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Chapter 5

LUMPED-PARAMETER
ELECTROMECHANICAL DYNAMICS

5.0 INTRODUCTION

The representation of lumped-parameter electromechanical systems by means of mathematical models has been the subject of the preceding chapters. Our objective in this chapter is to study their dynamical behavior. Mathematically, we are interested in the solution of differential equations of motion for given initial conditions and with given driving sources. Physically, we are interested in important phenomena that occur in electromechanical systems.

It is clear from previous examples that the differential equations that describe electromechanical systems are in most cases nonlinear. Consequently, it is impossible to develop a concise and complete mathematical theory, as is done for linear circuit theory. We shall find many systems for which we can assume "small-signal" behavior and linearize the differential equations. In these cases we have available to us the complete mathematical analysis developed for linear systems. If exact solutions are required for nonlinear differential equations, each situation must be considered separately. Machine computation is often the only efficient way of obtaining theoretical predictions. Some simple cases however, are amenable to direct integration. The physical aspects of a given problem often motivate simplifications of the mathematical model and lead to meaningful but tractable descriptions. Hence in this chapter we are as much concerned with illustrating approximations that have been found useful as with reviewing and expanding fundamental analytical techniques.

Lumped-parameter systems are described by ordinary differential equations. The partial differential equations of continuous or distributed systems are often solved by a reduction to one or more ordinary differential equations. Hence many concepts used here will prove useful in the chapters that follow.

Similarly, the physical behavior of a distributed system is sometimes most easily understood in terms of lumped parameter concepts. Examples discussed in this chapter are in many cases motivated by the physical background that they provide for more complicated interactions to be considered later.

Because the mathematics of linear systems is comparatively simple, we begin our study of the dynamic behavior of lumped-parameter electro- mechanical systems by considering the several types of system for which a linear model provides an adequate description. We shall then consider the types of system that are basically nonlinear and for which the differential equations can be integrated directly.

5.1 LINEAR SYSTEMS

We have stated that electromechanical systems are not usually described by linear differential equations. Many devices, however, called incremental- motion transducers, are designed to operate *approximately* as linear systems. Moreover, meaningful descriptions of the basic properties of nonlinear systems can often be obtained by making small-signal linear analyses. In the following sections we develop and illustrate linearization techniques, linearized models, and the dynamical behavior of typical systems.

5.1.1 Linear Differential Equations

First, we should recall the definition of a linear ordinary differential equation.* An nth-order equation has the form

$$\frac{d^n x}{dt^n} + A_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + A_n(t)x = f(t), \qquad (5.1.1)$$

where the order is determined by the highest derivative. Note that the coefficients $A_i(t)$ can in general be functions of the *independent* variable t. If, however, the coefficients were functions of the *dependent* (unknown) variable $x(t)$, the equation would be nonlinear. The "driving function" $f(t)$ is a known function of time.

The "homogeneous" form of (5.1.1) is provided by making $f(t) = 0$. There are n independent solutions $x_n(t)$ to the homogeneous equation. The general solution to (5.1.1) is a linear combination of these homogeneous solutions, plus a particular solution $x_p(t)$ to the complete equation:

$$x(t) = c_1 x_1(t) + \cdots + c_n x_n(t) + x_p(t). \qquad (5.1.2)$$

Although (5.1.1) is linear, it has coefficients that are functions of the

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* A review of differential equations can be found in such texts as L. R. Ford, *Differential Equations*, 2nd Ed., McGraw-Hill, New York, 1955.

independent variable and this can cause complications; for example, if $f(t)$ is a steady-state sinusoid of a given frequency, the solution may contain all harmonics of the driving frequency. Alternatively, if $f(t)$ is an impulse, the response varies with the time at which the impulse is applied. These complications are necessary in some cases; most of our linear systems, however, are described by differential equations with constant coefficients. For now we limit ourselves to the case in which the coefficients $A_i = a_i =$ constant, and (5.1.1) becomes

$$\frac{d^n x(t)}{dt^n} + a_1 \frac{d^{n-1} x(t)}{dt^{n-1}} + \cdots + a_n x(t) = f(t). \tag{5.1.3}$$

The solution to equations having this form is the central theme of circuit theory.* The solutions $x_i(t)$ to the homogeneous equation, when the coefficients are constant, are exponentials e^{st}, where s can in general be complex; that is, if we let

$$x(t) = \sum_{i=1}^{n} c_i e^{s_i t} \tag{5.1.4}$$

and substitute it in the homogeneous equation, we obtain

$$(s_i^n + a_1 s_i^{n-1} + \cdots + a_n) \sum_{i=1}^{n} c_i e^{s_i t} = 0 \tag{5.1.5}$$

and (5.1.4) is a solution, provided that the complex frequencies satisfy the condition

$$s_i^n + a_1 s_i^{n-1} + \cdots + a_n = 0. \tag{5.1.6}$$

Here we have an nth-order polynomial in s, hence a condition that defines the n possible values of s required in (5.1.4). The frequencies s_i that satisfy (5.1.6) are called the *natural frequencies* of the system and (5.1.6) is sometimes called the *characteristic equation*.†

Many commonly used devices are driven in the sinusoidal steady state. In this case the driving function $f(t)$ has the form

$$f(t) = \text{Re}\,[\hat{F} e^{j\omega t}]. \tag{5.1.7}$$

Here $\hat{F}$ is in general complex and determines the phase of the driving signal; for example, if $\hat{F} = 1$, $f(t) = \cos \omega t$, but, if $\hat{F} = -j$, $f(t) = \sin \omega t$. To find

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* See, for example, E. A. Guillemin, *Theory of Linear Physical Systems*, Wiley, New York, 1963 (especially Chapter 7).
† If the characteristic equation has repeated roots, the solution must be modified slightly, see, for example, M. F. Gardner and J. L. Barnes, *Transients in Linear Systems*, Wiley, New York, 1942, pp. 159–163.

the particular solution with this drive we assume

$$x_p(t) = \text{Re}\,[\hat{X}e^{j\omega t}] \qquad (5.1.8)$$

and substitute into (5.1.3) to obtain

$$\text{Re}\,\{e^{j\omega t}[\hat{X}((j\omega)^n + a_1(j\omega)^{n-1} + \cdots + a_n) - \hat{F}]\} = 0. \qquad (5.1.9)$$

It follows that (5.1.8) is the particular solution if

$$\hat{X} = \frac{\hat{F}}{(j\omega)^n + a_1(j\omega)^{n-1} + \cdots + a_n}. \qquad (5.1.10)$$

Note that the natural frequencies (5.1.6) are the values of $j\omega$ in (5.1.10) which lead to the possibility of a finite response $\hat{X}$ when $\hat{F} = 0$; thus the term natural frequency.

The general solution is the sum of the homogeneous solution and the driven solution (5.1.4) and (5.1.8):

$$x(t) = \sum_{i=1}^{n} c_i e^{s_i t} + \text{Re}\left[\frac{\hat{F}e^{j\omega t}}{(j\omega)^n + a_1(j\omega)^{n-1} + \cdots + a_n}\right]. \qquad (5.1.11)$$

Given n initial conditions [e.g., $x(0)$, $(dx/dt)(0)$, . . . , $(d^{n-1}x/dt^{n-1})(0)$], the constants c_i can be evaluated. The first term in (5.1.11) is the transient part of the solution and the second term is the driven or steady-state part. If the system is stable (i.e., if all the s_i have negative real parts), the transient term in (5.1.11) will damp out. After a long enough time the first term will become small enough to be neglected. Then the system is said to be operating in the sinusoidal steady state and the response is given by the second term alone. When we wish to calculate the sinusoidal steady-state response, we find only the particular solution.

5.1.2 Equilibrium, Linearization, and Stability

We have already stated that useful informaton can be obtained about many electromechanical systems by making small-signal linear analyses in the vicinity of equilibrium points. In this section we introduce the concept of equilibrium and illustrate how to obtain small-signal, linear equations. In the process we shall study the nature of the small-signal behavior and define two basic types of instability that can occur in the vicinity of an equilibrium point.

5.1.2a *Static Equilibrium and Static Instability*

In general, the term *equilibrium* is used in connection with a dynamical system to indicate that the motion takes on a particularly simple form; for example, a mass M, constrained to move in the x-direction and subject

to a force $f(x)$ will have a position $x(t)$ predicted by the equation [see (2.2.10) of Chapter 2]

$$M \frac{d^2 x}{dt^2} = f(x). \tag{5.1.12}$$

We say that the mass is in equilibrium at any point $x = X$ such that $f(X) = 0$. Physically, we simply mean that at the point $x = X$ there is no external force to accelerate the mass, hence it is possible for the mass to retain a static position (or be in equilibrium) at this point.

The word equilibrium is used to refer not only to cases in which the dependent variables (x) take on static values that satisfy the equations of motion but also to situations, such as uniform motion, in which the general (nonlinear) equations of motion are satisfied by the equilibrium solution. Equilibria of this type were of primary interest in Chapter 4, in which the steady-state behavior of rotating magnetic field devices was considered.

Small perturbations from the equilibrium positions are predicted approximately by linearized equations of motion, which are found by assuming that the dependent variables have the form

$$x(t) = X + x'(t), \tag{5.1.13}$$

where X is the equilibrium position and $x'(t)$ is the small perturbation. It is then possible to expand nonlinear terms in a Taylor series* about the equilibrium values; for example, $f(x)$ in (5.1.12) can be expanded in the series

$$f(x) = f(X) + x' \frac{df}{dx}(X) + \tfrac{1}{2}x'^2 \frac{d^2 f}{dx^2}(X) + \cdots. \tag{5.1.14}$$

Now, if x' is small enough, it is likely that the first two terms will make the most significant contributions to the series, hence the remaining terms can be ignored. Recall that by definition $f(X) = 0$ and 5.1.12 has the form,

$$\frac{d^2 x'}{dt^2} + \omega_o^2 x' = 0, \tag{5.1.15}$$

where

$$\omega_o^2 = -\frac{1}{M}\frac{df}{dx}(X).$$

The resulting equation is linear and can be solved as described in Section 5.1.1. The solution has the form

$$x' = c_1 e^{j\omega_o t} + c_2 e^{-j\omega_o t}. \tag{5.1.16}$$

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* F. B. Hildebrand, *Advanced Calculus for Engineers*, Prentice-Hall, New York, 1949, p. 125.
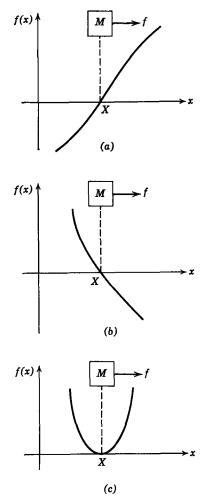
(a)

(b)

(c)

Fig. 5.1.1 Graphical representation of a force $f(x)$ which acts on the mass M with a static equilibrium at $x = X$: (a) unstable; (b) stable; (c) nonlinear.

We see that if $(df/dx)(X)$ is positive ω_0 is imaginary and any small displacement of the mass (as will inevitably be supplied by noise) will lead to a motion that is unbounded. In this case we say that the equilibrium position X is unstable, and we call this type of pure exponential instability a *static* instability. We can interpret this situation physically by reference to Fig. 5.1.1a which shows a plot of $f(x)$ with a positive slope at $x = X$. If the mass moves a small distance to the right of the equilibrium point, the force f becomes positive and tends to increase the displacement still further. Thus a static experiment will reveal the presence of the instability. Although our

solution cannot be trusted for long after the start of a transient, the small-signal instability provides the essential information that the mass will not remain at the equilibrium point. If df/dx is negative at $x = X$, ω_o is real and the mass will execute a sinusoidal motion about the equilibrium point with the angular frequency ω_o, as can be seen from (5.1.15) and (5.1.16). This result is also easily understood physically. Figure 5.1.1b shows a plot of force f having a negative slope at the equilibrium point. If the mass moves slightly to the right, the force becomes negative and tends to return the mass to $x = X$. When the mass reaches the equilibrium point it has finite velocity and overshoots. In the absence of further external disturbances the mass will oscillate sinusoidally about the equilibrium point with constant amplitude. We call this motion stable because the response is bounded. If there were damping in the system, the amplitude of the oscillation would decay until the mass came to rest at the equilibrium point.

Once a solution has been found to the linearized equations of motion, it is possible to check the accuracy of the prediction by considering the significance of the terms that were dropped in (5.1.14), compared with the second term. An extreme case in which linearized equations would not adequately describe the motion is illustrated in Fig. 5.1.1c, where the slope of $f(x)$ is also zero at the equilibrium point. In this case the lowest order, nonzero term in (5.1.14) must be retained.

Example 5.1.1. To illustrate some of these ideas we consider again the magnetic field transducer shown schematically in Fig. 5.1.2. The electric terminal relations were calculated in Example 2.1.1 and the equations of motion were written in Example 3.2.1.
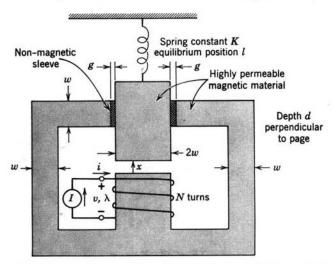

Fig. 5.1.2 Magnetic field transducer used to demonstrate linearization techniques in Example 5.1.1.

The electric terminal relation, as described by (d) and (e) of Example 2.1.1, is

$$\lambda = \frac{L_o i}{1 + x/g} \tag{a}$$

where

$$L_o = \frac{\mu_0 N^2 (2wd)}{g} \tag{b}$$

is the inductance with the air gap closed ($x = 0$). The force of electric origin is given by (d) of Example 3.2.1 and is

$$f^e = - \frac{L_o i^2}{2g(1 + x/g)^2}. \tag{c}$$

Because the electrical excitation is a current source, the equation for the electrical part of the system is not of interest. In the mechanical part we neglect damping; consequently, the equation of motion for the mechanical node (x) is [see (e) of Example 3.2.1]

$$M \frac{d^2 x}{dt^2} = - \frac{L_o I^2}{2g(1 + x/g)^2} - K(x - l). \tag{d}$$

This equation has the form of (5.1.12):

$$f(x) = - \frac{L_o I^2}{2g(1 + x/g)^2} - K(x - l) = f^e(x) + f^s(x). \tag{e}$$

In static equilibrium at $x = X$, (d) becomes

$$f(X) = - \frac{L_o I^2}{2g(1 + X/g)^2} - K(X - l) = 0. \tag{f}$$

This is a cubic equation in X which cannot be solved easily. Its properties, however, can be investigated by sketching the two terms as shown in Fig. 5.1.3. In this figure the negative of f^e has been plotted as a function of X on the same scale as a plot of f^s. Hence the intersections represent solutions to (f); that is, the points X_1 and X_2 are positions at which the
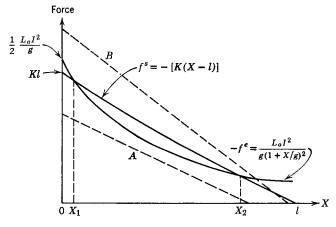

Fig. 5.1.3   Sketch for determining equilibrium points for system of Fig. 5.1.2.

plunger experiences no external forces, hence it can be in static equilibrium. There are, of course, three roots to the cubic equation, but one is not physical, since it requires that X be less than zero or that the plunger extend into the magnetic yoke. The relative values of the parameters can be such that there are no possible equilibrium points, as illustrated by the dashed curve A in Fig. 5.1.3, or there may be only one equilibrium point, as indicated by the dashed curve B. Note that all equilibrium points are such that $X < l$. Physically, this is expected, since the force f^e always tends to pull the plunger into the yoke, hence to extend the length of the spring.

