PROBLEMS

8.1. An identity $\nabla \cdot (\psi \mathbf{A}) = \psi \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla \psi$ is given, where ψ = scalar, $\mathbf{A}$ = vector. Show by means of index notation that this identity is valid.

8.2. Show, by means of index notation, that the following vector equation is valid: $\mathbf{B} \cdot \nabla(\psi \mathbf{A}) = \psi \mathbf{B} \cdot \nabla \mathbf{A} + \mathbf{A}\mathbf{B} \cdot \nabla \psi$; ψ = scalar and $\mathbf{A}$ and $\mathbf{B}$ are vectors.

8.3. Consider two orthogonal coordinate systems (x_1, x_2, x_3) and (x_1', x_2', x_3'). The primed coordinate system is related to the unprimed system as follows: $x_3' = x_3$; the x_1'-axis makes an angle of 60° with the x_1-axis as shown in Fig. 8P.3.

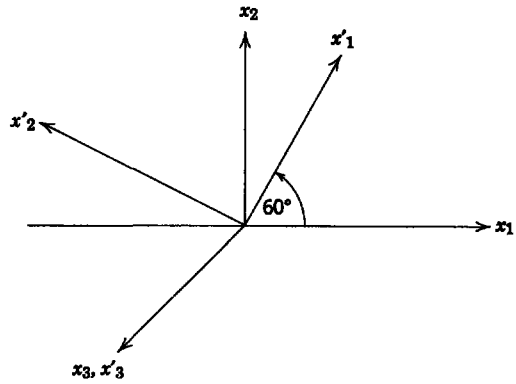

Fig. 8P.3

(a) You are given the components of a vector $\mathbf{A}$ in the (x_1, x_2, x_3) system: $A_1 = 1$; $A_2 = 2$; $A_3 = -1$. Find the components of $\mathbf{A}$ in the (x_1', x_2', x_3') system by using the transformation law for vectors, $A_i' = a_{ik}A_k$, where a_{ik} is the rotation matrix between (x_1, x_2, x_3) and (x_1', x_2', x_3').

(b) A tensor T_{mn} in the (x_1, x_2, x_3) system has elements $T_{11} = 1$, $T_{22} = 2$, $T_{12} = T_{21} = 3$, $T_{33} = 1$, and $T_{13} = T_{31} = T_{23} = T_{32} = 0$. Find the elements of T_{mn} in the (x_1', x_2', x_3') system by using $T_{ij}' = a_{ik}a_{jl}T_{kl}$.

8.4. A system has a stress tensor

$$T_{ij} = \begin{bmatrix} \frac{P_0^2}{2a^2}(x_1^2 - x_2^2) & -\frac{P_0^2}{a^2}x_1x_2 & 0 \\ -\frac{P_0^2}{a^2}x_1x_2 & \frac{P_0^2}{2a^2}(x_2^2 - x_1^2) & 0 \\ 0 & 0 & \frac{P_0^2}{2a^2}(-x_1^2 - x_2^2) \end{bmatrix}.$$

Find the volume force density that results from this stress tensor.

8.5. A flat plate of infinite extent is parallel to the x_3-axis and intersects the x_1 and x_2-axes, as shown in Fig. 8P.5. In region (2) $\mathbf{E} = 0$, whereas in region (1) the electric field is given by $\mathbf{E} = E_0(\frac{3}{2}\mathbf{i}_1 + \mathbf{i}_2)$. Find the x_1, x_2, and x_3 components of the force on the section of the

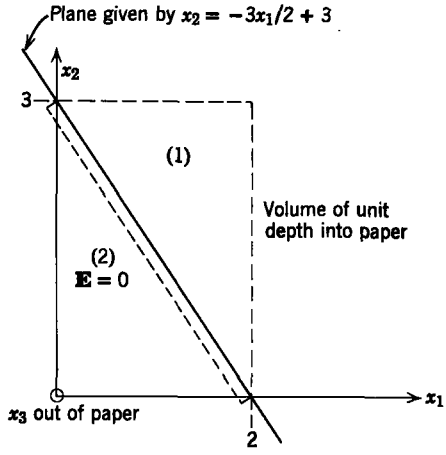

Fig. 8P.5

plate (per unit depth in the x_3-direction) that extends from the x_1- to the x_2-axes. Do this by integrating the Maxwell stress tensor over the surface of the volume shown in Fig. 8P.5, which encloses this section of the plate.

