded perfectly conducting electrode at $y = 0$, and P_o is a given constant. Assume that the configuration extends to infinity in the y direction and find Φ in regions (a) and (b), respectively, outside and inside the groove.

6.3.7   The system shown in cross-section in Fig. P6.3.7 extends to infinity in the $\pm x$ and $\pm z$ directions. The electrodes at $y = 0$ and $y = a + b$ are shorted. Given P_o and the dimensions, what is $\mathbf{E}$ in regions (a) and (b)?

6.3.8* In the two-dimensional configuration shown in Fig. P6.3.8, a perfectly conducting circular cylindrical electrode at $r = a$ is grounded. It is coaxial with a rotor of radius b which supports the polarization density $\mathbf{P} = \nabla[P_o r \cos(\phi - \alpha)]$.

   (a) Show that the polarization charge density is zero inside the rotor.
   (b) Show that the potential functions Φ^I and Φ^{II} respectively in the regions outside and inside the rotor are

$$\Phi^I = \frac{P_o b^2}{2\epsilon_o} \left(\frac{1}{r} - \frac{r}{a^2} \right) \cos(\phi - \alpha) \qquad (a)$$
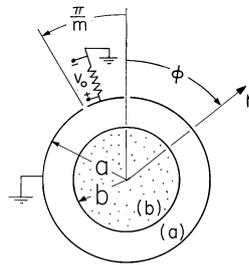
Fig. P6.3.9

$$\Phi^{II} = \frac{P_o(a^2 - b^2)}{2\epsilon_o a^2} r \cos(\phi - \alpha) \qquad\qquad (b)$$

    (c) Show that if $\alpha = \Omega t$, where Ω is an angular velocity, the field rotates in the ϕ direction with this angular velocity.

6.3.9   A circular cylindrical material having radius b has the polarization density $\mathbf{P} = \nabla[P_o(r^{m+1}/b^m)\cos m\phi]$, where m is a given positive integer. The region $b < r < a$, shown in Fig. P6.3.9, is free space.

    (a) Determine the volume and surface polarization charge densities for the circular cylinder.

    (b) Find the potential in regions (a) and (b).

    (c) Now the cylinder rotates with the constant angular velocity Ω. Argue that the resulting potential is obtained by replacing $\phi \rightarrow (\phi - \Omega t)$.

    (d) A section of the outer cylinder is electrically isolated and connected to ground through a resistance R. This resistance is low enough so that, as far as the potential in the gap is concerned, the potential of the segment can still be taken as zero. However, as the rotor rotates, the charge induced on the segment is time varying. As a result, there is a current through the resistor and hence an output signal v_o. Assume that the segment subtends an angle π/m and has length l in the z direction, and find v_o.

6.3.10* Plane parallel electrodes having zero potential extend to infinity in the $x - z$ planes at $y = 0$ and $y = d$.

    (a) In a first configuration, the region between the electrodes is free space, except for a segmented electrode in the plane $x = 0$ which constrains the potential there to be $V(y)$. Given $V(y)$, what is the potential distribution in the regions $0 < x$ and $x < 0$, regions (a) and (b), respectively?

    (b) Now the segmented electrode is removed and the region $x < 0$ is filled with a permanently polarized material having $\mathbf{P} = P_o\mathbf{i_x}$, where P_o is a given constant. What continuity conditions must the potential satisfy in the $x = 0$ plane?

(c) Show that the potential is given by

$$\Phi = \frac{dP_o}{\epsilon_o} \sum_{n=1}^{\infty} \frac{[1-(-1)^n]}{(n\pi)^2} \sin\frac{n\pi}{d} y \exp\left(\mp \frac{n\pi}{d}x\right); \qquad x \gtrless 0 \qquad (a)$$

(The method used here to represent Φ is used in Example 6.6.3.)

6.3.11 In Prob. 6.1.1, there is a perfect conductor in the plane $y = 0$ and the region $d < y$ is free space. What are the potentials in regions (a) and (b), the regions where $d < y$ and $0 < y < d$, respectively?