We now assume that the conditions represented by the solid curves of Fig. 5.1.3 have been established and consider the dynamics for small excursions from the equilibrium points; for example, about X_1,

$$x(t) = X_1 + x'(t)$$

and (d) becomes (see 5.1.15)

$$\frac{d^2x'}{dt^2} + \omega_o^2 x' = 0,\tag{g}$$

where

$$\omega_o^2 = -\frac{1}{M}\frac{df}{dx}(X_1) = \frac{1}{M}\left[K - \frac{L_o I^2}{g^2(1 + X_1/g)^3}\right].\tag{h}$$

It should be clear from (5.1.15) and (5.1.16), and the associated discussion, that the relative magnitudes of the two terms in brackets determines whether this system is stable at the operating point X_1.

Although we could proceed in a formal mathematical way to study the stability at the equilibrium points, we shall pursue the subject with some qualitative study of the curves in Fig. 5.1.3.

At the point X_1 the magnitude of df^e/dX is larger than the magnitude of df^s/dX. Hence the derivative df/dX at X_1 has the sign of df^e/dX; that is,

$$\frac{df}{dX}(X_1) > 0,\tag{i}$$

and we conclude that this equilibrium point is unstable. We have found mathematically that a small excursion of the mass to the right of $X = X_1$ (Fig. 5.1.3) subjects the plunger to a force dominated by the spring force, which tends to force the plunger further to the right. (Remember that f^s is defined as a force that acts in the $+x$-direction if it is positive.)

Similarly, at X_2 the slope df^s/dX has a larger magnitude than df^e/dX, hence the sign of df/dX is negative at X_2,

$$\frac{df}{dX}(X_2) < 0\tag{j}$$

and the equilibrium point X_2 is stable.

The process of using the first two terms in a Taylor series expansion to make a linear approximation has been described for the case in which there is a single dependent variable (x). It can be generalized to an arbitrary number of dependent variables. Suppose there are M variables $x_1, x_2, \ldots, x_M$, in terms of which a general function is expressed as $f(x_1, x_2, \ldots, x_M)$. If there is an equilibrium point $(X_1, X_2, X_3, \ldots, X_M)$ about which we wish to obtain a linear approximation to f by using a Taylor series expansion, we express each variable as

$$x_k = X_k + x_k'.$$

Then we write the fixed and linearly varying parts of the Taylor series about the fixed point as

$$f(x_1, x_2, \ldots, x_M) \approx f(X_1, X_2, \ldots, X_M)$$

$$+ \sum_{k=1}^{M} \frac{\partial f}{\partial x_k} (X_1, X_2, \ldots, X_M) x_k'. \quad (5.1.17)$$

The range of x_k' over which this approximation is valid within specified limits of error must be evaluated by using higher order terms in the Taylor series.* In Section 5.2.1 we consider the errors that result from using a linear approximation.

5.1.2b Dynamic or Steady-State Equilibrium

In the preceding section we studied small-signal operation about a static equilibrium. In many cases it is desirable to analyze devices as they are perturbed from a steady-state dynamic condition. This often occurs in rotating devices that are commonly designed to operate with constant angular velocity (as discussed in Chapter 4). Changes in external constraints (excitation or load) produce changes in the angular velocity that can be described as perturbations from the steady-state condition.

The linearization techniques that are the subject of this section are also applicable to many situations containing continuous media. The steady-state equilibrium may involve a moving medium such as a fluid, which has a constant velocity at a given point in space. The dynamics that result from perturbations from this steady flow could be described in a way similar to that developed here and illustrated in Chapter 10.

In the following example a synchronous magnetic field machine is used to illustrate the ideas involved in studying linearized motions about a dynamic equilibrium. The steady-state behavior of this type of device was studied in Section 4.1.6a and the example picks up the equations of motion developed there.

Example 5.1.2. A synchronous machine is modeled by the system of three coils shown schematically in Fig. 5.1.4. The physical arrangement of these coils may be as shown in Fig. 4.1.10.

The magnetic torque on the rotor, as a function of the stator currents i_{as} and i_{bs}, the rotor current i_r, and the rotor angle θ, is (4.1.38)

$$T^e = Mi_r(i_{bs} \cos \theta - i_{as} \sin \theta). \tag{a}$$

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* For a discussion of this expansion see, for example, F. B. Hildebrand, *Advanced Calculus for Engineers*, Prentice-Hall, New York, 1949, p. 353.
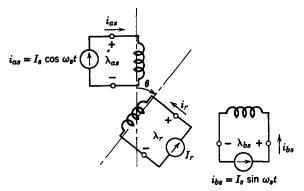
Fig. 5.1.4 Schematic representation of synchronous machine showing two fixed (stator) and one rotatable (rotor) coils.

To obtain steady-state synchronous conditions the stator and rotor terminals are excited by the current sources

$$i_r = I_r,$$ (b)

$$i_{as} = I_s \cos \omega_s t,$$ (c)

$$i_{bs} = I_s \sin \omega_s t,$$ (d)

where it is helpful for purposes of discussion to consider I_r, I_s, and ω_s as positive constants. In addition to the torque T^e, the shaft is subject to an inertial torque, a friction torque, and a load torque. We represent the total moment of inertia about the axis of rotation as J [see (2.2.27)], the friction torque as linear with coefficient B [see (2.2.6)], and the load torque as a driving function $T_m(t)$. Thus we can write the mechanical equation of motion for the angular deflection θ of the rotor as

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + MI_rI_s(\cos \omega_s t \sin \theta - \sin \omega_s t \cos \theta) = T_m(t).$$ (e)

To establish a dynamic equilibrium we assume $T_m = 0$ and constant angular velocity

$$\frac{d\theta}{dt} = \Omega = \text{constant}$$ (f)

and write

$$\theta = \Omega t + \gamma_o,$$ (g)

where γ_o is a constant to be determined. We substitute (g) into (e) with $T_m = 0$ to obtain the equilibrium equation

$$B\Omega + MI_rI_s[\cos \omega_s t \sin (\Omega t + \gamma_o) - \sin \omega_s t \cos (\Omega t + \gamma_o)] = 0.$$ (h)

The use of a trigonometric identity to simplify the term in brackets yields

$$B\Omega = MI_rI_s \sin [(\omega_s - \Omega)t - \gamma_o].$$ (i)

The left side of this equation is constant; consequently, the equation can be satisfied for all t only if

$$\Omega = \omega_s.$$ (j)

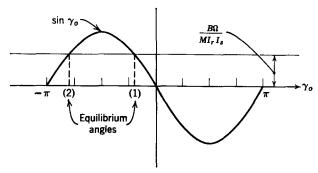This is the synchronous speed at which the rotor can run in dynamic equilibrium. Using

Fig. 5.1.5 The equilibrium values of angle γ_o.

(j) in (i), we obtain an expression for γ_o:

$$\sin \gamma_o = \frac{-B\Omega}{MI_rI_s}. \tag{k}$$

Dynamic equilibrium is defined for the system of Fig. 5.1.4 with the electrical excitations of (b) to (d) by (g), (j), and (k). Note that there is a limited range of parameters over which a dynamic equilibrium is possible because

$$-1 < \sin \gamma_o < 1.$$

Also, for any value of

$$\left| \frac{B\Omega}{MI_rI_s} \right| < 1$$

there are basically two different solutions for γ_o, as indicated in Fig. 5.1.5.

We now assume operation in dynamic equilibrium and describe perturbations from this equilibrium by $\gamma'(t)$; thus

$$\theta(t) = \Omega t + \gamma_o + \gamma'(t). \tag{l}$$

Substitution of this expression into (e), use of appropriate trigonometric identities, and retention of only linear terms yields after some simplification

$$J\frac{d^2\gamma'}{dt^2} + B\frac{d\gamma'}{dt} + K\gamma' = T_m(t), \tag{m}$$

where $K = MI_rI_s \cos \gamma_o$ is the effective spring constant of the magnetic torque. In writing this equation we have subtracted out the equilibrium equation represented by (k).

To study the dynamic behavior we assume that the load torque T_m is a small step occurring at $t = 0$:

$$T_m(t) = Tu_{-1}(t), \tag{n}$$

where T is a constant and $u_{-1}(t)$ is the unit step occurring at $t = 0$.

The method of solution reviewed in Section 5.1.1 can be used here. A particular solution is

$$\gamma' = \frac{T}{K}. \tag{o}$$

The initial conditions are $\gamma'(0) = 0$ and $(d\gamma'/dt)(0) = 0$. The complete solution therefore is

$$\gamma'(t) = \frac{T}{K}\left(1 + \frac{s_2}{s_1 - s_2}e^{s_1t} + \frac{s_1}{s_2 - s_1}e^{s_2t}\right), \tag{p}$$

where

$$s_1 = -\frac{B}{2J} + \left[\left(\frac{B}{2J}\right)^2 - \frac{K}{J}\right]^{1/2},$$

$$s_2 = -\frac{B}{2J} - \left[\left(\frac{B}{2J}\right)^2 - \frac{K}{J}\right]^{1/2}$$

are the natural frequencies that satisfy the characteristic equation

$$s^2 + \frac{B}{J}s + \frac{K}{J} = 0. \tag{q}$$

We refer now to the two equilibrium points illustrated in Fig. 5.1.5. At equilibrium point (2) the effective spring constant K is negative because $\cos \gamma_o$ is negative. Consequently, s_1 is positive, the response is unbounded, and the machine has a static instability at equilibrium point (2).

At equilibrium point (1) in Fig. 5.1.5 $\cos \gamma_o$ is positive, K is positive, and the real parts of s_1 and s_2 are both negative. Consequently, this equilibrium point is stable. When we assume the system to be underdamped,

$$\frac{K}{J} > \left(\frac{B}{2J}\right)^2,$$

we can write (p) in the form

$$\gamma'(t) = \frac{T}{K}\left[1 - e^{-\alpha t}\left(\cos \omega t + \frac{\alpha}{\omega}\sin \omega t\right)\right], \tag{r}$$

where

$$\alpha = \frac{B}{2J},$$

$$\omega = \left[\frac{K}{J} - \left(\frac{B}{2J}\right)^2\right]^{1/2}.$$

The response of (r) is plotted as a function of time for two values of α/ω in Fig. 5.1.6. Two
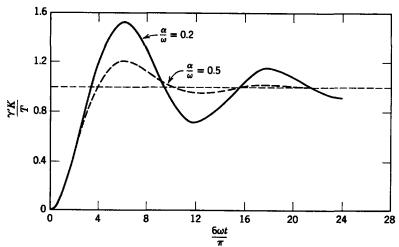

Fig. 5.1.6 Hunting transient of a synchronous machine.

general features of this "hunting" transient should be noted. First the initial parts of both transients are the same because, as (e) indicates, the initial part of the transient is dominated by the angular acceleration term $J(d^2\theta/dt^2)$, which is the same for both cases. Second, the damping retards the phase as indicated by the second term in parentheses in (r). The normalization for the time axis is different in the two cases because ω differs by a small amount.

5.1.2c   Overstability or Dynamic Instability

In Section 5.1.2a we described a static instability characterized by a pure exponential growth in time. There is a second basic type of instability, called *overstability* or *dynamic instability*, which we now illustrate.

The equilibrium points of the system described by (5.1.12) are not changed if the mass is subject to an additional force proportional to velocity. The equation of motion has the form

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} = f(x), \qquad (5.1.18)$$

and in static equilibrium the additional term makes no contribution. The linearized equation, however, is

$$\frac{d^2x'}{dt^2} + \frac{B}{M}\frac{dx'}{dt} + \omega_o{}^2 x' = 0 \qquad (5.1.19)$$

and the natural frequencies of the system are

$$s_{\binom{1}{2}} = \frac{-B}{2M} \pm \left(\frac{B^2}{4M^2} - \omega_o{}^2\right)^{\!\frac{1}{2}}. \qquad (5.1.20)$$

If the term $B(dx/dt)$ is due to viscous damping, B will be positive (see Section 2.2.1b). In this case, if $\omega_o{}^2 < 0$, one of the natural frequencies will be positive, and the result will be a static instability that can be detected by the static experiment described in Section 5.1.2a. On the other hand, if $\omega_o{}^2 > 0$, the system will be stable, regardless of the magnitude of $\omega_o{}^2$.

As we show in Example 5.1.3, feedback can be used to make $B < 0$ in (5.1.18). If $\omega_o{}^2 < 0$, one of the natural frequencies (5.1.20) will be real and positive and will result in a pure exponential growth that is a static instability detectable with a static experiment. Alternatively, if

$$0 < \omega_o{}^2 < \frac{B^2}{4M^2}, \qquad B < 0, \qquad (5.1.21)$$

the natural frequencies will be real and both will be positive, thus indicating an instability that cannot be detected by our static argument. If

$$0 < \frac{B^2}{4M^2} < \omega_o{}^2, \qquad B < 0, \qquad (5.1.22)$$

the radical in (5.1.20) will be imaginary and the natural frequencies will be

complex with positive real parts. In this case the transient is an exponentially growing sinusoid. This is *overstability** or *dynamic instability*.

Note that under conditions of overstability $\omega_o{}^2 > 0$ and our static experiment of a small displacement will result in a force that tends to return the mass toward the equilibrium point; but this force is reinforced by the negative damping and the system overshoots the equilibrium point and reaches a larger displacement in the opposite direction.

It is clear that the static argument we used in Section 5.1.2a to detect a static instability will not detect overstability and one type of exponential growth. Thus we must regard the static argument as a sufficient, but not a necessary, condition for instability.

We have discussed stability in Section 5.1.2a and in this section in terms of a system describable by a second-order differential equation. Many systems have differential equations of an order higher than 2. Whenever a system goes unstable, however, the instability is usually caused by one or two natural frequencies. Because the instability dominates the dynamical behavior, the system differential equation can sometimes be approximated by a first- or second-order differential equation for studying the instability.