8.6. A pair of parallel insulating sheets is shown in Fig. 8P.6. The sheet at $y = d$ supports a surface charge density $-\sigma_f$, whereas the sheet at $y = 0$ supports the image surface charge density σ_f. Hence the electric field between the plates *due to the charges* is $(\sigma_f/\epsilon_0)\mathbf{i}_y$. External

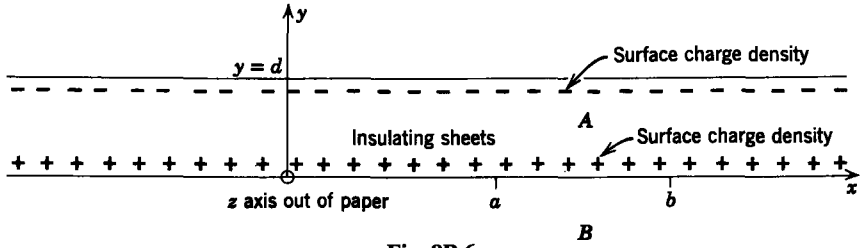

Fig. 8P.6

electrodes are used to impose an additional uniform electric field given everywhere by $\mathbf{E} = E_0\mathbf{i}_x + E_0\mathbf{i}_y$, where E_0 is a constant.

(a) Write the components of the Maxwell stress tensor at points A and B in terms of σ_f and E_0.

(b) Use the Maxwell stress tensor to find the total electric force in each of the coordinate directions on the section of the lower sheet between $x = a$ and $x = b$ having depth D in the z-direction.

8.7. Two perfectly conducting plates are arranged as shown in Fig. 8P.7. A magnetic field trapped between the plates is established in such a way that it does not penetrate the perfectly conducting plates. Also $H_3 = 0$ and $\partial/\partial x_3 = 0$. Under the assumption that $b \ll L$, find the x_1-component of the force per unit x_3 on the section of the lower plate between $x_1 = L$ and $x_1 = -L$. You may assume that, when $x_1 = -L$, $\mathbf{H} = H_0\mathbf{i}_1$, where H_0 is a known constant.

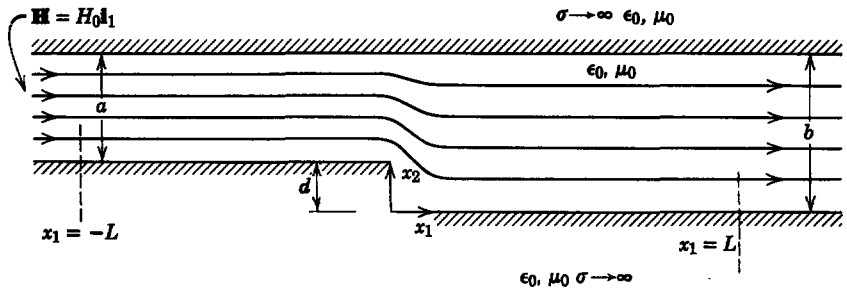

Fig. 8P.7

8.8. Three perfectly conducting plates are arranged as shown in Fig. 8P.8. A potential difference V_0 between the middle electrode and the outer electrodes is shown. Under the assumption that $a \ll l$, $b \ll l$, use the Maxwell stress tensor to find the force on the middle plate in the x-direction. Be sure to give all of your arguments.

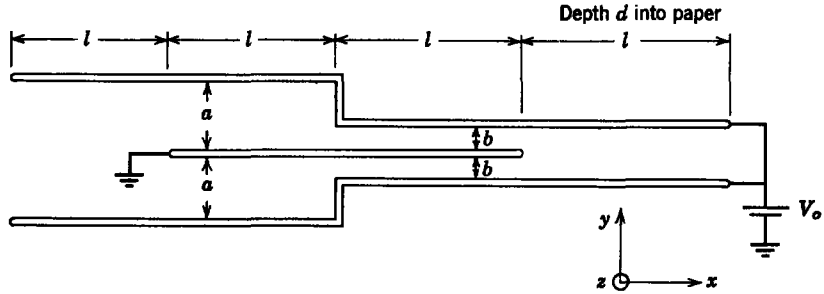

Fig. 8P.8

8.9. Capacitor plates with depth d (into the paper), length l, and spacing s are arranged as shown in Fig. 8P.9. Many of the plates are distributed along the x_1-axis. The plates have, alternately, the potentials $+V_0$ and $-V_0$, as shown, so that an electric field exists between each pair of them. You are to find the force in the x_2-direction on the section of plate enclosed by the volume V, which has a depth $w \ll d$ into the paper and encloses a section of the plate centered between its x_3 extremes.