6.4 Polarization Constitutive Laws

6.4.1 Suppose that a solid or liquid has a mass density of $\rho = 10^3 kg/m^3$ and a molecular weight of $M_o = 18$ (typical of water). [The number of molecules per unit mass is Avogadro's number ($A_o = 6.023 \times 10^{26}$ molecules/kg-mole) divided by M_o.] This material has a permittivity $\epsilon = 2\epsilon_o$ and is subject to an electric field intensity $E = 10^7 v/m$ (approaching the highest field strength that can be sustained without breakdown on scales of a centimeters in liquids and solids). Assume that each molecule has a polarization $q\mathbf{d}$ where $q = e = 1.6 \times 10^{-19}$ C, the charge of an electron). What is $|d|$?

6.5 Fields in the Presence of Electrically Linear Dielectrics

6.5.1[*] The plane parallel electrode configurations of Fig. P6.5.1 have in common the fact that the linear dielectrics have dielectric "constants" that are functions of x, $\epsilon = \epsilon(x)$. The systems have depth c in the z direction.

(a) Show that regardless of the specific functional dependence on x, $\mathbf{E}$ is uniform and simply $\mathbf{i_y}v/d$.
(b) For the system of Fig. P6.5.1a, where the dielectric is composed of uniform regions having permittivities ϵ_a and ϵ_b, show that the capacitance is

$$C = \frac{c}{d}(\epsilon_b b + \epsilon_a a) \qquad (a)$$

(c) For the smoothly inhomogeneous capacitor of Fig. P6.5.1b, $\epsilon = \epsilon_o(1 + x/l)$. Show that

$$C = \frac{3\epsilon_o cl}{2d} \qquad (b)$$

6.5.2 In the configuration shown in Fig. P6.5.1b, what is the capacitance C if $\epsilon = \epsilon_a(1 + \alpha \cos\beta x)$, where $0 < \alpha < 1$ and β are given constants?
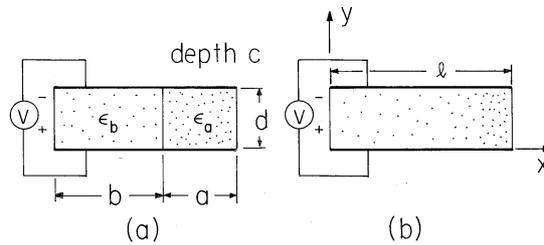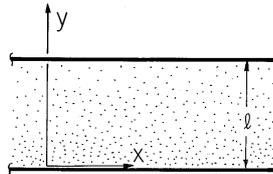
Fig. P6.5.1

Fig. P6.5.3

6.5.3* The region of Fig. P6.5.3 between plane parallel perfectly conducting electrodes in the planes $y = 0$ and $y = l$ is filled by a uniformly inhomogeneous dielectric having permittivity $\epsilon = \epsilon_o[1 + \chi_a(1 + y/l)]$. The electrode at $y = 0$ has potential v relative to that at $y = l$. The electrode separation l is much smaller than the dimensions of the system in the x and z directions, so the fields can be regarded as not depending on x or z.

    (a) Show that D_y is independent of y.
    (b) With the electrodes having area A, show that the capacitance is

$$C = \frac{\epsilon_o A}{l} \chi_a / ln\Big[\frac{1 + 2\chi_a}{1 + \chi_a}\Big] \qquad\qquad (a)$$

6.5.4 The dielectric in the system of Prob. 6.5.3 is replaced by one having permittivity $\epsilon = \epsilon_p \exp(-y/d)$, where ϵ_p is constant. What is the capacitance C?

6.5.5 In the two configurations shown in cross-section in Fig. P6.5.5, circular cylindrical conductors are used to make coaxial capacitors. In Fig. P6.5.5a, the linear dielectric has a wedge shape with interfaces with the free space region that are surfaces of constant ϕ. In Fig. P6.5.5b, the interface is at $r = R$.