It is worthwhile to establish the physical significance of the three modes of instability. For this purpose, suppose that the mass is given an initial position $x' = x_o$ with no initial velocity. Then the appropriate solution to (5.1.19) is [in terms of the roots s_1 and s_2 defined by (5.1.20)],

$$x'(t) = \frac{x_o}{s_2 - s_1}(s_2 e^{s_1 t} - s_1 e^{s_2 t}). \qquad (5.1.23)$$

This solution is shown in Fig. 5.1.7 for the three cases of instability that have been illustrated. Further insight is provided by the following example.

Example 5.1.3.   In many situations it is desirable to support an object with a magnetic field; for example, in a wind tunnel effects of the mechanical structure (stinger) supporting the model under study introduce errors in drag and lift measurements. One solution to this problem is to use a magnetic field.† Then, if the fluid is an ordinary nonconducting gas, the magnetic field will not interfere with the flow. To support a large mass it is desirable to use a ferromagnetic core in the model so that magnetic forces will be of a useful magnitude. It is familiar to anyone who has held a piece of magnetic material near a magnet, that any static equilibrium achieved with these forces is unstable. The example undertaken here shows how feedback can be used to stabilize an inherently unstable equilibrium. Without feedback the equilibrium exhibits a static instability. This is obviated by the introduction of feedback, but then dynamic instability comes into play. This second type of instability is removed by additional feedback.

* For additional discussion of the terminology we use for describing instabilities see S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, Oxford University Press, London, 1961, pp. 1–3.
† J. E. Chrisinger et al., "Magnetic Suspension and Balance System for Wind Tunnel Application," *J. Roy. Aeron. Soc.*, **67**, 717–724 (1963).
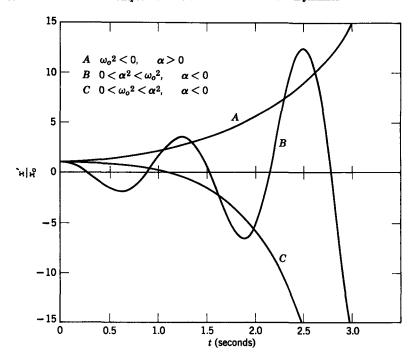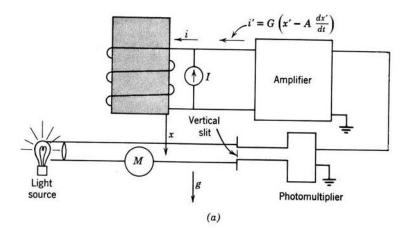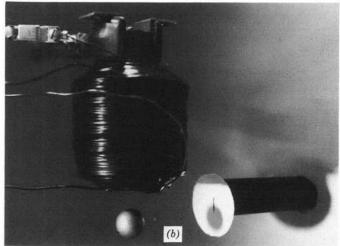
Fig. 5.1.7 Three modes of instability for a second-order system. The normalized damping α is negative for cases B and C ($\alpha = B/2M$).

The simple example to be considered here is shown schematically in Fig. 5.1.8a and an operating system of this type is illustrated in Fig. 5.1.8b. This system has the basic ingredients of systems constructed to levitate a mass M to be used in gyroscopes and accelerometers. The spherical particle is magnetic and is therefore attracted upward by the magnetic field induced by I. Hence there is a position $x = d$ at which the mass is supported against gravity by the magnetic field. With no feedback ($i' = 0$) the equilibrium is unstable, for, as the ferromagnetic mass approaches the inductor, the upward force increases, whereas the gravitational force remains constant. Feedback is introduced by using the optical system to detect the position of the mass. The photomultiplier and amplifier are adjusted to give a current i' as nearly as possible proportional to the deflection x' from the equilibrium position $x = d$. Hence with feedback there is an addition to the magnetic force proportional to the deflection x'. By adjustment of the loop gain it is possible to make the effective spring constant introduced by the feedback large enough so that the equilibrium will appear to be stable on the basis of a static experiment of displacing the sphere from equilibrium and finding a restoring force. The amplifier, however, is not an ideal current source, and the effect of the coil inductance with finite amplifier output impedance causes the equilibrium to be dynamically unstable. A feedback signal proportional to sphere velocity is then added to stabilize the equilibrium completely. This discussion characterizes the design process used with the system shown in Fig. 5.1.8b.*

* The analysis was used in a student laboratory project to achieve the stable suspension of the sphere shown.
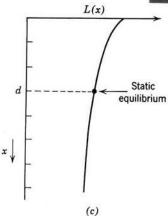
$$i' = G\left(x' - A\frac{dx'}{dt}\right)$$

Amplifier

i

I

Vertical
slit

x

Light
source

M

g

Photomultiplier

(a)

(b)

$L(x)$

d

Static
equilibrium

x

(c)

Fig. 5.1.8 (a) Mass M levitated by a magnetic field. The optical system provides a signal that is fed back to stabilize the static equilibrium of the mass at $x = d$; (b) view of laboratory project shown schematically by (a); (c) dependence of coil inductance on the mass displacement.

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Measurement of the inductance L as a function of the position of the mass M would produce a curve like that shown in Fig. 5.1.8c. The inductance has its largest value when the ferromagnetic sphere is next to the coil and decreases to a constant as the sphere is removed to $x = \infty$. For the present purposes we take this dependence as

$$L(x) = L_1 + \frac{L_o}{1 + x/a}, \tag{a}$$

where L_1, L_o, and a are positive constants. Then the methods of Chapter 3 provide the force on the mass M; that is, the coenergy (since the electrical terminal relation is linear) is

$$W'(i, x) = \frac{1}{2}\left(L_1 + \frac{L_o}{1 + x/a}\right)i^2 \tag{b}$$

and the force of electrical origin follows as

$$f = \frac{\partial W'}{\partial x} = -\frac{1}{2a}\frac{L_o}{(1 + x/a)^2}i^2. \tag{c}$$

When the sphere is in static equilibrium, the gravitational force is balanced by this force:

$$Mg = \frac{1}{2a}\frac{L_o}{(1 + d/a)^2}I^2. \tag{d}$$

Given the current I, the equilibrium position d is determined. Perturbations x' from the equilibrium lead to a perturbation current i'; that is,

$$\begin{aligned} x &= d + x', \\ i &= I + i'. \end{aligned} \tag{e}$$

To linear terms in the perturbation quantities the force of (c) becomes

$$f = -\frac{L_o}{2a}\left[\frac{I^2}{(1 + d/a)^2} - \frac{2x'I^2}{a(1 + d/a)^3} + \frac{2Ii'}{(1 + d/a)^2}\right]. \tag{f}$$

It follows that the incremental equation of motion is

$$M\frac{d^2x'}{dt^2} - \frac{L_oI^2}{a^2(1 + d/a)^3}x' + \frac{L_oIi'}{a(1 + d/a)^2} = 0, \tag{g}$$

where use has been made of (d) to cancel out the constant part of the force equation.

In the absence of feedback ($i' = 0$) it is clear from (g) that the equilibrium is statically unstable. To consider first the effect of ideal feedback, assume that the output voltage of the photomultiplier is linear with x' and that the amplifier is a perfect current source feeding the coil. In this case

$$i' = Gx', \tag{h}$$

where G is a constant, including the amplifier gain. Substitution of this expression into (g) yields

$$M\frac{d^2x'}{dt^2} + \left[\frac{GL_oI}{a(1 + d/a)^2} - \frac{L_oI^2}{a^2(1 + d/a)^3}\right]x' = 0. \tag{i}$$

This system has natural frequencies determined from [see (5.1.15)]

$$\omega_o^2 = \frac{L_oI}{aM(1 + d/a)^2}\left(G - \frac{I}{a + d}\right). \tag{j}$$

Thus the response is bounded and the system is stable when the feedback gain is raised high enough to make

$$G > \frac{I}{a + d}. \qquad \text{(k)}$$

No amplifier is absolutely ideal; consequently, we consider next the system performance in which the amplifier has a large but finite output impedance. In this case the amplifier produces a voltage proportional to sphere displacement x' and this voltage is applied to the series RL circuit illustrated in Fig. 5.1.9 in which the resistance R includes the internal resistance of the amplifier. The relation between x' and i' is now determined from the equation

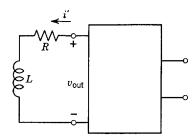$$Gx' = \frac{1}{R} \frac{d(Li')}{dt} + i'. \qquad \text{(l)}$$

Fig. 5.1.9 The driving current of Fig. 5.1.8a is produced by a voltage amplifier connected in series with a large resistance R. The output voltage is proportional to the displacement x'. The inductance L is the equilibrium inductance of the coil shown in Fig. 5.1.8a.

Expansion of the derivative and retention of only linear terms yields

$$Gx' = \frac{1}{R}\left(L_1 + \frac{L_o}{1 + d/a}\right)\frac{di'}{dt} - \frac{L_oI}{Ra(1 + d/a)^2}\frac{dx'}{dt} + i'. \qquad \text{(m)}$$

Using the exponential forms

$$x' = \hat{x}e^{st} \quad \text{and} \quad i' = \hat{i}e^{st}, \qquad \text{(n)}$$

we rewrite (m) in the form

$$\hat{i} = G\hat{x}\left[\frac{1 + \dfrac{L_oI}{GRa(1 + d/a)^2}s}{1 + \dfrac{1}{R}\left(L_1 + \dfrac{L_o}{1 + d/a}\right)s}\right]. \qquad \text{(o)}$$

For relatively high gain G and relatively high amplifier output impedance we assume

$$\frac{L_oI}{GRa(1 + d/a)^2}s \ll 1,$$

$$\frac{1}{R}\left(L_1 + \frac{L_o}{1 + d/a}\right)s \ll 1,$$

and approximate (o) by the form

$$\hat{i} = G\hat{x}(1 - As), \qquad \text{(p)}$$

where

$$A = \frac{1}{R}\left[\left(L_1 + \frac{L_o}{1 + d/a}\right) - \frac{L_oI}{Ga(1 + d/a)^2}\right]. \qquad \text{(q)}$$

It is evident that when the inequality k is satisfied $A > 0$.

We rewrite (p) in differential form as

$$i' = Gx' - GA\frac{dx'}{dt} \tag{r}$$

and substitute this result in (g) to obtain

$$M\frac{d^2x'}{dt^2} - \frac{GAL_oI}{a(1 + d/a)^2}\frac{dx'}{dt} + \frac{L_oI}{a(1 + d/a)^2}\left(G - \frac{I}{a+d}\right)x' = 0. \tag{s}$$

This has the form of (5.1.19) with $B < 0$; thus the system is overstable or dynamically unstable when the nonideal nature of the amplifier is included. The response of this system to a small disturbance will be like curve B or curve C of Fig. 5.1.7, depending on the relative sizes of the coefficients in (s) [see (5.1.19) to (5.1.22)].

We can interpret the three curves of Fig. 5.1.7 physically with reference to this example. For curve A the electrical force due to the bias current I exceeds the restoring force induced by the feedback. Thus, when the mass is released from rest with a small initial displacement, the displacement increases exponentially. This occurs regardless of whether the damping is negative or positive, the only effect of the damping being to change the *rate* of exponential growth.

Curve B of Fig. 5.1.7 represents the situation in which the feedback force dominates the force due to bias current I to provide a restoring force but the derivative term due to electrical feedback is negative. Also, the relative parameter values are such that the system is oscillatory. When the mass is released from rest with an initial displacement, the feedback force immediately accelerates the mass back toward equilibrium. The negative damping force adds to this feedback force to cause the position to overshoot equilibrium by more than the magnitude of the initial displacement. The process repeats periodically as the amplitude of the oscillation grows exponentially.

Curve C of Fig. 5.1.7 represents the situation in which the feedback force dominates the electrical force due to the bias current I to provide a static restoring force. The negative damping due to the electrical feedback is large enough to make the system's natural frequencies s real. When the mass is released from rest with an initial displacement, the feedback force accelerates the mass back toward equilibrium. As the mass starts moving, however, the negative damping adds a force to accelerate the mass further toward equilibrium. As the mass passes through equilibrium, the negative damping force dominates to accelerate the mass further along a rising exponential in a direction opposite that of the initial displacement.

To stabilize the equilibrium with a nonideal amplifier it is necessary to modify the amplifier signal so that its output current contains a component proportional to (dx'/dt) with the proper sign. This process is called compensation.* The simplest method of compensation is achieved by using the RC circuit of Fig. 5.1.10 between the photomultiplier output and the amplifier input. Implicit in what follows is the assumption that the internal impedance of the photomultiplier is very low and the amplifier input impedance is very high.

In terms of complex exponentials

$$v_2 = \hat{v}_2 e^{st} \quad \text{and} \quad v_1 = \hat{v}_1 e^{st},$$

* Compensating networks and their use in automatic control systems are discussed in such texts as J. J. D'Azzo and C. H. Houpis, "Feedback Control System Analysis and Synthesis." McGraw-Hill, New York, 2nd ed., 1966 p. 158.
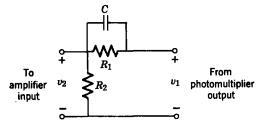
Fig. 5.1.10 Compensating network to provide stabilization of dynamic instability caused by finite amplifier output impedance.

the transfer function of the compensating network is

$$\frac{\hat{v}_2}{\hat{v}_1} = \frac{R_2}{R_1 + R_2}\left[\frac{R_1Cs + 1}{[R_1R_2/(R_1 + R_2)]Cs + 1}\right]. \tag{t}$$

For proper operation of the network it is conventional to set $R_2 \ll R_1$; thus

$$\frac{R_1R_2}{R_1 + R_2}Cs \ll R_1Cs$$

and for low frequencies we can approximate (t) by the form

$$\hat{v}_2 = \frac{R_2}{R_1 + R_2}(R_1Cs + 1)\hat{v}_1$$

or in differential form as

$$v_2 = \frac{R_2}{R_1 + R_2}\left(R_1C\frac{dv_1}{dt} + v_1\right) \tag{u}$$

Because v_1 is the output voltage of the photomultiplier and is therefore proportional to x', we now rewrite (m) as

$$G_1x' + G_1R_1C\frac{dx'}{dt} = \frac{1}{R}\left(L_1 + \frac{L_o}{1 + d/a}\right)\frac{di'}{dt} - \frac{L_oI}{Ra(1 + d/a)^2}\frac{dx'}{dt} + i', \tag{v}$$

where

$$G_1 = \frac{R_2}{R_1 + R_2}G.$$

Using the exponential forms of (n), we solve for i to obtain

$$i = G_1\hat{x}\left[\frac{1 + R_1Cs + \dfrac{L_oI}{G_1Ra(1 + d/a)^2}s}{1 + \dfrac{1}{R}\left(L_1 + \dfrac{L_o}{1 + d/a}\right)s}\right] \tag{w}$$

Making the assumption that parameter values and frequency s are such that we need retain

only first-order terms in s, (w) becomes

$$i = G_1\hat{x}[1 + (R_1C - A)s],\qquad\text{(x)}$$

where A is defined in (q) and $A > 0$ when inequality k is satisfied. Writing (x) in differential form, we have

$$i' = G_1x' + G_1(R_1C - A)\frac{dx'}{dt}.\qquad\text{(y)}$$

Substitution of this expression into (g) yields

$$M\frac{d^2x'}{dt^2} + \frac{L_oIG_1(R_1C - A)}{a(1 + d/a)^2}\frac{dx'}{dt} + \frac{L_oI}{a(1 + d/a)^2}\left(G_1 - \frac{I}{a + d}\right)x' = 0.\qquad\text{(z)}$$

It is clear from this equation that the compensation circuit has added positive damping to the system and that the system is completely stable when

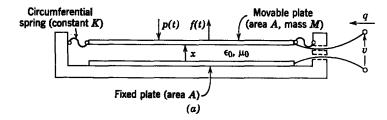$$G_1 > \frac{I}{a + d}\quad\text{and}\quad R_1C > A.$$

Note that because $G_1 = [R_2/(R_1 + R_2)]G$ and $R_2 \ll R_1$ for proper compensation the amplifier gain must be greater when compensation is used. This is a principal consequence of compensation—that amplifier gain can be traded for a change in dynamic system behavior.