8.10. Figure 8P.10 shows an electromechanical electrostatic voltmeter for measuring the relaxation time in liquids with very long relaxation times. The two outer conducting plates are fixed. The middle plate is constrained by a spring that is relaxed when $x = a$ but otherwise free to move in the x-direction. This plate (mass M) moves in a liquid dielectric of uniform conductivity σ and permittivity ϵ (the ϵ/σ to be measured). The liquid fills the region between the plates.

(a) Use Maxwell's stress tensor to find the total electric force on the middle plate in the x-direction as a function of the potential v of the middle plate and the position x. (Your answer should be exact, as $[s/(a - x)] \to 0$.)

(b) Use the energy method to check the result of part (a).

(c) The switch S has been closed for a long enough time to establish the middle plate in static equilibrium. Write the equations of motion for the plate position $x(t)$ (as many equations as unknowns) after the switch is opened.

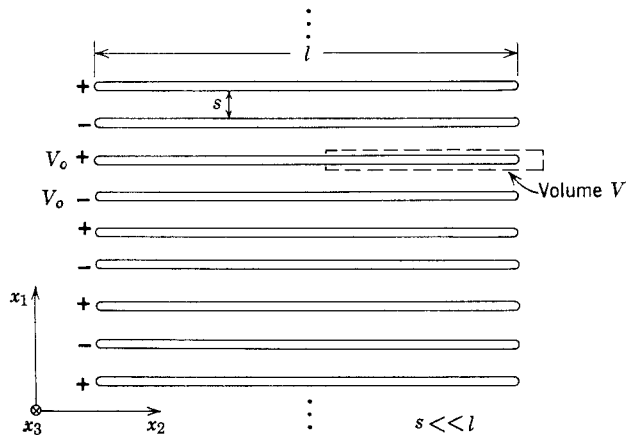

Fig. 8P.9

(d) Assume that the inertial force on the plate can be ignored (that the plate moves very slowly) and find $x(t)$. Is your assumption that the inertial force can be ignored consistent with the liquid having a very long relaxation time?
(e) How would you use this device to measure the relaxation time of the liquid?

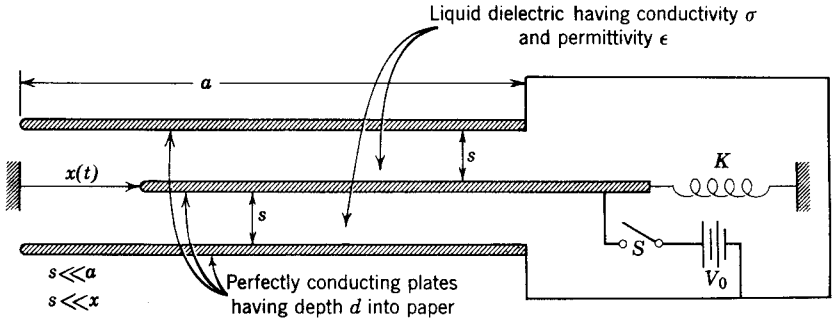

Fig. 8P.10

8.11. Two parallel conducting plates with a potential difference V_0 are shown in Fig. 8P.11. Assuming that $c < b < a \ll l \ll D$ and that the fringing fields are zero at the extreme points A and B, find the force in the x_1-direction on the lower plate.

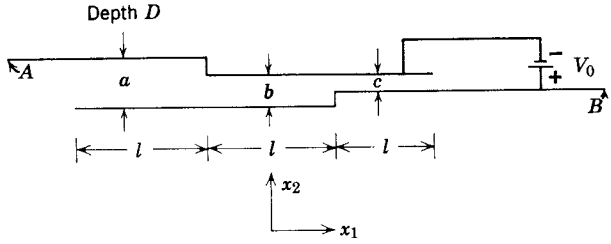

Fig. 8P.11

8.12. In Fig. 8P.12 two parallel perfectly conducting electrodes extend from $x_1 = 0$ to $x_1 = \infty$ and are infinite in the x_3-direction. The separation of the electrodes in the x_2-direction is a. A potential $\phi = \phi_0 \sin(\pi/a)x_2$ is established along the x_2-axis at $x_1 = 0$.