    (a) Determine $\mathbf{E}(\mathbf{r})$ in regions (1) and (2) in each configuration, showing that simple fields satisfy all boundary conditions on the electrode surfaces and at the interfaces between dielectric and free space.
    (b) For lengths l in the z direction, what are the capacitances?

6.5.6* For the configuration of Fig. P6.5.5a, the wedge-shaped dielectric is replaced by one that fills the gap (over all ϕ as well as over the radius
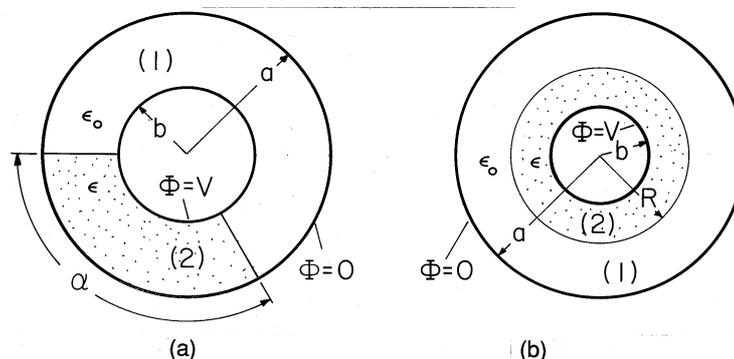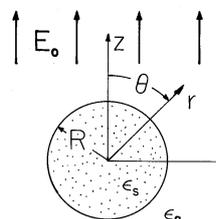
Fig. P6.5.5

Fig. P6.6.1

$b < r < a$) with material having the permittivity $\epsilon = \epsilon_a + \epsilon_b \cos^2 \phi$, where ϵ_a and ϵ_b are constants. Show that the capacitance is

$$C = (2\epsilon_a + \epsilon_b)\pi l / ln(a/b) \qquad (a)$$

6.6 Piece-Wise Uniform Electrically Linear Dielectrics

6.6.1* An insulating sphere having radius R and uniform permittivity ϵ_s is surrounded by free space, as shown in Fig. P6.6.1. It is immersed in an electric field $E_o(t)\mathbf{i_z}$ that, in the absence of the sphere, is uniform.

(a) Show that the potential is

$$\Phi = E_o(t)\begin{cases} -r\cos\theta + R^3 A\frac{\cos\theta}{r^2}; & R < r \\ Br\cos\theta; & r < R \end{cases} \qquad (a)$$

where $A = (\epsilon_s - \epsilon_o)/(\epsilon_s + 2\epsilon_o)$ and $B = -3\epsilon_o/(\epsilon_s + 2\epsilon_o)$.

(b) Show that, in the limit where $\epsilon_s \to \infty$, the electric field intensity tangential to the surface of the sphere goes to zero. Thus, the surface becomes an equipotential.

(c) Show that the same solution is obtained for the potential outside the sphere as in the limit $\epsilon_s \to \infty$ if this boundary condition is used at the outset.
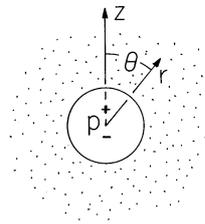
Fig. P6.6.2

6.6.2   An electric dipole having a z-directed moment p is situated at the origin, as shown in Fig. P6.6.2. Surrounding it is a spherical cavity of free space having radius a. Outside of the radius a is a linearly polarizable dielectric having permittivity ϵ.

   (a) Determine Φ and $\mathbf{E}$ in regions (a) and (b) outside and inside the cavity.
   (b) Show that in the limit where $\epsilon \rightarrow \infty$, the electric field intensity tangential to the interface of the dielectric goes to zero. That is, in this limit, the effect of the dielectric on the interior fields is the same as if the dielectric were a perfect conductor.
   (c) Show that the same interior potential is obtained as in the limit $\epsilon \rightarrow \infty$ if this boundary condition is used at the outset.