5.1.2d   Steady-State Sinusoidal Response

Many incremental-motion transducers, such as speakers, microphones, and electromechanical filters, are designed to operate approximately as linear systems. One of the most important design factors is the driven response. A transducer may be used to convert a mechanical signal (pressure for example) to an electrical form, in which case the electrical signal is the response to a driving force. It may also convert the output of an amplifier to an acoustic signal, and thus the pressure or velocity response to a driving voltage is of interest. The most commonly used and convenient driven response for a linear system with constant coefficients is the sinusoidal steady state. Many systems operate largely in a sinusoidal steady-state condition, but even for those that do not the techniques of Fourier transforms and Fourier series are available. The response to an arbitrary signal can be synthesized from the response to sinusoidal driving signals ranging over the frequency spectrum of interest.*

In this section we show, by means of an example, how the techniques of Section 5.1.1 can be used to find the steady-state response to a sinusoidal excitation. The example also serves to illustrate characteristic dynamical behavior and impedance levels in an electric field system.

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* See, for example, S. J. Mason and H. J. Zimmermann, *Electronic Circuits, Signals, and Systems*, Wiley, New York, 1960, Chapter 7.
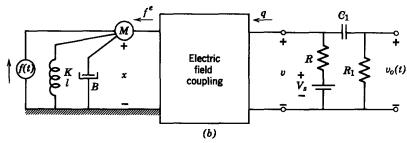
Fig. 5.1.11 Electric field transducer that can be used as a pressure sensor (microphone) or pressure source (speaker): (a) cross section of device (side view); it would appear circular from above; (b) equivalent electromechanical circuit.

Example 5.1.4. The simple variable capacitor shown schematically in Fig. 5.1.11 illustrates the basic construction of an electric field transducer that can be used as a microphone or as a speaker.* A circular movable plate with a total mass M is mounted on a peripheral bellows spring which has an equivalent constant K. The plate moves against a damping force (primarily caused by the surrounding air), which we assume is linear, with a coefficient B. When the device operates as a microphone, the differential pressure $p(t)$ acting over the area of the moving plate exerts a force $f(t)$ (defined as positive in the positive x-direction) which moves the plate and changes the capacitance, as seen from the electrical terminals. When the electrical terminals are biased with a constant voltage V_s (see Fig. 5.1.11b), a change in capacitance due to the motion induces a current dq/dt in the circuit. This current, at least in part, flows through resistance R_1 and produces a voltage $v_o(t)$ at the output terminals.

In this example we wish to analyze the behavior of the device of Fig. 5.1.11 for steady-state sinusoidal operation as a microphone. We assume that the driving force (pressure) is a sinusoidal function of time with constant amplitude and ask for the steady-state response of the output voltage. In the process we make approximations appropriate for the analysis of devices of this type.

Because the general properties of the coupling network should be described without taking into account the external elements, we begin with the electric field coupling network. We neglect fringing fields at the edges of the plates and describe the capacitance of this

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* L. L. Beranek, *Acoustic Measurements,* Wiley, New York, 1949, pp. 173–176 and pp. 211–224.

electrically linear system as

$$C(x) = \frac{\epsilon_0 A}{x}.$$ (a)

Thus the electrical terminal relation for the coupling network is

$$q(v, x) = C(x)v = \frac{\epsilon_0 A v}{x}.$$ (b)

The system is conservative, hence we obtain the force f^e as

$$f^e = \frac{\partial W_e'(v, x)}{\partial x}.$$ (c)

Because the system is electrically linear, we express the electric coenergy as

$$W_e' = \tfrac{1}{2} C(x)v^2 = \frac{\epsilon_0 A v^2}{2x}.$$ (d)

We use this expression with (c) to evaluate the force

$$f^e = -\frac{\epsilon_0 A v^2}{2x^2}.$$ (e)

The equation of motion for the mechanical node (x) is written by referring to the circuit of Fig. 5.1.11b:

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + K(x - l) = f^e + f(t).$$ (f)

We use (e) in this expression to rewrite (f) as

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + K(x - l) + \frac{\epsilon_0 A v^2}{2x^2} = f(t).$$ (g)

Before equations for the electric circuit shown in Fig. 5.1.11b are written, we shall make some appropriate approximations to simplify the problem. The circuit comprised of capacitance C_1 and resistance R_1 is used to isolate the output terminals from the bias voltage V_s and is not intended to affect the dynamic behavior of the system in normal operation. Consequently, with a driving frequency ω we assume that the values of R_1 and C_1 satisfy the inequalities

$$\frac{1}{\omega C_1} \ll R_1 \quad \text{and} \quad R_1 \gg R$$ (h)

over the frequency range of interest. Thus the output voltage $v_o(t)$ is essentially the time-varying component of v and the current through R_1 can be neglected compared with the current through R.

Using the inequality (h) we now write the node equation for the electric circuit as

$$\frac{dq}{dt} = \frac{V_s - v}{R} = \frac{d}{dt}\left(\frac{\epsilon_0 A v}{x}\right)$$ (i)

and describe the output voltage as

$$-v_o(t) = R\frac{dq}{dt} = R\frac{d}{dt}\left(\frac{\epsilon_0 A v}{x}\right);$$ (j)

(g), (i), and (j) are the general equations from which we can calculate the output voltage $v_o(t)$ once the driving force $f(t)$ is specified. Note that these equations are nonlinear.

This capacitor microphone is representative of a class of devices constructed and operated purposely to behave as linear devices. Nonlinear effects cause distortion and loss of fidelity. Linearization techniques are especially meaningful because they are appropriate under conditions that must be fulfilled in construction and operation to achieve linearity.

We now use the technique presented in Section 5.1.2 to linearize (g), (i), and (j) for small-signal operation about a static equilibrium. We define the static equilibrium by requiring that all time derivatives and $f(t)$ be zero. Thus from (j) the equilibrium value for $v_o(t)$ is zero. Defining the equilibrium values of x and v as X and V, respectively, we find the relations from (g) and (i):

$$K(X - l) + \frac{\epsilon_0 A V^2}{2X^2} = 0, \tag{k}$$

$$V_s - V = 0. \tag{l}$$

It is clear that the equilibrium value of the terminal voltage is the bias voltage V_s and that the equilibrium position X is determined from the cubic equation. This equation and the properties of the equilibria are similar to those studied in Example 5.1.1. For our purposes here it suffices to state that we select the solution of (k) that represents a stable equilibrium position X.

We now assume the two variables x and v to be perturbed from their equilibrium values by small time-dependent functions $x'(t)$ and $v'(t)$. Thus

$$x(t) = X + x'(t),$$

$$v(t) = V_s + v'(t).$$

We substitute these variables into (g), (i), and (j), subtract out equilibrium terms, and retain only linear terms in the perturbation variables to obtain

$$M\frac{d^2 x'}{dt^2} + B\frac{dx'}{dt} + K_o x' + C_o E_o v' = f(t), \tag{m}$$

$$C_o\frac{dv'}{dt} - C_o E_o\frac{dx'}{dt} = -\frac{v'}{R}, \tag{n}$$

$$v_o(t) = v'(t), \tag{o}$$

where we have defined the following constants:

$C_o = \dfrac{\epsilon_0 A}{X}$ is the capacitance at equilibrium,

$E_o = \dfrac{V_s}{X}$ is the magnitude of the electric field intensity between the plates at equilibrium,

$K_o = K - C_o E_o^2$ is the effective (net) spring constant and is positive for the stable equilibrium we are using.

Our interest here is in the steady-state response of the system to a sinusoidal driving force. Thus (see Section 5.1.1 and Fig. 5.1.11) we assume that

$$f(t) = -A\, p(t) = \text{Re}\, (Fe^{j\omega t}) = F\cos \omega t, \tag{p}$$

where F and ω are positive real constants. The system equations are linear with constant coefficients; thus we assume solutions of the form

$$x'(t) = \text{Re} \ (\hat{X}e^{j\omega t}),$$

$$v_o(t) = v'(t) = \text{Re} \ (\hat{V}e^{j\omega t}),$$

where $\hat{X}$ and $\hat{V}$ are complex amplitudes. We substitute these forms of the dependent variables into (m) and (n), cancel the $e^{j\omega t}$, and drop the Re to obtain the algebraic equations.

$$[(j\omega)^2 M + j\omega B + K_o]\hat{X} + C_o E_o \hat{V} = F, \tag{q}$$

$$\left(j\omega C_o + \frac{1}{R}\right)\hat{V} = j\omega C_o E_o \hat{X}. \tag{r}$$

We solve these two equations to find the complex amplitude of the output voltage $\hat{V}$ as a function of the amplitude of the driving force F:

$$\hat{V} = \left[\frac{j\omega C_o R E_o}{(K_o - \omega^2 M + j\omega B)(j\omega C_o R + 1) + j\omega (C_o E_o)^2 R}\right] F. \tag{s}$$

This expression could be used to determine the time response $v_o(t)$ for any set of parameters, value of frequency, and amplitude of drive. It is customary, however, to describe the steady-state sinusoidal response by plotting the magnitude (and often the phase) of the complex amplitude as a function of frequency for a constant input amplitude. Such a plot could be made for (s), but the denominator is quite complex and in its general form obscures the fact that different physical phenomena predominate in different frequency ranges.

We plot the amplitude of the transfer function $\hat{V}/F$ as a function of frequency by making approximations to simplify the expression in three frequency ranges. Our approximations are those made to achieve good microphone design.*

We consider first the low frequency behavior of the microphone and set the limit of (s) as $\omega \rightarrow 0$ to obtain

$$\left(\frac{\hat{V}}{F}\right)_{\text{lf}} = \frac{j\omega C_o R E_o}{K_o}. \tag{t}$$

The operation at very low frequencies can be interpreted as follows: first, the velocity and acceleration are so small that the inertia and damping forces are much smaller than the spring force in (g) and can be neglected. Next, the perturbation voltage $\hat{V}$ has negligible effect in the force equation and the term $C_o E_o \hat{V}$ can be neglected in (q). These two assumptions lead to the result in (t) if we recognize that the first term is small compared to the second on the left hand side of (r).

The fact that we can neglect the $\hat{V}$ term in (q) indicates that from a mechanical viewpoint the microphone capacitance is constrained to constant voltage. Thus the spring constant K_o includes the electrical spring constant due to the constant (bias) voltage [see (m)].

In summary, in the low-frequency limit the microphone capacitance operates at constant voltage and the mechanical system behaves as a spring, which includes electrical "spring" effects.

The low-frequency approximation breaks down when the frequency becomes large enough that

$$\omega C_o R \approx 1.$$

* *Ibid.*, pp. 211–218 and especially Fig. 5.30a.

This occurs in conventional microphones around a frequency of 10 Hz.* In the region of transition between the low-frequency and the mid-frequency ranges the transfer function is

$$\frac{\hat{V}}{F} = \frac{j\omega C_o R E_o}{K_o(j\omega C_o R + 1) + j\omega (C_o E_o)^2 R} \cdot \tag{u}$$

We next define the mid-frequency range as starting at a frequency such that

$$\omega \gg \frac{1}{RC_o}$$

and continuing until inertia and damping forces become appreciable. This results in a mid-frequency transfer function

$$\left(\frac{\hat{V}}{F}\right)_{mf} = \frac{E_o}{K} \cdot \tag{v}$$

Note that the denominator is the spring constant K alone, without the electrical spring force [see (m)]. This is an indication that the microphone capacitance is operating at constant charge because a parallel plate capacitor with negligible fringing fields and constant charge will have a force that is independent of plate spacing. The constraint of constant charge results because the resistance R is so large ($R \gg 1/\omega C_o$) that appreciable charge cannot flow on or off the plates. We can obtain this mid-frequency solution by neglecting the $1/R$ term in (r) and using that result to eliminate the term involving $\hat{V}$ in (q).

The mid-frequency range is the normal operating range of the microphone. The amplitude and phase of the transfer function are constant over this range; thus the output voltage is an exact replica of the input force and high fidelity is obtained.

The approximate transfer function for the mid-frequency range (v) breaks down when inertia and damping forces become appreciable. In practice, the mechanical system is lightly damped and a resonance occurs. In fact, elaborate means are used to provide additional mechanical damping to reduce the size of the resonance peak.† With a resonance, the mid-frequency transfer function breaks down when the frequency becomes high enough so that

$$\omega^2 M \approx K.$$

The frequency corresponding to this transition is usually around 10,000 Hz.‡

In the transition between the mid-frequency and high-frequency ranges the transfer function is

$$\frac{\hat{V}}{F} = \frac{E_o}{K - \omega^2 M + j\omega B} \cdot \tag{w}$$

Note that in this region the microphone capacitance is still operating at constant charge, as evidenced by the presence of only the mechanical spring constant K in the denominator.

The high-frequency range is defined by the condition

$$\omega^2 M \gg K \quad \text{and} \quad \omega^2 M \gg \omega B.$$

Thus the high-frequency transfer function is

$$\left(\frac{\hat{V}}{F}\right)_{hf} = -\frac{E_o}{\omega^2 M} \cdot \tag{x}$$

* *Ibid.*
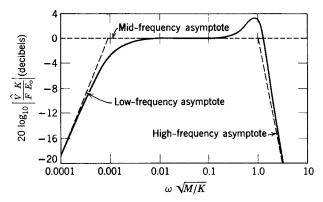† *Ibid.*, pp. 217 and 220.
‡ *Ibid.*, p. 220.

Fig. 5.1.12   Frequency response of capacitor microphone.

In this case the microphone capacitance is still operating at constant charge and the mechanical dynamics are determined completely by the mass M.