(a) Find the electric field intensity **E** everywhere between the plates and sketch.
(b) Find the total force on the bottom plate per unit depth in the x_3-direction.
(c) Find the total force on the top plate per unit depth in the x_3-direction.

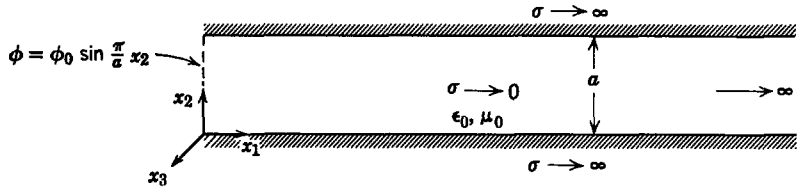

Fig. 8P.12

8.13. In the system in Fig. 8P.13 the geometry of two equipotentials is defined. These equipotentials are maintained at a potential difference V_0 by the battery, and the upper conductor has a movable section $(-a < x_1 < a)$, as indicated. The system has a large width w in the x_3-direction; thus we neglect any variations with x_3 and approximate the potential in the region between the conductors with the expression

$$\phi = \frac{V_0}{3a^2}(x_2^2 - x_1^2 - a^2);$$

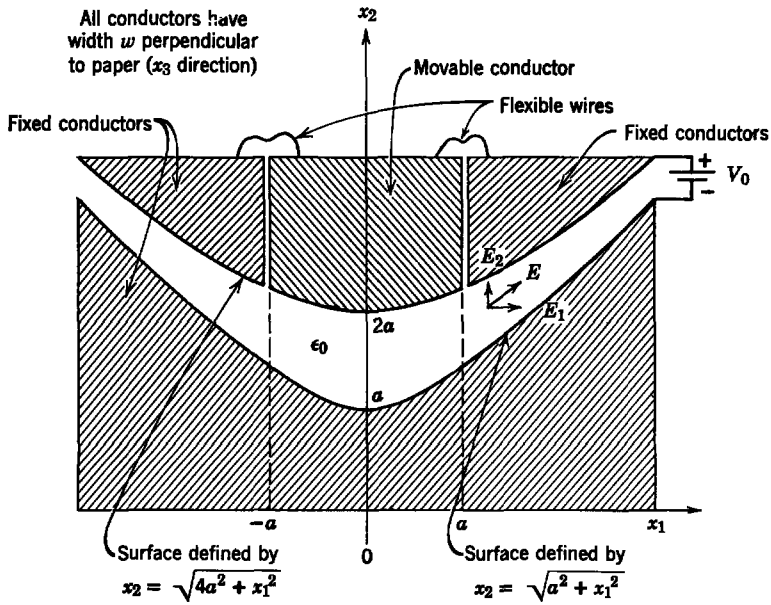

Fig. 8P.13

the two nonzero components of electric field intensity are then

$$E_1 = \frac{2V_0x_1}{3a^2}; \qquad E_2 = -\frac{2V_0x_2}{3a^2}.$$

(a) Find the components T_{22} and T_{21} of the Maxwell stress tensor between the conductors in terms of ϵ_0, V_0, a, x_1, and x_2.
(b) Use the stress tensor to find the component f_2 of the force applied to the movable section of the upper conductor ($-a < x_1 < a$) by the electric field. Assume that the movable conductor is held in equilibrium in the position shown by externally applied forces.
(c) Prove that $f_1 = 0$ by using the stress tensor.
(d) Find f_2 by using the surface force density written in terms of the surface charge density σ_f (see Section 8.4.2).