6.6.3*   In Example 6.6.1, an artificial dielectric is made from an array of perfectly conducting spheres. Here, an artificial dielectric is constructed using an array of rods, each having a circular cross-section with radius R. The rods run parallel to the capacitor plates and hence perpendicular to the imposed electric field intensity. The spacing between rod centers is s, and they are in a square array. Show that, for s large enough so that the fields induced by the rods do not interact, the equivalent electric susceptibility is $\chi_c = 2\pi(R/s)^2$.

6.6.4   Each of the conducting spheres in the artificial dielectric of Example 6.6.1 is replaced by the dielectric sphere of Prob. 6.6.1. Again, with the understanding that the spacing between spheres is large enough to justify ignoring their interaction, what is the equivalent susceptibility of the array?

6.6.5*   A point charge finds itself at a height h above an infinite half-space of dielectric material. The charge has magnitude q, the dielectric has a uniform permittivity ϵ, and there are no unpaired charges in the volume of the dielectric or on its surface. The Cartesian coordinates x and z are in the plane of the dielectric interface, while y is directed perpendicular to the interface and into the free space region. Thus, the charge is at $y = h$. The field in the free space region can be taken as the superposition of a particular solution due to the point charge and a homogeneous solution due to a charge q_b at $y = -h$ below the interface. The field in the dielectric can be taken as that of a charge q_a at $y = h$.

(a) Show that the potential is given by

$$\Phi = \frac{1}{4\pi\epsilon_o} \begin{cases} (q/r_+) - (q_b/r_-); & 0 < y \\ q_a/r_+; & y < 0 \end{cases}$$

where $r_\pm = \sqrt{x^2 + (y \mp h)^2 + z^2}$ and the magnitudes of the charges turn out to be

$$q_a = \frac{2q}{\left(\frac{\epsilon}{\epsilon_o} + 1\right)}; \qquad q_b = \frac{q\left(\frac{\epsilon}{\epsilon_o} - 1\right)}{\left(\frac{\epsilon}{\epsilon_o} + 1\right)} \qquad (b)$$

(b) Show that the charge is attracted to the dielectric with the force

$$f = q\frac{q_b}{16\pi\epsilon_o h^2} \qquad (c)$$

6.6.6 The half-space $y > 0$ is filled by a dielectric having uniform permittivity ϵ_a, while the remaining region $0 > y$ is filled by a dielectric having the uniform permittivity ϵ_b. Running parallel to the interface between these dielectrics along the line where $x = 0$ and $y = h$ is a uniform line charge of density λ. Determine the potentials in regions (a) and (b), respectively.

6.6.7* If the permittivities are nearly the same, so that $(1 - \epsilon_a/\epsilon_b) \equiv \kappa$ is small, the qualitative approach to determining the field distribution given in connection with Fig. 6.6.7 can be made quantitative. That is, if κ is small, the polarization charge induced by the imposed field can be determined to a good approximation and that charge, in turn, used to find the change in the applied field. Consider the following approximate approach to finding the fields in and around the dielectric cylinder of Example 6.6.2.

  (a) In the limit where κ is zero, the field is equal to the applied field, both inside and outside the cylinder. Write this field in polar coordinates.
  (b) Show that this field gives rise to $\sigma_{sp} = \epsilon_b E_o \kappa \cos\phi$ at the surface of the cylinder.
  (c) Find the field due to this induced polarization surface charge and add it to the imposed field to show that, with the first-order contribution of the induced polarization surface charge, the field is

$$\Phi = -RE_o \begin{cases} \left(\frac{r}{R} - \frac{\kappa R}{2r}\right)\cos\phi; & r > R \\ \frac{r}{R}\left(1 - \frac{\kappa}{2}\right)\cos\phi; & r < R \end{cases} \qquad (a)$$

  (d) Expand the exact fields given by (21) and (22) to first order in κ and show that they are in agreement with this result.