The amplitude of the transfer function $\hat{V}/F$ is plotted as a function of frequency in Fig. 5.1.12. The approximate solutions, commonly called asymptotes, given by (t), (v), and (x) are shown as dashed lines. The more exact solutions in the transition regions (u) and (w) are shown as solid lines. The curves are plotted for the following relations among parameters:

$$\frac{B}{\sqrt{MK}} = 0.7; \qquad \frac{K_o}{K} = 0.9; \qquad C_o R \left(\frac{K}{M}\right)^{1/2} = 1000.$$

Note that in Fig. 5.1.12 both the amplitude and frequency scales are logarithmic. This is a Bode plot, used for plotting frequency-response data.*

5.1.3   Physical Approximations

There are two indices by which the usefulness of an engineering model can be measured. First, there is the degree to which it represents the essential features of the physical situation. Second, there is the amount of effort required to use it for an analytical study. Obviously, these two considerations are in conflict and the choice of a model represents a compromise.

The selection of an appropriate model demands an awareness of the interplay between physical approximations and mathematical techniques; for example, if dissipation mechanisms are not significant in a given situation, it may be a simple matter to describe the nonlinear dynamics. In the opposite extreme, if dissipation dominates the dynamics, it may also be possible to include nonlinear effects. In the intermediate case of moderate damping nonlinear effects may be included only with a great deal of effort.

In this section a simple example is used to illustrate how the mathematical model can be simplified by recognizing the important physical effects at the

* F. E. Nixon, *Principles of Automatic Controls*, Prentice-Hall, Englewood Cliffs, N.J., 1953, pp. 165–174.

outset. This is often done in circuit theory. Suppose that the voltage v in the LR circuit of Fig. 5.1.13 is given and the current i is to be computed. We can, of course, solve this problem with little trouble. This allows us to see that if interest is confined to the current at very low frequencies
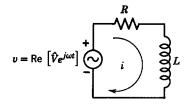

Fig. 5.1.13   *L-R* circuit to illustrate appropriate electrical approximations when the period of excitation is extreme compared with the time constant L/R.

$$v \simeq iR; \qquad (5.1.24)$$

that is at low frequencies virtually all of the voltage drop is across the resistance. By contrast, at high frequencies the inductive reactance greatly exceeds the resistance and virtually all the voltage drop is across the inductance. In this limit

$$v \simeq L\frac{di}{dt}. \qquad (5.1.25)$$

The frequency is considered to be low or high, depending on the relationship between the period of excitation $2\pi/\omega$ and the time constant L/R of the circuit. This is normally expressed as a ratio of inductive reactance and resistance. Thus, when $\omega L/R \ll 1$, (5.1.24) can be used; and, when $\omega L/R \gg 1$, (5.1.25) can be used. When $\omega L/R \approx 1$, neither approximation is appropriate.

By contrast with the circuit of Fig. 5.1.13, most electromechanical problems are represented by nonlinear equations unless the dynamics are limited to incremental motions. In these situations approximations analogous to those represented by (5.1.24) and (5.1.25) are useful. The electromechanical approximation, however, is more subtle because the frequency or characteristic time constant of the system is often not known until after the problem has been solved. In Fig. 5.1.13 we knew at the outset that the current i had the same frequency as the driving voltage. With the circuit coupled to a mechanical system and natural or free motions of the system under consideration (not the sinusoidal steady state resulting from a given driving function), the temporal behavior of the system is at least in part determined by mechanical effects. Hence the characteristic frequencies of the response to initial conditions can be low or high, compared with the natural frequencies of the electrical system.

The pair of coils shown in Fig. 5.1.14 provides a concrete example of the physical consequences of making electrical approximations in a magnetic field system. A pair of fixed coils is driven by the constant current $i_2 = I$ and arranged to give a uniform magnetic flux density B_o in the region of a pivoted coil with the angular deflection θ. The rotatable coil is short-circuited but has a
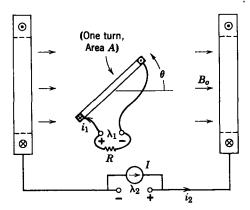
Fig. 5.1.14 A pivoted coil is free to rotate with the angular deflection θ in a uniform magnetic field produced by i_2.

resistance R, which is represented by a resistance connected to the terminals. We wish to study the mechanical response of the coil when it is given an initial angular velocity $d\theta/dt = \Omega$ at the angular position $\theta = 0$.

The equations of motion are found by first writing the electrical terminal relations as

$$\lambda_1 = L_1 i_1 + M i_2 \sin \theta, \tag{5.1.26}$$

$$\lambda_2 = M i_1 \sin \theta + L_2 i_2, \tag{5.1.27}$$

where L_1, M, and L_2 are constants. The dependence of the mutual inductance on θ should be evident from Fig. 5.1.14, and in the absence of magnetic materials the self-inductances are independent of θ. For this electrically linear system the coenergy follows from (5.1.26) and (5.1.27) as

$$W' = \tfrac{1}{2} L_1 i_1{}^2 + M i_1 i_2 \sin \theta + \tfrac{1}{2} L_2 i_2{}^2. \tag{5.1.28}$$

Hence the electrical torque is

$$T^e = \frac{\partial W'}{\partial \theta} = M i_1 i_2 \cos \theta. \tag{5.1.29}$$

The rotatable coil has a moment of inertia J, so that if mechanical damping is ignored the mechanical equation of motion is $(i_2 M = IM = AB_o)$.

$$J \frac{d^2\theta}{dt^2} = AB_o i_1 \cos \theta. \tag{5.1.30}$$

The electrical equation requires that $-i_1 R = d\lambda_1/dt$, which, in view of (5.1.26) and the fact that $M i_2 = AB_o$, is

$$-i_1 R - L_1 \frac{di_1}{dt} = AB_o \cos \theta \frac{d\theta}{dt}. \tag{5.1.31}$$

The voltage on the right-hand side of this equation is induced by the motion of the coil through the magnetic induction B_o. The equation expresses the fact that this "speed voltage" is absorbed by the self-inductance of the coil and by the resistance R. Note that the relative magnitudes of these terms on the left are determined by the same considerations discussed in connection with Fig. 5.1.13. Now, however, the current i_1 has a temporal behavior that depends on the mechanical deflection of the coil. From (5.1.31) it is clear that the moment of inertia plays a part in determining whether the inductive reactance or the resistance (or both) are significant.

We approach the problem here by assuming at the outset that one or the other of the terms on the left in (5.1.31) dominates, investigating the analytical consequences, and returning to check the validity of the initial assumption by using the predicted response. Suppose first that

$$|i_1R| \gg L_1 \left| \frac{di_1}{dt} \right|. \tag{5.1.32}$$

Then, i_1 can be found explicitly from (5.1.31) and substituted into (5.1.30). The result, after some trigonometric manipulation, has the form

$$\frac{d}{dt}\left[J\frac{d\theta}{dt} + \frac{(AB_o)^2}{2R}(\cos\theta \sin\theta + \theta) \right] = 0. \tag{5.1.33}$$

Here one derivative has been factored to show that the quantity in brackets is constant. The initial conditions that $d\theta/dt = \Omega$ when $\theta = 0$ fix this constant so that (5.1.33) can be integrated.

$$J\frac{d\theta}{dt} + \frac{(AB_o)^2}{2R}(\cos\theta \sin\theta + \theta) = J\Omega. \tag{5.1.34}$$

This first-order equation can be integrated to find $\theta(t)$ without approximations concerning the amplitude of the angular deflection θ. This approach to nonlinear dynamics is the subject of Section 5.2.2. It serves our purpose here to establish the physical significance of the approximation by limiting consideration to small amplitude (linearized) deflections about $\theta = 0$, in which case (5.1.34) is approximated by

$$\frac{d\theta}{dt} + \frac{\theta}{\tau_o} = \Omega, \tag{5.1.35}$$

where

$$\tau_o = \frac{RJ}{(AB_o)^2}.$$

In view of the initial conditions, this linear equation has the solution
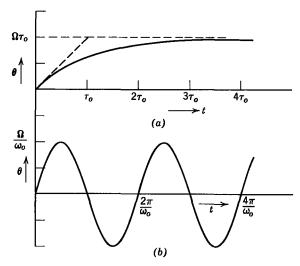
$$\theta = \Omega\tau_o(1 - e^{-t/\tau_o}), \tag{5.1.36}$$

Fig. 5.1.15   Response in angular deflection of the rotatable coil of Fig. 5.1.14 to an initial angular velocity Ω at $\theta = 0$: (a) motion dominated by electrical resistance $(\tau_o \gg L_1/R)$; (b) flux λ_1 conserved at zero $(2\pi/\omega_o \ll L_1/R)$ as it would be if the resistance R were very small.

which is sketched in Fig. 5.1.15a. Remember that this expression is valid only if the inequality of (5.1.32) is satisfied. Use of (5.1.36) shows that the inequality requires

$$\tau_o \gg \frac{L_1}{R} . \tag{5.1.37}$$

Note that the electromechanical time constant τ_o is proportional to the moment of inertia J. When τ_o is large enough to satisfy (5.1.37), it simply means that the inertial effect slows the motion to the point at which the inductive reactance (which depends on the rate of change of i_1) is of negligible influence.

This approximation is typical of those used in the analysis of large, magnetic-field type devices such as rotating machines. Mechanical and electro-mechanical time constants are so long compared with electrical time constants that mechanical and electromechanical transients are assumed to occur with the electrical system always operating in the steady state $[L_1(di_1/dt)$ neglected in the example just completed]. Conversely, electrical transients are so fast that they are assumed to occur with the mechanical system operating at constant speed.

It is important to see that an equation of motion in the form of (5.1.35) would be obtained if the magnetic induction B_0 were absent but the coil rotated in a viscous fluid. In the limit in which the reactance of the coil can be ignored the magnetic field and short-circuited coil combine to act as a mechanical damper. This is the limit used in synchronous machines when

short-circuited damper windings are added to the rotor to assist in damping electromechanical oscillations (see Section 4.1.6a and Example 5.1.2).

It is evident from the solution (5.1.36) that there are no oscillations (as would be expected from "springlike" torques). The reactance represents the contribution of self-currents to the total magnetic field. When $L_1 di_1/dt$ is ignored in (5.1.31), it means that we are ignoring the magnetic field induced by the current i_1. This kind of physical approximation is useful in dealing with continuum interactions (see Chapter 10). We shall also find that in the limit in which electrical dissipation dominates, media tend to "ooze" rather than "bounce."

In both lumped parameter and continuum electromechanics it is often meaningful to model a conducting medium as "perfectly" conducting. This model is illustrated here by taking an extreme (to that so far considered) in which characteristic times of the electromechanical system are short enough to warrant neglecting the drop across the resistance compared with that across the reactance; that is, the first term in (5.1.31) is ignored compared with the second. That expression can then be integrated to give

$$i_1 = - \frac{AB_o}{L_1} \sin \theta. \tag{5.1.38}$$

Here, we assume that when the motion is initiated at $t = 0$, not only do $\theta = 0$ and $d\theta/dt = \Omega$ but $i_1 = 0$. Note that (5.1.38) requires that the flux λ_1 linking the rotatable coil be conserved. The initial conditions require that this flux be conserved at $\lambda_1 = 0$.

It follows from (5.1.38) and (5.1.30) that the equation of motion is

$$J \frac{d^2\theta}{dt^2} = - \frac{(AB_o)^2}{L_1} \sin \theta \cos \theta. \tag{5.1.39}$$

This nonlinear expression can be integrated without further approximations. For now, we delay this nonlinear problem until Section 5.2.1 and illustrate the physical consequences of the approximation by considering small amplitude deflections about $\theta = 0$. Then (5.1.39) becomes

$$\frac{d^2\theta}{dt^2} + \omega_o^2 \theta = 0, \tag{5.1.40}$$

where

$$\omega_o = \frac{AB_o}{\sqrt{JL_1}}.$$

The solution to this equation, which satisfies the initial conditions, is

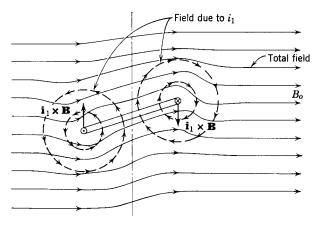$$\theta = \frac{\Omega}{\omega_o} \sin \omega_o t. \tag{5.1.41}$$

Fig. 5.1.16 Distortion of the initially uniform magnetic field caused by angular deflection of coil shown in Fig 5.1.14. The flux of the moveable coil is constrained to be zero.

This expression is plotted in Fig. 5.1.15b, where it can be compared with the loss-dominated case.

When the electrical dissipation can be ignored, the magnetic torque has the same effect on the motion as a torsional spring (in the nonlinear case, a nonlinear torsional spring). The reason for this can be seen physically with the help of Fig. 5.1.16. Remember that in this limit the total flux λ_1 through the rotatable coil is constrained to be zero. With the angular deflection shown in Fig. 5.1.16, the flux density B_o links the coil, thus contributing to λ_1. This flux must be canceled by a flux induced by the current i_1. The deflection shown in Fig. 5.1.16 is accompanied by the currents, as indicated, which induce a flux that cancels that from B_o. The total magnetic field is distorted to remain tangential to the plane of the coil. Note that the magnetic force $\mathbf{i}_1 \times \mathbf{B}$ tends to restore the coil to the angle $\theta = 0$. Because the induced current is proportional to the angular deflection (and not to its rate of change), the magnetic torque is similar to that of a spring.

The assumption that the inductive reactance is of primary importance to the dynamics is equivalent to recognizing that the effect of the magnetic field induced by the motion is on the same order as that of the imposed magnetic field. More is said on this point in Section 7.0, in which the same physical arguments appear in the context of a distributed interaction. In the context of lumped parameters perfectly conducting media (in a magnetic field system) behave in a "springlike" or "bouncing" fashion. In continuous media (e.g., a "perfectly" conducting fluid) the same approximation leads to the possibility of wavelike motions, as illustrated in Section 12.2.3.

Our remarks in this section have been limited to electrical approximations that are appropriate in magnetic field systems. We could further illustrate the

role of electrical dissipation in electric field systems. This was done implicitly however, in Example 5.1.4, in which it was shown that the electrical variables of a capacitor microphone were essentially constrained to constant potential and constant charge in the low and high-frequency ranges, respectively; that is, if the capacitor plate in Fig. 5.1.11 responded at a low frequency, the $R \, dq/dt$ drop across the resistance R could be ignored and the potential on the plate taken as the constant V_g. In the opposite extreme rapid variations in the capacitance of the microphone meant that there was little chance of charge leaking off through the resistance R. The result was an essentially constant charge on the movable plate. The critical parameter that determined which approximation was valid was $\omega R C_o$ or again essentially the ratio of the electrical time constant and the period of the mechanical response. The simplifying feature of Example 5.1.4 was the known response frequency ω. We could, however, easily envision a situation like the one considered in this section, in which the characteristic dynamic time would not be known until after the problem had been solved.