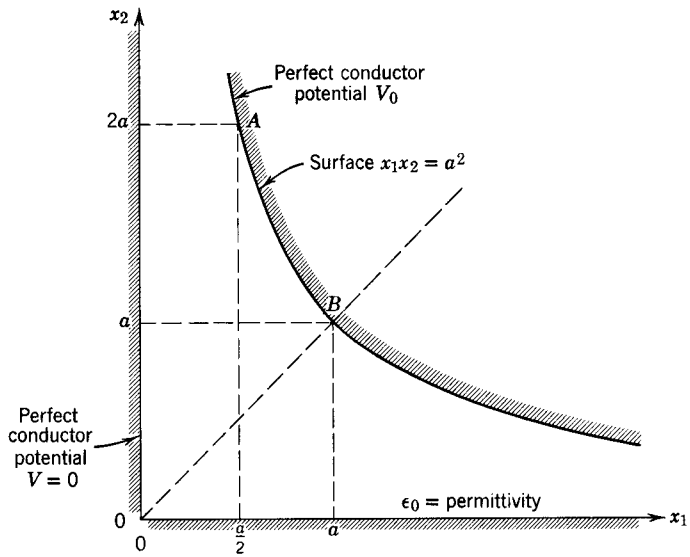

Fig. 8P.14

8.14. Figure 8P.14 shows two equipotential surfaces that are very long in the x_3-direction. The electric potential is

$$\phi = \frac{V_0}{a^2}x_1x_2,$$

where $\mathbf{E} = -\nabla\phi$.

(a) Evaluate all elements of the stress tensor for the region between the perfect conductors.
(b) Find the total force applied by the field to the segment of the curved conductor between points A and B and having depth D in the x_3-direction.

8.15. A conducting block moves with the velocity V between plane-parallel, perfectly conducting electrodes, short-circuited as shown in Fig. 8P.15. A uniform magnetic field $\mathbf{H}_0$ is imposed. Ignore the magnetic field induced by currents flowing in the block,

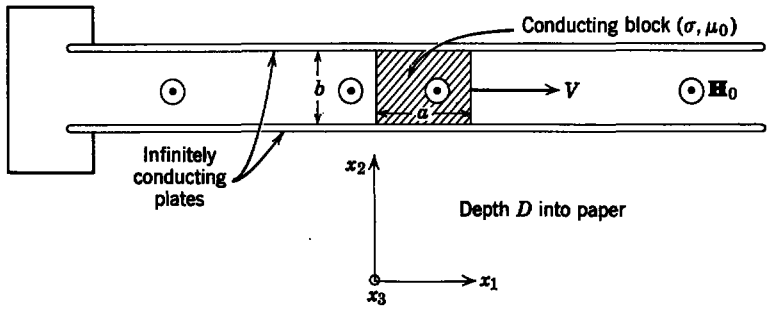

Fig. 8P.15

(a) Compute the total force on the block using $\mathbf{J} \times \mu_0\mathbf{H}$.
(b) Show that in this case the Maxwell stress tensor gives zero force on the block.
(c) Why do the results of (a) and (b) differ?

8.16. Figure 8P.16 shows a block of conducting material free to slide between two perfectly conducting plates that extend to infinity on the right. The conductivity of the block may be taken as $\sigma = \sigma_0(1 + \sin \pi x/2L)$ and the permeability as $\mu = \mu_0$. The conductivity σ_0 is a

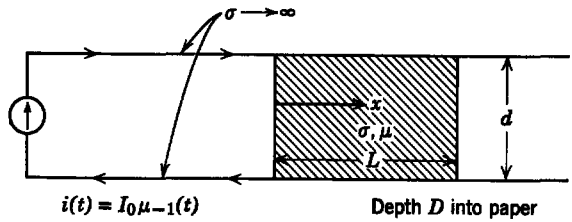

Fig. 8P.16

positive constant and x is the distance from the left-hand edge of the block. Find the total force of electromagnetic origin on the block as a function of time. Assume $d \ll D$, $d \ll L$.

8.17. A slab of conducting material (e.g., graphite) is sandwiched between perfectly conducting plates, as shown in Fig. 8P.17. The dimension a is much smaller than D and the x-dimension of the slab. In addition, the x-dimension is much larger than the skin depth at the frequency ω.

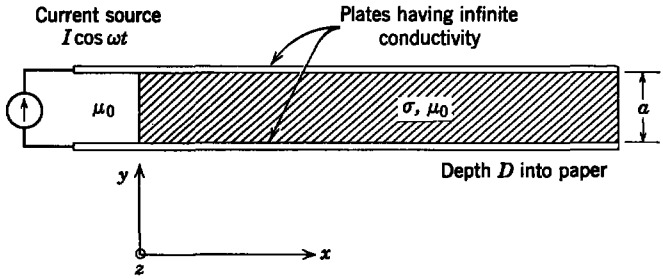

Fig. 8P.17

(a) Find the steady-state magnetic field H and current density J in the slab.
(b) Compute the total force on the slab in the x-direction by integrating $\mathbf{J} \times \mu_0\mathbf{H}$ over a volume.
(c) Compute the total force on the slab in the x-direction by integrating the Maxwell stress tensor over a surface.