6.6.8 As an illustration of how identification of the induced polarization charge can be used in a qualitative determination of the fields, consider the fields
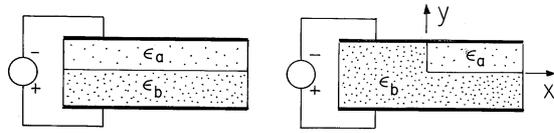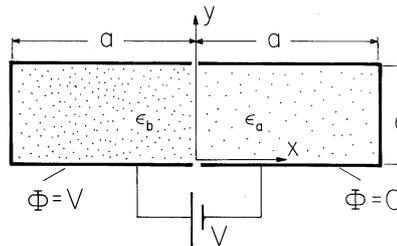
Fig. P6.6.8

Fig. P6.6.9

between the plane parallel electrodes of Fig. P6.6.8. In Fig. P6.6.8a, there are two layers of dielectric.

(a) In the limit where $\kappa = (1 - \epsilon_a/\epsilon_b)$ is zero, what is the imposed **E**?

(b) What is the σ_{sp} induced by this field at the interface between the dielectrics.

(c) For $\epsilon_a > \epsilon_b$, sketch the field lines in the two regions. (You should be able to see, from the superposition of the fields induced by this σ_{sp} and that imposed, which of the fields is the greater.)

(d) Now consider the more complicated geometry of Fig. 6.6.8b and carry out the same steps. Based on your deductions, draw a sketch of σ_{sp} and **E** for the case where $\epsilon_b > \epsilon_a$.

6.6.9   The configuration of perfectly conducting electrodes and perfectly insulating dielectrics shown in Fig. P6.6.9 is similar to that shown in Fig. 6.6.8 except that at the left and right, the electrodes are "shorted" together and the top electrode is also divided at the middle. Thus, the $\supset$ shaped electrode is grounded while the $\subset$ shaped one is at potential V.

(a) Determine Φ in regions (a) and (b).

(b) With the permittivities equal, sketch Φ and **E**. (Use physical reasoning rather than the mathematical result.)

(c) Assuming that the permittivities are nearly equal, use the result of (b) to deduce σ_{sp} on the interface between dielectrics in the case where ϵ_a/ϵ_b is somewhat greater than and then somewhat less than 1. Sketch **E** deduced as the sum of the fields induced by these surface charges and the imposed field.

(d) With ϵ_a much greater that ϵ_b, draw a sketch of Φ and **E** in region (b).

(e) With ϵ_a much less than ϵ_b, sketch Φ and **E** in both regions.

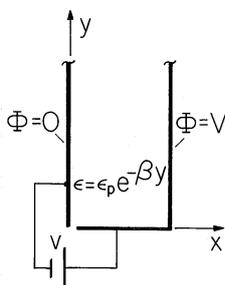**6.7 Smoothly Inhomogeneous Electrically Linear Dielectrics**

Fig. P6.7.1

6.7.1* For the two-dimensional system shown in Fig. P6.7.1, show that the potential in the smoothly inhomogeneous dielectric is

$$\Phi = \frac{Vx}{a} + \sum_{n=1}^{\infty} \left(\frac{2V}{n\pi}\right) e^{\beta y/2}$$
$$\exp\left[-\sqrt{(\beta/2)^2 + (n\pi/a)^2}\, y\right] \sin\left(\frac{n\pi}{a}x\right) \tag{a}$$

6.7.2 In Example 6.6.3, the dielectrics to right and left, respectively, have the permittivities $\epsilon_a = \epsilon_p \exp(-\beta x)$ and $\epsilon_b = \epsilon_p \exp(\beta x)$. Determine the potential throughout the dielectric regions.

6.7.3 A linear dielectric has the permittivity

$$\epsilon = \epsilon_a\{1 + \chi_p \exp[-(x^2 + y^2 + z^2)/a^2]\} \tag{a}$$

An electric field that is uniform far from the origin (where it is equal to $E_o \mathbf{i_y}$) is imposed.

(a) Assume that ϵ/ϵ_o is not much different from unity and find ρ_p.
(b) With this induced polarization charge as a guide, sketch $\mathbf{E}$.