5.2 NONLINEAR SYSTEMS

As we have seen, most lumped-parameter electromechanical devices are described in general by nonlinear differential equations. Section 5.1 was devoted to showing that for many purposes these equations can be approximated by linearized equations. There are cases in which the nonlinear dynamics are essential, and indeed nonlinear interactions represent possibilities for engineering applications that are not available within the framework of linear systems. Unfortunately, there is no general mathematical theory to cover the solution of all types of nonlinear differential equation. This is not surprising, since nonlinear equations include all types that are not linear, as defined in Section 5.1.1.

The most direct way of obtaining numerical answers to nonlinear problems is to use machine computation, either analog or digital. In some simple cases it is possible to integrate the equations of motion. In the following two sections we illustrate two classes of these simple systems and the analytical techniques that are useful in obtaining solutions. Our objective is not only to study techniques for describing nonlinear systems but to gain a deeper physical insight into electromechanical dynamics.

5.2.1 Conservative Systems

In mechanics, if the energy remains constant throughout the motion of a system, the system is said to be conservative; that is, although the velocity and position of a mass change with time, the total energy is conserved at its
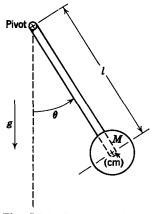
initial value. As we show in this section, this provides a basis for finding the motions of many electromechanical systems.

A simple mechanical system that illustrates the approach while allowing considerable physical insight is the simple pendulum of Fig. 5.2.1. It consists of a mass M whose center of mass is connected by a rigid, weightless rod of length l to a frictionless pivot. We consider the motion in which there is no externally applied torque except that due to gravity, which acts downward as shown.

The torque equation is therefore

Fig. 5.2.1 Simple pendulum with an angular deflection $\theta(t)$ in the gravitational field g.

$$J \frac{d^2\theta}{dt^2} = T_g, \qquad (5.2.1)$$

where the moment of inertia $J = Ml^2$ and T_g, the torque due to gravity, can be written as

$$T_g = -\frac{\partial V}{\partial \theta}; \qquad V = -Mgl\cos\theta. \qquad (5.2.2)$$

It is useful to write the torque as the derivative of the potential V because (5.2.1) can then be written as

$$\frac{d}{dt}\left[\frac{J}{2}\left(\frac{d\theta}{dt}\right)^2 + V\right] = 0. \qquad (5.2.3)$$

The best way to see that this is true is to take the first time derivative in (5.2.3) and see that (5.2.1) is recovered. It follows from (5.2.3) that the quantity in brackets is constant, or conserved. If we call this constant E,

$$\frac{J}{2}\left(\frac{d\theta}{dt}\right)^2 = E - V(\theta). \qquad (5.2.4)$$

To understand the physical significance of this equation it is helpful to think in terms of the potential plot shown in Fig. 5.2.2. The constant E, which is the sum of the kinetic and potential energies, is independent of θ. According to (5.2.4), the kinetic energy, hence the square of the angular velocity, is proportional to the difference between E and $V(\theta)$. This is shown graphically in Fig. 5.2.2. It is apparent from the diagram that at points (*a*) and (*b*), at which $|\theta| = \theta_m$, the angular velocity is zero, whereas at $\theta = 0$ the magnitude of the angular velocity has its largest value. The kinetic energy $J(d\theta/dt)^2/2$ is always positive. Hence, given the value of E, we can picture the angular
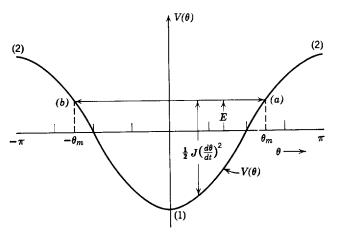
Fig. 5.2.2 Potential well $V(\theta)$ for the pendulum of Fig. 5.2.1 and for the rotatable coil of Fig. 5.2.4. If the pendulum is released from a stationary state at $\theta = \theta_m$, it will have an excursion between the points (a) and (b). The square of the angular velocity at any given position is proportional to the vertical distance between the constant E line and the potential $V(\theta)$.

deflection as limited to those regions of the potential plot in which the constant E line is above the potential $V(\theta)$. For the value of E shown in Fig. 5.2.2 the pendulum oscillates between the angles $\theta = \theta_m$ and $\theta = -\theta_m$.

The invariant E is established by the initial conditions. Suppose that at $t = 0$, $\theta = \theta_m$ and $d\theta/dt = 0$; that is, the pendulum is released from an initial static deflection $\theta = \theta_m$. Then from (5.2.4) $E = V(\theta_m)$ and

$$\frac{J}{2}\left(\frac{d\theta}{dt}\right)^2 = V(\theta_m) - V(\theta). \tag{5.2.5}$$

Similarly, the pendulum could be given an initial velocity $(d\theta/dt)_m$ at $\theta = 0$, and it follows from (5.2.4) that $E = (J/2)(d\theta/dt)_m^2 + V(0)$. If the initial kinetic energy exceeds $V(\pi) - V(0)$, the line of constant E in Fig. 5.2.2 never intersects the potential curve and the pendulum deflection increases monotonically. This simply means that, given a large enough initial energy, the pendulum rotates continuously on its pivot, moving rapidly at $\theta = 0$, 2π, 4π, . . . and slowing down at $\theta = \pi$, 3π, ...

Note that at the angles $\theta = 0$, π, . . . the pendulum can be in static equilibrium, for at these points $\partial V/\partial\theta = 0$ $(T_g = 0)$. The question whether these equilibria are stable can be answered in terms of the potential plot. Suppose that the pendulum is given an initial static position $\theta = 0$. This establishes the constant E line in Fig. 5.2.2 as passing through point (1). To see if the equilibrium is stable, the pendulum is given a slight kinetic energy, which

raises the constant E line just above point (1) and shows that small amplitude oscillations result. In the sense defined in Section 5.1.2a we say that this equilibrium is stable. Similarly, if the pendulum is placed in static equilibrium at (2), the constant E line just grazes the peaks of the potential curve. To test for stability the pendulum is given a slight additional kinetic energy (greater E). It is clear that the pendulum does not tend to return to position (2), which is a point of unstable static equilibrium. This is not surprising to someone who has tried to balance a broom on his finger. The pendulum is upside down at $\theta = \pi$.

As a consequence of our deductions concerning Fig. 5.2.2, we can interpret the system behavior as if the moment of inertia were a particle that slides without friction on a physical "hill" with the shape $V(\theta)$. Motions within a *potential well* are bounded and therefore stable.

It is worthwhile to place the linear stability theory of Section 5.1.2a in perspective by relating it to the potential well. The torque T_g can be expanded about a point of static equilibrium $\theta = \Theta$:

$$T_g = -\frac{\partial V}{\partial \theta}(\Theta) - (\theta - \Theta)\frac{\partial^2 V}{\partial \theta^2}(\Theta). \tag{5.2.6}$$

Because the equilibrium is static, the first term on the right is zero. Hence (5.2.1) has been linearized and is

$$J\frac{d^2\theta}{dt^2} + \left[\frac{\partial^2 V}{\partial \theta^2}(\Theta)\right](\theta - \Theta) = 0. \tag{5.2.7}$$

From this constant coefficient linear equation it follows that the solutions will be oscillatory, hence stable, at points of zero slope on the potential plot at which

$$\frac{\partial^2 V}{\partial \theta^2} > 0. \tag{5.2.8}$$

Our linearized techniques of Section 5.1.2a tell us that small perturbations about a point in the bottom of a potential well (1) are stable, whereas those from the peaks (2) are unstable. We shall see in the examples that follow cases in which even though small amplitude deflections are unstable large deflections remain bounded.

We return to this example of the mechanical pendulum to show how (5.2.4) can be integrated to determine the detailed temporal behavior. Before doing so, however, it would be more to the point to see how these ideas can be extended into the context of electromechanics.

In Fig. 5.2.3 an electromechanical system is represented schematically by a mechanical system connected to an electromechanical coupling network.
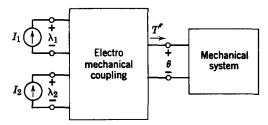
Fig. 5.2.3 Electromechanical system in which conservation of an energy function can be used to determine the dynamics. The mechanical system is composed of lossless elements and one of each of the electrical terminal variables is held constant.

This network, as defined in Chapter 3, does not contain elements that dissipate energy, but, of course, it in turn couples to an electrical system that does contain such elements. If, however, the electrical terminals are constrained so that a variable at each of the terminal pairs is held fixed, we expect to find a constant of the motion for the total electromechanical system. This is true because any additions to the total electromechanical energy of the system must be made through the electrical terminals. If a terminal variable is fixed, there is either no flow of energy or no flow of coenergy through that terminal pair; for example, in the case shown in Fig. 5.2.3 an increment of coenergy at the (i_1, λ_1) terminal pair is $\lambda_1 \, di_1$, which is constrained to zero as long as $I_1 = $ constant. Now we know from Chapter 3 that the torque of electrical origin T^e can be found as the derivative of an "energy function" written as a function of θ and one of each of the electrical terminal variables (the "independent" variables): for example, in Fig. 5.2.4

$$T^e = \frac{\partial W'}{\partial \theta} (i_1, i_2, \theta), \qquad (5.2.9)$$

where W' is the coenergy function (see, for example, Section 3.1.2b). If i_1 and i_2 are constrained to be constant, then T^e is the derivative of a known function of θ:

$$T^e = \frac{\partial W'}{\partial \theta} (I_1, I_2, \theta). \qquad (5.2.10)$$

Because the mechanical system is composed of elements that do not dissipate energy, the mechanical torque can also be written as the derivative with respect to θ of an
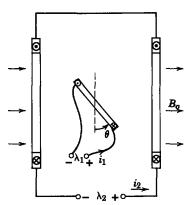

Fig. 5.2.4 A pair of fixed coils is excited in series by the current i_2. A pivoted coil has the angular deflection θ and the electrical terminal variables (i_1, λ_1). When both coils are driven by constant current sources, the potential well is as shown in Fig. 5.2.2.

energy function, which we might call $U(\theta)$

$$T^m = -\frac{\partial U}{\partial \theta}.$$ (5.2.11)

It follows that the torque equation can be written as

$$J\frac{d^2\theta}{dt^2} + \frac{\partial V}{\partial \theta} = 0,$$ (5.2.12)

where

$$V(\theta) = U(\theta) - W'(I_1, I_2, \theta),$$ (5.2.13)

which now takes on the same form as (5.2.3).

Example 5.2.1 is based on this generalization of the potential well to include the electromechanical energy function.

Example 5.2.1. The electromechanical system shown in Fig. 5.2.4 has the schematic description of Fig. 5.2.3. The pivoted coil has the terminal variables (i_1, λ_1), whereas the fixed coils are connected in series, with the terminal variables (i_2, λ_2). There is no magnetic material in the problem, so that self-inductances remain constant and the electrical terminal relations can be written as

$$\lambda_1 = L_1 i_1 + M(\theta) i_2,$$ (a)

$$\lambda_2 = M(\theta) i_1 + L_2 i_2,$$ (b)

where for this particular case $M(\theta) = M_o \cos\theta$, L_1, L_2, and M_o are constants. From the terminal relations the coenergy $W'(i_1, i_2, \theta)$ follows as

$$W' = \tfrac{1}{2}L_1 i_1{}^2 + \tfrac{1}{2}L_2 i_2{}^2 + M(\theta) i_1 i_2.$$ (c)

The first two terms in this expression are constant and can be absorbed in the constant of the motion E. Hence from (5.2.13) we have

$$V = -I_1 I_2 M(\theta).$$ (d)

For the particular case of Fig. 5.2.4

$$V = -I_1 I_2 M_o \cos\theta.$$ (e)

This potential has the same form as the mechanical pendulum of Fig. 5.2.1; hence the developments relevant to dynamics of the pendulum are equally applicable here. For quantitative purposes Mgl is replaced by $I_1 I_2 M_o$.

Note that the state of stable static equilibrium at (1) in Fig. 5.2.2 now corresponds to the situation in which the magnetic field generated by i_1 at the center of the pivoted coil is aligned with the field produced by i_2. If either I_1 or I_2 is made negative, the potential well in Fig. 5.2.2 is inverted, with $\theta = 0$ becoming an unstable static equilibrium and point (2) becoming a stable equilibrium.

Now that we have discussed the basic considerations involved in using potential-well techniques for describing electromechanical problems it would be interesting to embark on examples that illustrate characteristic dynamic behavior. It is not often that an engineer is concerned with the detailed temporal behavior of a system. Example 5.2.2 illustrates how a knowledge of the

electrical terminal relations can be used to establish the significant features of dynamic behavior under a variety of electrical constraints. This approach is extremely useful because the relevant features of the electrical terminal relations can be found quantitatively by simple measurements or qualitatively by sketching the electric or magnetic fields.

Example 5.2.2. A coil (mass M) is mounted on a massless pendulum of length R, as shown in Fig. 5.2.5, to allow it to swing through the magnetic field generated by a pair of series-connected fixed coils. We wish to study the dynamical consequences of energizing the fixed coils with a current source I_2 and constraining the electrical terminals of the movable coil in two different ways. First, the pendulum motions that result when $i_1 = I_1$ or the current through the moving coil is constant provide us with an opportunity to illustrate how a combination of electrical and mechanical potentials is handled, a situation that can then be contrasted with the second case to be considered in which the terminals (i_1, λ_1) are constrained to zero flux linkage. This is the physical result if the terminals of the moving coil are short-circuited and the resistance of the coil is "small." This limit is discussed in Section 5.1.3, in which it is shown that the flux can be considered essentially constant if the current i_1 is limited by the self-reactance of the coil and not by the resistance. This demands that the characteristic time constant of the motion be short compared with the L/R time constant of the moving coil.

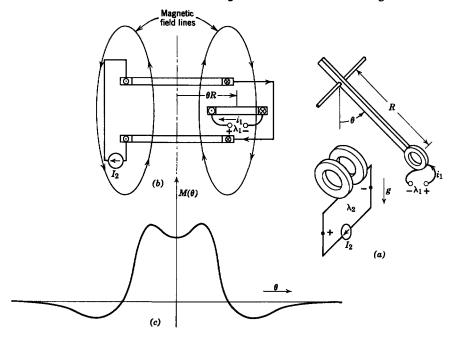The mutual inductance between the moving and fixed coils is shown in Fig. 5.2.5c. The

Fig. 5.2.5 (a) A coil is attached to a pendulum in such a way that when $\theta = 0$ it is directly between a pair of fixed coils energized by the current I_2; (b) top view of (a) showing the magnetic field produced by the fixed coils; (c) mutual inductance between the fixed and movable coils of (b).

dependence on θ can be understood qualitatively by considering Fig. 5.2.5b. When θR is very large, it is clear that the magnetic field generated by the fixed coils does not link the movable coil. As the movable coil comes into the vicinity of the fixed coils, it links a magnetic field having the opposite direction to that linked when it is directly between the coils. Hence the mutual inductance first becomes negative and then positive. If the diameter of the fixed coils is large compared with their spacing, the field, which tends to concentrate near the currents, will be smaller at the center ($\theta = 0$) than at off-center angles where the movable coil is adjacent to the fixed windings. Hence the mutual inductance is shown with a dip in the vicinity of $\theta = 0$.