8.18. A rigid, perfectly conducting body of arbitrary shape is positioned between two perfectly conducting infinite plates, as shown in Fig. 8P.18. The plates are at a potential

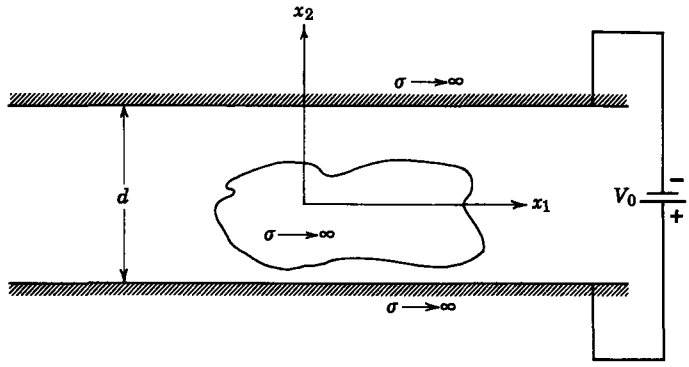

Fig. 8P.18

difference V_0. Take advantage of the fact that far from the body $\mathbf{E} = \mathbf{i}_2(V_0/d)$ to calculate the x_1-directed force on the body.

8.19. A pair of wires carries the constant current I as shown in Fig. 8P.19. The spacing $2a$ of the wires is much larger than the radius of either wire.

(a) Use the force density $\mathbf{J}_f \times \mathbf{B}$ to determine the force on a unit length of the right wire in the x_1-direction.

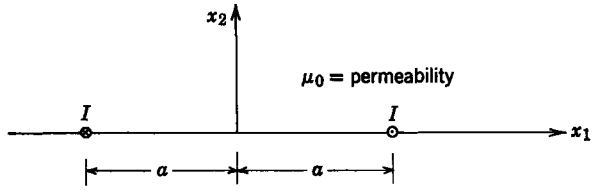

Fig. 8P.19

(b) Now enclose this section of wire with a convenient surface and integrate the Maxwell stress tensor over the surface to find the force in the x_1-direction. Compare your answer with that found in (a).
Hint. A "convenient" surface might take advantage of the fact that the fields go to zero as x_1 and $x_2 \to \infty$ and that $x_1 = 0$ is a plane of symmetry.

8.20. Two line charges of strength $\pm\lambda$ per unit x_3 are located at $x_2 = +a$ and $x_2 = -a$ (see Fig. 8P.20). The line charges extend to $\pm\infty$ in the x_3-direction.

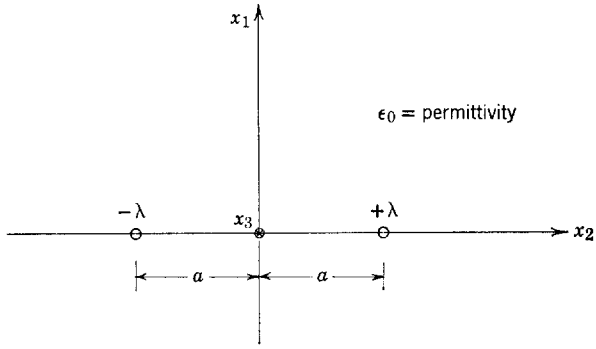

Fig. 8P.20

(a) Use the Maxwell stress tensor to find the force in the x_2-direction per unit depth in the x_3-direction exerted by the electric field on the line charge at $x_2 = +a$.
(b) Can you think of any other way of computing this force? If so, check it with part (a).

8.21. In Problem 7.14 a vehicle system was proposed in which a magnetic field provided both suspension (i.e., levitation) and propulsion forces. There it was assumed that the condition $ks \ll 1$ is valid and, to calculate the volume force density, $\mathbf{J} \times \mathbf{B}$ was applied. The Maxwell stress tensor provides an alternate and useful method for the calculation of

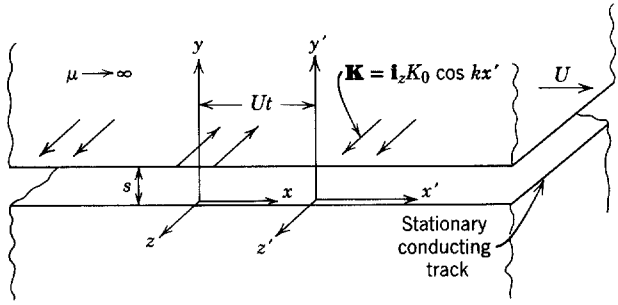

Fig. 8P.21

the forces per unit area (Fig. 8P.21). The solution for the magnetic field in the region $-\infty < y < 0$ is

$$B_x = \text{Re}\,[\mu_0 K_0 e^{\alpha y} e^{jk(x-Ut)}]$$

and

$$B_y = \text{Re}\left[\frac{-jk\mu_0 K_0}{\alpha} e^{\alpha y} e^{jk(x-Ut)}\right],$$

where

$$\alpha = k\left(1 - j\frac{\mu_0 \sigma U}{k}\right)^{1/2}.$$

(a) Write the components of the Maxwell stress tensor explicitly in terms of B_x and B_y. Present your results in matrix form.
(b) Using the stress tensor, compute the time average force per unit area (in the x-z plane) that holds the vehicle up. Take advantage of the periodic variation with x to define a suitable surface.
(c) Again using the stress tensor, compute the time average force per unit area (x-z) that tends to propel the train.

8.22. A pair of perfectly conducting plane-parallel electrodes is shorted by a bar of metal with conductivity σ (a constant) (Fig. 8P.22). A source of constant current I_0 (amperes) is

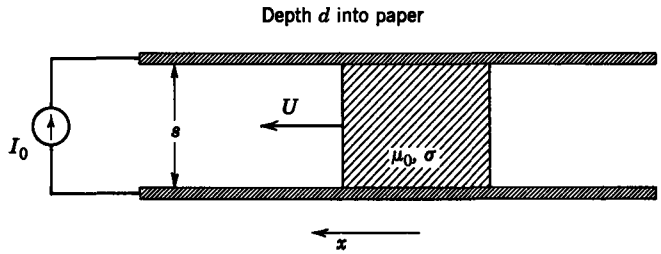

Fig. 8P.22

distributed along the left edges of the plates, and the block moves with the velocity U in the x-direction. What is the magnetic force on the block in the x-direction? Your answer should include the possibility that the magnetic Reynolds number is large or small.

8.23. A pair of perfectly conducting plane parallel electrodes "sandwich" a slab of lossy dielectric of thickness b and a region of free space of thickness $(a - b)$, as shown in Fig. 8P.23. The conductivity of the slab varies in the x-direction, and σ_0 and σ_1 are constants.

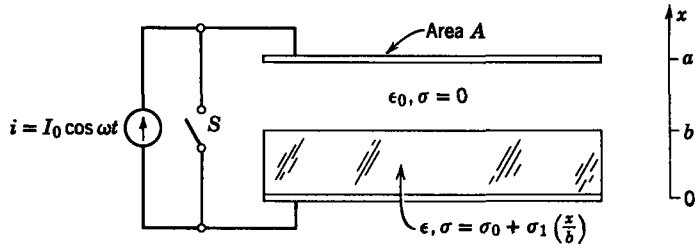

Fig. 8P.23

When $t < 0$, the switch S is closed and no electric fields exist between the plates. When $t = 0$, the switch S is opened. Neglect fringing fields and find the force in the x-direction on the upper plate as a function of time.

8.24. A pair of planar, diverging, perfectly conducting plates has a constant potential difference V_0 and the dimensions shown in Fig. 8P.24. What is the total electrical force on the lower plate in the x-direction? (Note that x is the radial direction half-way between the plates.)