This inductance could be measured by exciting the fixed coil with a sinusoidal current and measuring the induced voltage in the movable coil as a function of position. Therefore we can regard $M(\theta)$ as determined either qualitatively or quantitively. The electric terminal relations have the form of (a) and (b) in Example 5.2.1.

<center>CONSTANT CURRENT CONSTRAINTS</center>

First, consider the consequences of driving both fixed and movable coils by constant-current sources. The total potential is then the sum of a gravitational potential (5.2.2) and a potential due to the magnetic field [(d) of Example 5.2.1]:

$$V = -Mgl \cos \theta - I_1 I_2 M(\theta). \tag{a}$$

There are two possibilities. Either $I_1 I_2 > 0$, in which case the total potential appears as shown in Fig. 5.2.6a, or $I_1 I_2 < 0$, and $V(\theta)$ is as shown in Fig. 5.2.6b.
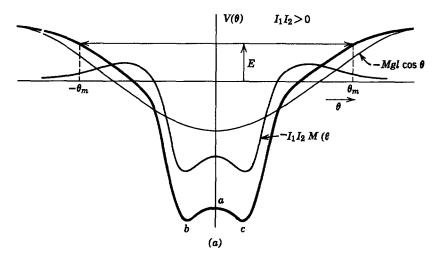
Consider first the case in which both currents are positive. If the pendulum holding the coil is released from an initial static position at $\theta = \theta_m$, the line of constant E appears as shown in Fig. 5.2.6a. The pendulum swings completely through the region of the fixed coils. The effect of the negative slope of the mutual inductance is not sufficient to decelerate it as it approaches them. Once the movable coil is between the fixed coils, the $\mathbf{I} \times \mathbf{B}$ force tends to accelerate it toward the center, except very near $\theta = 0$. The effect of the constant current constraints with both currents positive is to make the potential well centered on $\theta = 0$ even deeper than it would be without the field.

A similar experiment with one of the currents reversed results in motions characterized by the constant E line shown in Fig. 5.2.6b. For this case the pendulum released from the initial angle θ_m is reflected by the magnetic interaction with the fixed coil. This is expected, since the $\mathbf{I} \times \mathbf{B}$ force on the moving coil as it nears the origin is now in the direction required to retard the motion. Of course, given enough initial energy, the pendulum will pass on through the interaction region. Note that the pendulum could be trapped in a region near the origin.

A significant feature of the constant current dynamics is its dependence on the sign of the excitation current. If one of the currents is reversed, that part of the potential due to the magnetic field is turned upside down; for example, in Fig. 5.2.6b, in which one of the currents is reversed, the static equilibrium at the origin (a) is stable, and there are two additional points (d and e) at which the pendulum can be in stable static equilibrium. The equilibria (b and c) that were stable in Fig. 5.2.6a are replaced by unstable equilibria (b and c) in Fig. 5.2.6b.

<center>CONSTANT-CURRENT CONSTANT-FLUX CONSTRAINTS</center>

We now embark on describing the motions when the pendulum coil of Fig. 5.2.5 is constrained to constant (zero) flux. As is evident from the development, the electro-mechanical coupling must now be represented by a hybrid energy function, for neither all of the currents nor all of the fluxes are constrained to be constant.
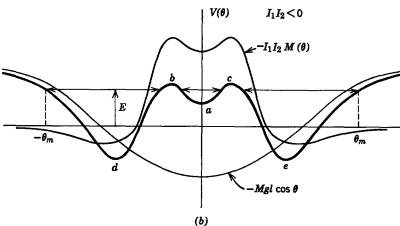
$V(\theta)$ $I_1 I_2 > 0$

E

$-Mgl \cos \theta$

θ_m

θ

$-I_1 I_2 M (\theta$

a

b c

$-\theta_m$

(a)

$V(\theta)$ $I_1 I_2 < 0$

$-I_1 I_2 M (\theta)$

b c

E

a

$-\theta_m$ θ_m

d e

$-Mgl \cos \theta$

(b)

Fig. 5.2.6   Potential plots for the system of Fig. 5.2.5 constrained to constant current: (a) both coil currents positive; (b) one coil current positive and one negative.

A schematic representation of the terminal constraints is shown in Fig. 5.2.7. To use the potential techniques, we require a function $W''(\theta)$ from which we can find the electrical torque by taking a derivative:

$$T^e = \frac{\partial W''}{\partial \theta}. \tag{b}$$

This function, with the terminations of Fig. 5.2.3, was simply the coenergy W', since in the coenergy function the currents are used as independent variables. For our present purposes it is helpful to recall that any of the energy functions are derived from a statement of conservation of energy for the electromechanical coupling.

$$i_1 \, d\lambda_1 + i_2 \, d\lambda_2 = dW + T^e \, d\theta; \tag{c}$$
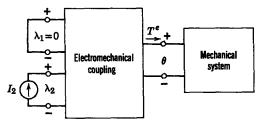
Fig. 5.2.7 Schematic representation of the system shown in Fig. 5.2.5a when the pendulum coil is constrained to zero flux.

for example, in the case in which the currents were held constant it was appropriate to transform this expression to one involving the currents i_1 and i_2 as independent variables (as discussed in Section 3.1.2b). For the present situation λ_1 and i_2 are fixed and therefore should be used as independent variables. Hence the second term in (c) is rewritten as

$$i_2 \, d\lambda_2 = d(i_2 \lambda_2) - \lambda_2 \, di_2 \tag{d}$$

so that (c) becomes

$$\lambda_2 \, di_2 - i_1 \, d\lambda_1 = dW'' - T^e \, d\theta, \tag{e}$$

where

$$W'' = i_2 \lambda_2 - W.$$

From (e) it follows that the electrical torque is given by (b). As for the energy and coenergy functions, W'' is found by integrating a form of the energy equation, which is now (e). In carrying out this integration, it must be remembered that i_2 and λ_1 are independent variables and must be used to express λ_2 and i_1 in (e). Thus (a) and (b) of Example 5.2.1 are written as

$$i_1 = \frac{\lambda_1}{L_1} - \frac{M(\theta)}{L_1} i_2, \tag{f}$$

$$\lambda_2 = \left(L_2 - \frac{M^2(\theta)}{L_1} \right) i_2 + \frac{M(\theta)}{L_1} \lambda_1. \tag{g}$$

Now, if the integration of (e) is carried out in the usual way (Section 3.1.1), we obtain

$$W'' = \frac{1}{2}\left(L_2 - \frac{M^2}{L_1} \right) i_2^2 - \frac{1}{2}\frac{\lambda_1^2}{L_1} + \frac{M}{L_1} i_2 \lambda_1. \tag{h}$$

The terminal constraints require that $i_2 = I_2$ and that $\lambda_1 = 0$. (The initial conditions determine the constant λ_1 the short-circuited pendulum coil will retain.) For the present purposes we assume that the coil is initially outside the magnetic field, where a short circuit establishes the flux $\lambda_1 = 0$. We have established the function W'' to be used in (b):

$$W'' = \frac{1}{2}\left(L_2 - \frac{M^2}{L_1} \right) I_2^2. \tag{i}$$

The potential V, which includes both the effects of the magnetic field and gravity, is [from (5.2.2)]

$$V = -Mgl \cos \theta + \frac{I_2^2}{2L_1} M^2(\theta). \tag{j}$$

Here, a constant has been absorbed in the constant of the motion E [which appears in (5.2.4)].

The potential well for the flux-constrained coil is shown in Fig. 5.2.8 as the superposition of the same gravitational potential used before and a magnetic potential that is proportional to the square of $M(\theta)$, as given in Fig. 5.2.5c. At the outset two observations are of physical significance; the potential well is unaltered by reversing the direction of the field due to the fixed coils [I_2 is squared in (j)] and the interaction does not depend on the sign of the mutual inductance. This is by contrast with the current constrained situation in which the potential well could take on the alternative forms shown in Fig. 5.2.6. These observations reflect the fact that the current i_1 is *induced* in a direction that cancels any flux due to i_2 linking the pendulum coil. This type of dynamics is familiar from Section 5.1.3 and is evident here from the potential diagram.

The field generated by the fixed coils represents a magnetic barrier to the pendulum coil; for example, if the pendulum is given the initial velocity required to establish the constant of the motion as E_1 in Fig. 5.2.8, the moving coil will bounce off the potential barrier set up by the magnetic field. In fact, at an energy E_2, the pendulum coil oscillates between a maximum deflection magnitude determined by gravity and a minimum determined by the magnetic interaction.

At an energy E_3 it is possible to trap the coil in a well created solely by the imposed magnetic field. This magnetic trapping is a lumped-parameter illustration of how a magnetic field can be shaped to "bottle up" a highly conducting continuum such as a plasma. We can think of the pendulum coil as replaced by a highly conducting "blob" of material, which in turn can be modeled by many arbitrarily oriented, perfectly conducting loops. Each of these loops tends to behave as described here.

The effects of a finite coil resistance were considered in Section 5.1.3 to place the dynamics as found here in perspective. The zero flux constraint is a meaningful model for the actual
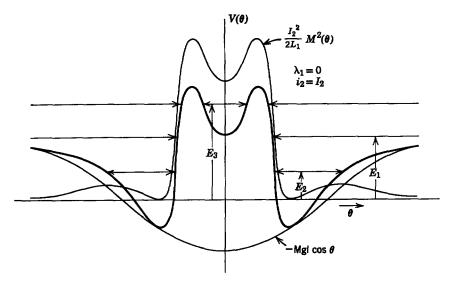
Fig. 5.2.8 Potential well for the system of Fig. 5.2.5a when the pendulum coil is constrained to zero flux. The potential has this shape regardless of the sign of the current I_2.

physical situation only as long as characteristic times for the motion are short compared with the L/R time constant of the pendulum coil.

So far in this discussion of nonlinear conservative systems we have not concerned ourselves with detailed temporal behavior. Sometimes nonlinear periods of oscillation or deflections as a function of time are required and (5.2.4) must be integrated. For this purpose it is written as

$$\frac{d\theta}{dt} = \pm\left\{\frac{2}{J}\left[E - V(\theta)\right]\right\}^{\frac{1}{2}}. \qquad (5.2.14)$$

The plus and minus signs indicate that the pendulum can be moving in either direction at a given angle θ.

If we specify that when $t = 0$ the angle $\theta = \theta_m$, (5.2.14) can be integrated.

$$t = \pm\int_{\theta_m}^{\theta} \frac{d\theta'}{\left\{\frac{2}{J}\left[E - V(\theta')\right]\right\}^{\frac{1}{2}}}. \qquad (5.2.15)$$

The parameter θ' is a running variable of integration. Whether (5.2.15) can be integrated analytically depends on the form of $V(\theta)$. In any case, given $V(\theta)$, numerical integration is a straightforward matter.

In the case of the mechanical pendulum of Fig. 5.2.1, $V(\theta)$ is given by (5.2.2) and the integration of (5.2.15) can be carried out. Suppose that the pendulum is released from rest at $\theta = \theta_m$. Then from (5.2.4) $E = V(\theta_m)$ and the integral of (5.2.15) takes the standard form of an elliptic integral* for which solutions are tabulated. One fourth of a cycle of oscillation is shown as a function of time in Fig. 5.2.9, where for (*a*) the pendulum is released from an initial "small" angle $\theta_m = 20°$ and for (*b*) the initial amplitude is 90°. For these plots the time is normalized to the frequency $\omega = \sqrt{g/l}$. This is the frequency of oscillation for small amplitudes, as can be seen by combining and linearizing (5.2.1) and (5.2.2) to obtain

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0. \qquad (5.2.16)$$

For the initial conditions considered in Fig. 5.2.9 the solution to this equation is

$$\theta = \theta_m \cos \omega t. \qquad (5.2.17)$$

This response, predicted by the linearized equation of motion, is shown as dashed curves in Fig. 5.2.9. For an amplitude of $\theta = 20°$ the results from the linear and nonlinear models are almost identical, although there is some

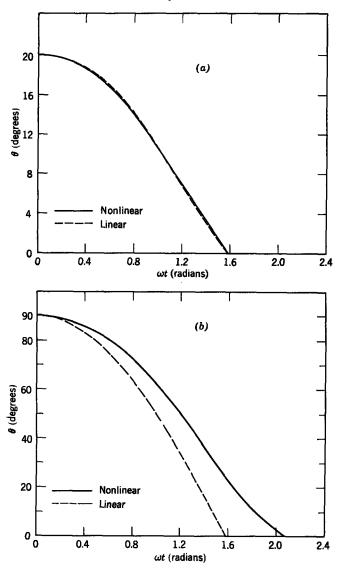* P. Franklin, *Methods of Advanced Calculus*, McGraw-Hill, New York, 1944, Chapter VII.

Fig. 5.2.9   Oscillation of a simple pendulum: (*a*) small amplitude; (*b*) large amplitude.

discrepancy between the predictions for $\theta = 90°$. This comparison between the exact and linearized solutions should help to place the methods of Section 5.1 in perspective. Of course, the adequacy of a linearized model will depend greatly on the nature of the nonlinearity. The following example is one in which a linearized model would be difficult to make and would be of doubtful usefulness.
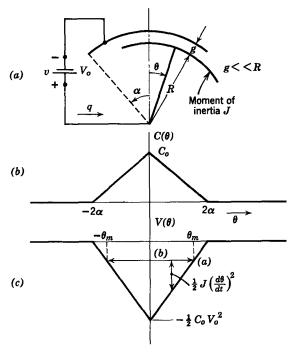
Fig. 5.2.10 (a) Capacitor plates just overlap when $\theta = 0$ and are constrained to constant potential V_o; (b) capacitance as a function of θ; (c) potential for the system of (a).

Example 5.2.3. The electric field system shown in Fig. 5.2.10 illustrates how the period of oscillation, computed from (5.2.15), can have nonlinear behavior. A capacitor is constructed from fixed and rotatable plates with the shape of sections from coaxial cylinders. At $\theta = 0$ the plates are aligned and the capacitance has the maximum value C_0. At $\theta = \pm 2\alpha$ there is no overlap between the plates and the capacitance is essentially zero. Hence $C(\theta)$ depends on θ essentially as shown in Fig. 5.2.10b. With the plates constrained to have the constant potential difference V_o, we expect that the rotatable plate can be at rest at $\theta = 0$, for the induced charges will tend to make the plates attract one another. We wish to determine the period of oscillation that results when the plate is deflected from this equilibrium.