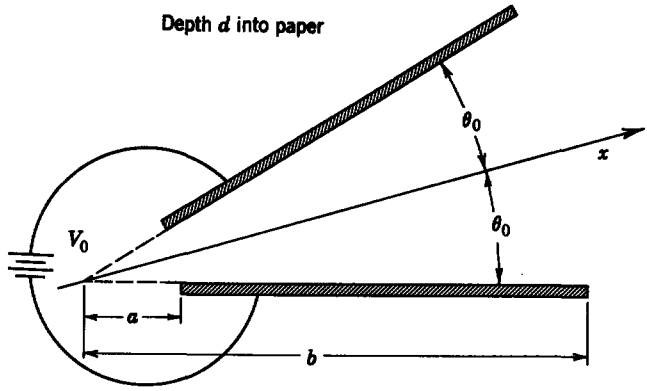

Fig. 8P.24

8.25. The dielectric slab shown in Fig. 8P.25 is free to slide in the x_1-direction. The upper and lower surfaces of the slab are in contact with perfectly conducting plates. The remaining volume is free space. Find the x_1-component of force on the slab. Use the Maxwell stress tensor.

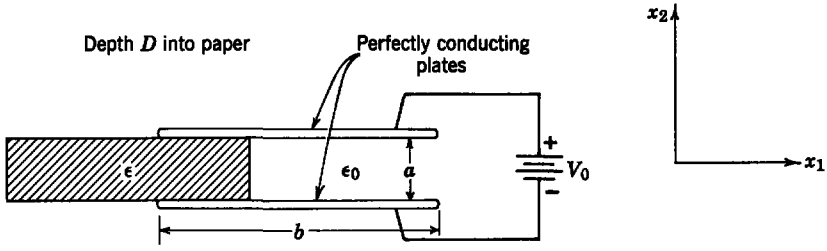

Fig. 8P.25

8.26. An elastic material is placed between two equipotential surfaces with its left-hand edge fixed to a rigid insulating wall, as shown in Fig. 8P.26. The right-hand edge of the elastic bulk is free and the permittivity of the material is a function of its mass density $\epsilon_1 = \epsilon_1(\rho)$. Free space fills the remaining volume. A potential difference (V_0) exists between the two plates.

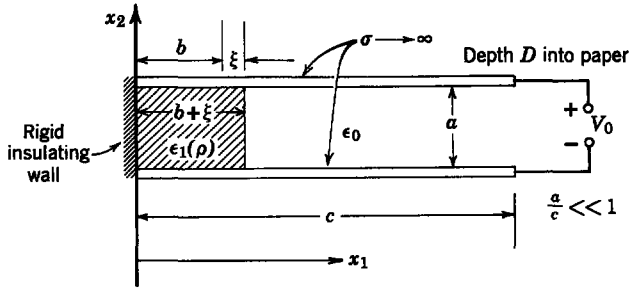

Fig. 8P.26

(a) Using the Maxwell stress tensor for polarizable material, find the force on the right-hand edge of the elastic bulk.
(b) Using energy methods, find the force on the right-hand edge of the elastic bulk.
(c) Compare the answers of parts (a) and (b).

8.27. The force density on a polarized fluid with permittivity $\epsilon(x_1, x_2, x_3, t)$ is $\mathbf{F} = -\frac{1}{2}\mathbf{E} \cdot \mathbf{E}\nabla\epsilon + \frac{1}{2}\nabla(\epsilon\mathbf{E} \cdot \mathbf{E}b)$, where the free charge $\rho_f = 0$, $\nabla \times \mathbf{E} = 0$, and $b = (\rho/\epsilon)(\partial\epsilon/\partial\rho)$ is a parameter that accounts for the electrostriction of the fluid (Fig. 8P.27). The planar

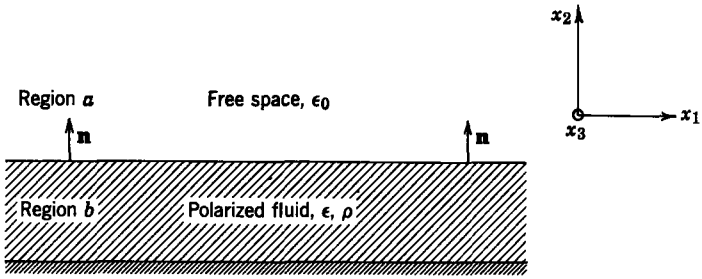

Fig. 8P.27

surface between dielectrics ϵ_0 and ϵ has a normal vector $\mathbf{n}$. Show that the polarization forces alone cannot exert a traction $\boldsymbol{\tau}$ on the interface between the two dielectrics which has a shear component. Remember that $T_m = [T_{mn}{}^a - T_{mn}{}^b]n_n$.

8.28. Use the electric force density of (8.5.45) to obtain the stress tensor of (8.5.46).