The terminals are constrained to constant potential; hence it is appropriate to write the electric torque in terms of the coenergy $W'(v, \theta)$ (see Section 3.1.2b):

$$T^e = \frac{\partial W'}{\partial \theta} (V_o, \theta). \tag{a}$$

Hence the potential V is [from (5.2.1) and (5.2.2)]

$$V = -W'(V_o, \theta), \tag{b}$$

which for this example is

$$V = -\tfrac{1}{2}V_o^2 C(\theta), \tag{c}$$

with $C(\theta)$ as shown in Fig. 5.2.10b. It follows that the potential produced by the electric field appears as shown in Fig. 5.2.10c. From this diagram it is clear that if the movable

plate (with the moment of inertia J but with no effect from gravity) is given a kinetic energy greater than

$$\tfrac{1}{2}J\left(\frac{d\theta}{dt}\right)^2 = \tfrac{1}{2}C_o V_o^2 \tag{d}$$

at $\theta = 0$ the motions will not be oscillatory. Instead, the plate will rotate continuously with constant velocity when the plates do not overlap and with a maximum velocity at $\theta = 0$.

We can compute the period of oscillation T from (5.2.15) when the initial conditions are such that the motions are oscillatory. For this purpose we call the peak deflection θ_m so that the line of constant energy is as shown in Fig. 5.2.10c. In terms of that figure the rotor moves from (a) to (b) in $T/4$ sec. Over this interval the potential can be written as

$$V(\theta) = \frac{C_o}{2\alpha} (\theta - 2\alpha) \frac{V_o^2}{2} . \tag{e}$$

For our purposes (5.2.15) becomes

$$\frac{T}{4} = -\int_{\theta_m}^{0} \frac{d\theta'}{\left\{\dfrac{2}{J} [V(\theta_m) - V(\theta')]\right\}^{1/2}} , \tag{f}$$

where from (e)

$$V(\theta_m) - V(\theta) = \frac{C_o}{2\alpha} (\theta_m - \theta) \frac{V_o^2}{2} . \tag{g}$$

Substitution of (g) into (f), followed by integration, gives the required period of oscillation:

$$T = 8 \left(\frac{2J\alpha}{C_o V_o^2}\right)^{1/2} \sqrt{\theta_m}; \qquad |\theta_m| < 2\alpha. \tag{h}$$

Of course, this result is limited to a range of θ_m in which the plates overlap. Beyond this range the motions are not bounded (oscillatory) and

$$T \rightarrow \infty; \qquad |\theta_m| > 2\alpha. \tag{i}$$

The dependence of the oscillation period on θ_m is shown in Fig. 5.2.11. In a linear system the period of oscillation is independent of amplitude. Hence the plot emphasizes the nonlinear character of the motion.

It should be recognized that the approximate function $C(\theta)$ is valid only if the plate spacing g is small compared with deflections θR of interest. In a more exact model the functional dependence of C would be smoothed in the region near $\theta = 0$ and $\theta = \pm 2\alpha$ in Fig. 5.2.10. This is true because the fringing fields would extend beyond the edge of the overlapping plates a distance on the order of g. Hence we cannot expect the period of oscillation given by (h) to be correct unless $\theta_m R \gg g$.

5.2.2 Loss-Dominated Systems

The approach to the analysis of nonlinear problems in Section 5.2.1 took advantage of the small mechanical and electrical energy dissipations. In this section we wish to illustrate how simplifying assumptions can be made valid when there is a *large* effect from damping mechanisms. We can illustrate briefly the notion involved by returning to the example of the pendulum used in Section 5.2.1 (Fig. 5.2.1). If there is viscous damping (with coefficient B),
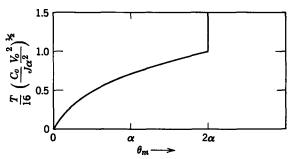
Fig. 5.2.11 Normalized period of oscillation for the electric pendulum of Fig. 5.2.10a as a function of peak amplitude.

(5.2.1) for the deflection of the pendulum is replaced by (see Section 2.2.1b for a discussion of torsional viscous dampers)

$$J \frac{d^2\theta}{dt^2} + B \frac{d\theta}{dt} = -Mgl \sin \theta. \tag{5.2.18}$$

In the analysis presented in Section 5.2.1 the viscous term was implicitly assumed small compared with the other terms in the equation. In the opposite extreme the viscous damping is so large that the inertial effects of the first term are ignorable. This would be the result of immersing the pendulum in heavy oil. Then we can approximate (5.2.18) by

$$B \frac{d\theta}{dt} = -Mgl \sin \theta, \tag{5.2.19}$$

which is a simple nonlinear expression to integrate. Rearranging, we have

$$\frac{d\theta}{\sin \theta} = -\frac{Mgl}{B} dt. \tag{5.2.20}$$

If, when $t = 0$, $\theta = \theta_m$, we can integrate this equation between θ_m and $\theta(t)$

$$\int_{\theta_m}^{\theta(t)} \frac{d\theta}{\sin \theta} = -\frac{Mgl}{B} \int_0^t dt. \tag{5.2.21}$$

We carry out this integration to obtain*

$$t \left(\frac{Mgl}{B} \right) = -\ln \left[\frac{\tan (\theta/2)}{\tan (\theta_m/2)} \right]. \tag{5.2.22}$$

This expression for θ is plotted in Fig. 5.2.12.

We now consider an example of a nonlinear, lossy system, for which we can make simplifying assumptions to allow analytical integration of nonlinear equations, and analyze a time-delay relay in which the dynamic

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* C. D. Hodgman, *Mathematical Tables from Handbook of Chemistry and Physics*, 9th ed., Chemical Rubber Publishing Co., Cleveland, Ohio.
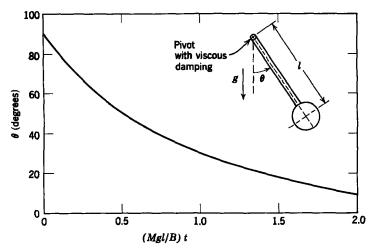
Fig. 5.2.12 Angular deflection θ of a pendulum heavily damped and released from $\theta_m = 90$.

behavior is controlled by mechanical losses. Although we treat a time-delay relay in this example, the approximations and analytical techniques are applicable to many other systems.

Example 5.2.4. In Fig. 5.2.13 we show schematically the basic actuator for a mechanically damped time-delay relay.* The basic operation to be analyzed is as follows: with switch S open, the spring pulls the plunger against a mechanical stop at $x = x_0$; when switch S is closed, current in the coil causes a magnetic force that pulls the plunger against the stop at $x = 0$. This displacement opens or closes relay contacts. The motion of the plunger when driven by the magnetic force is dominated by the mechanical damper. The damper, or dashpot, normally used in this application consists of a piston with a small orifice, moving in a cylinder filled with oil (see Fig. 2.2.10a). As discussed in Section 2.2.1b, a damper of this type is represented quite well by a damping force proportional to the square of the velocity [see (2.2.8)].

The electromechanical coupling of the configuration in Fig. 5.2.13 has been analyzed in Examples 2.1.1, 3.2.1, and 5.1.1. Neglecting fringing fields and assuming infinitely permeable magnetic material, we obtain the electrical and mechanical terminal relations from (a) and (c) of Example 5.1.1.

$$\lambda = \frac{L_o i}{1 + x/g} \tag{a}$$

$$f^e = -\frac{L_o i^2}{2g(1 + x/g)^2} , \tag{b}$$

where

$$L_o = \frac{\mu_0 N^2 (2wd)}{g} .$$

* A. E. Knowlton, ed., *Standard Handbook for Electrical Engineers*, 9th ed., McGraw-Hill, New York, 1957, Section 5–150.
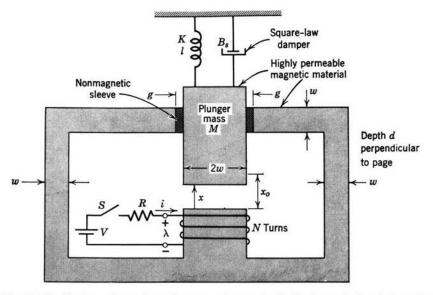
Fig. 5.2.13 Basic configuration of actuator for mechanically damped time-delay relay.

Thus, with switch S closed and the plunger between the stops

$$0 < x < x_o,$$

the equations of motion are

$$V = Ri + \frac{L_o}{1 + x/g} \frac{di}{dt} - \frac{L_o i}{g(1 + x/g)^2} \frac{dx}{dt},$$ (c)

$$M \frac{d^2x}{dt^2} \pm B_s \left(\frac{dx}{dt} \right)^2 + K(x - l) = - \frac{L_o i^2}{2g(1 + x/g)^2}.$$ (d)

The $+$ or $-$ sign on the damping force is chosen to make the damping force oppose the motion.

As already stated, we wish to analyze the transient that occurs when the switch S is closed at $t = 0$ with the initial conditions at $t = 0$

$$x = x_o, \qquad i = 0.$$ (e)

The mechanical damper dominates the mechanical motion and slows down the motion of the plunger so that the closing time can be of the order of seconds to minutes. Consequently, the speed voltage, the last term in (c), can be neglected throughout the analysis, and the transient in current is complete before the position x changes significantly from the initial position x_o. Thus we find the current from the simplified equation

$$V = Ri + \frac{L_o}{1 + x_o/g} \frac{di}{dt},$$ (f)

which has constant coefficients. The current transient is

$$i = \frac{V}{R}(1 - e^{-t/\tau_e}),\tag{g}$$

where

$$\tau_e = \frac{L_0}{R(1 + x_0/g)}.$$

The time constant τ_e is the electrical time constant with the gap fully open.

With the purely electrical transient completed before the mechanical motion starts and with the neglect of the speed voltage in (c) the mechanical motion occurs with the current essentially constant. Thus from the viewpoint of the mechanical motion the voltage source V and resistance R form an effective current source. This situation, in which electrical time constants are much shorter than mechanical time constants (see Section 5.1.3), occurs often in electromechanical transducers and is the source of significant simplification.

As stated before, the mechanical motion is dominated by the damper. This means that except for the very short time during which the mass initially accelerates, the damping force, the second term on the left of (d), is much greater than the acceleration force and the spring force. Consequently, when our interest is in the time required for the air gap to close, we can simplify (d) to

$$-B_s\left(\frac{dx}{dt}\right)^2 = -\frac{L_0(V/R)^2}{2g(1 + x/g)^2},\tag{h}$$

where we have chosen the minus sign because $B_s(dx/dt)^2$ must act to retard the motion of the plunger in the $-x$-direction. Solution of this expression for the velocity yields

$$\frac{dx}{dt} = -\frac{\sqrt{L_0/2gB_s}\,V/R}{1 + x/g}.\tag{i}$$

We have specified a minus sign because we know from the initial conditions and the physical nature of the problem that x is decreasing.

With the initial condition as specified by (e), we integrate (i) to find the time t at which the plunger is at position x as

$$\int_{x_0}^{x}\left(1 + \frac{x'}{g}\right)dx' = -\int_0^t \left(\frac{L_0}{2gB_s}\right)^{1/2}\frac{V}{R}\,dt'.\tag{j}$$

Integration of this expression yields

$$(x_0 - x) + \frac{(x_0^2 - x^2)}{2g} = \left(\frac{L_0}{2gB_s}\right)^{1/2}\frac{V}{R}t.\tag{k}$$

We find x as a function of time by solving this quadratic equation. The result is

$$x = -g + \left[(g + x_0)^2 - \left(\frac{2L_0g}{B_s}\right)^{1/2}\frac{V}{R}t\right]^{1/2}\tag{l}$$

Note that this expression satisfies the initial condition that at $t = 0$, $x = x_0$. Because the other root of (h) does not satisfy this condition, it is extraneous.
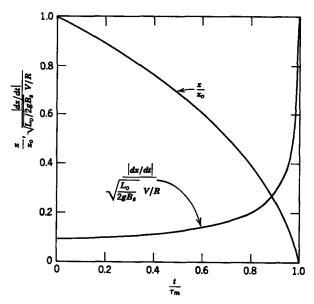
Fig. 5.2.14   Dynamic response of system of Fig. 5.2.13.

Alternatively, if we wish to evaluate the time τ_m necessary for the airgap to close, we set $x = 0$ in (k) and obtain

$$\tau_m = \left(\frac{2gB_s}{L_0}\right)^{1/2} \frac{R}{V} x_0 \left(1 + \frac{x_0}{2g}\right). \tag{m}$$

It is clear from this expression how the closing time can be controlled by the damping constant B_s or by the current V/R.

The response of the position as a function of time (l) is plotted in Fig. 5.2.14. For this purpose (l) is normalized in the following way:

$$\frac{x}{x_0} = -\frac{g}{x_0} + \left[\left(\frac{g}{x_0} + 1\right)^2 - \left(\frac{2g}{x_0} + 1\right)\frac{t}{\tau_m}\right]^{1/2}, \tag{n}$$

and we assume the ratio

$$\frac{g}{x_0} = 0.1.$$

Also plotted in Fig. 5.2.14 is the magnitude of the velocity, normalized, by using (i) and (n) in the form

$$\frac{|dx/dt|}{(L_0/2gB_s)^{1/2}V/R} = \left[\left(1 + \frac{x_0}{g}\right)^2 - \frac{x_0}{g}\left(2 + \frac{x_0}{g}\right)\frac{t}{\tau_m}\right]^{-1/2}. \tag{o}$$

We note from the curves of Fig. 5.2.14 that the plunger moves at almost constant velocity over most of the travel and then accelerates markedly. This results because the magnetic force (b) increases rapidly as $x \to 0$. A characteristic of this type is desirable in a time-delay relay to ensure that the contacts will close rapidly to avoid arcing. In any given situation,

however, the validity of ignoring the acceleration and speed voltage effects in this last region should be examined.

The curve of velocity in Fig. 5.2.14 indicates that this system behaves much like a linear statically unstable system (see curve A of Fig. 5.1.7). In some cases it may be desirable to approximate the equations of motion by a linear set with a static instability. This is especially true if the nonlinear expressions cannot be integrated analytically.

Although we have confined our attention in this section to discussing examples in which the damping is mechanical in nature, electrical damping can also dominate the dynamics. An example in which this is the case was discussed in Section 5.1.3. There the dynamic behavior of a coil rotating in a magnetic field was discussed in the limit at which the inductive reactance could be ignored [condition of (5.1.32)]. This made it possible to reduce the solution of the nonlinear motions to the problem of integrating (5.1.34), a procedure that is analogous to integrating (5.2.19).

5.3 DISCUSSION

In this chapter there have been two objectives. For the first, important types of dynamical behavior have been illustrated in which attention has been given to the relation of basic electromechanical interactions to mathematical models. For the second objective we have formed a basis on which to build an understanding of continuum interactions. In this regard both the mathematical techniques and physical approximations of this chapter are important in the chapters that follow.