PROBLEM 5.1

Part a

The capacitance of the system of plane parallel electrodes is

$$C = (L+x)d\varepsilon_o/s \qquad \text{(a)}$$

and since the co-energy W' of an electrically linear system is simply $\frac{1}{2}Cv^2$ (remember v is the terminal voltage of the capacitor, not the voltage of the driving source)

$$f^e = \frac{\partial W'}{\partial x} = \frac{1}{2}\frac{d\varepsilon_o}{s}v^2 \qquad \text{(b)}$$

The plates tend to increase their area of overlap.

Part b

The force equation is

$$M\frac{d^2x}{dt^2} = -Kx + \frac{1}{2}\frac{d\varepsilon_o}{s}v^2 \qquad \text{(c)}$$

while the electrical loop equation, written using the fact that the current dq/dt through the resistance can be written as Cv, is

$$V(t) = R\frac{d}{dt}[(L+x)\frac{d\varepsilon_o}{s}v] + v \qquad \text{(d)}$$

These are two equations in the dependent variables (x,v).

Part c

This problem illustrates the important point that unless a system involving electromechanical components is either intrinsically or externally biased, its response will not in general be a linear reproduction of the input. The force is proportional to the square of the terminal voltage, which in the limit of small R is simply $V^2(t)$. Hence, the equation of motion is (c) with

$$v^2 = V^2(t) = u_{-1}(t)\frac{V_o^2}{2}(1-\cos 2\omega t) \qquad \text{(e)}$$

where we have used the identity $\sin^2\omega t = \frac{1}{2}(1-\cos 2\omega t)$. For convenience the equation of motion is normalized

PROBLEM 5.1 (Continued)

$$\frac{d^2x}{dt^2} + \omega_o^2\, x = \alpha u_{-1}(t)(1-\cos 2\omega t) \tag{f}$$

where

$$\omega_o^2 = K/M \;;\; \alpha = V_o^2\, d\, \varepsilon_o/4sM$$

To solve this equation, we note that there are two parts to the particular solution, one a constant

$$x = \frac{\alpha}{\omega_o^2} \tag{g}$$

and the other a cosinusoid having the frequency 2ω. To find this second part solve the equation

$$\frac{d^2x}{dt^2} + \omega_o^2\, x = -\,\mathrm{Re}\,\alpha e^{2j\omega t} \tag{h}$$

for the particular solution

$$x = \frac{-\alpha \cos 2\omega t}{\omega_o^2 - 4\omega^2} \tag{i}$$

The general solution is then the sum of these two particular solutions and the homogeneous solution $t > 0$

$$x(t) = \frac{\alpha}{\omega_o^2} - \frac{\alpha \cos 2\omega t}{\omega_o^2 - 4\omega^2} + A \sin\omega_o t + B\cos\omega_o t \tag{j}$$

The constants A and B are determined by the initial conditions. At $t=0$, $dx/dt = 0$, and this requires that $A = 0$. The spring determines that the initial position is $x = 0$, from which it follows that

$$B = \alpha 4\omega^2/\omega_o^2(\omega_o^2 - 4\omega^2) \tag{k}$$

Finally, the required response is $(t > 0)$

$$x(t) = \frac{\alpha}{\omega_o^2}\left[1 - \frac{\cos 2\omega t}{[1-(\frac{2\omega}{\omega_o})^2]} + \frac{(\frac{2\omega}{\omega_o})^2 \cos \omega_o t}{[1-(\frac{2\omega}{\omega_o})^2]}\right] \tag{l}$$

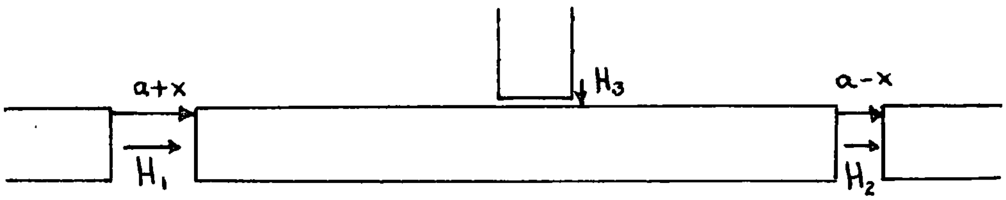
PROBLEM 5.1 (Continued)

Note that there are constant and double frequency components in this response, reflecting the effect of the drive. In addition, there is the response frequency ω_o reflecting the natural response of the spring mass system. No part of the response has the same frequency as the driving voltage.

PROBLEM 5.2

Part a

The field intensities are defined as in the figure



Ampere's law, integrated around the outside magnetic circuit gives

$$2N_1i_1 = H_1(a+x) + H_2(a-x) \qquad \text{(a)}$$

and integrated around the left inner circuit gives

$$N_1i_1 - N_2i_2 = H_1(a+x) - H_3a \qquad \text{(b)}$$

In addition, the net flux into the movable plunger must be zero

$$0 = H_1 - H_2 + H_3 \qquad \text{(c)}$$

These three equations can be solved for H_1, H_2 and H_3 as functions of i_1 and i_2. Then, the required terminal fluxes are

$$\lambda_1 = N_1\mu_o dW(H_1+H_2) \qquad \text{(d)}$$

$$\lambda_2 = N_2\mu_o dWH_3 \qquad \text{(e)}$$

Hence, we have

$$\lambda_1 = \frac{N_1\mu_o dW}{3a^2-x^2}\,[i_1 6aN_1 + i_2 2N_2 x] \qquad \text{(f)}$$

$$\lambda_2 = \frac{N_2\mu_o dW}{3a^2-x^2}\,[i_1 2N_1 x + i_2 2aN_2] \qquad \text{(g)}$$

Part b

To use the device as a differential transformer, it would be excited at a frequency such that

PROBLEM 5.2 (Continued)

$$\frac{2\pi}{\omega} << T \tag{h}$$

where T is a period characterizing the movement of the plunger. This means that in so far as the signal induced at the output terminals is concerned, the effect of the motion can be ignored and the problem treated as though x is a constant (a quasi-static situation, but not in the sense of Chap. 1). Put another way, because the excitation is at a frequency such that (h) is satisfied, we can ignore idL/dt compared to Ldi/dt and write

$$v_2 = \frac{d\lambda_2}{dt} = -\frac{\omega 2N_1N_2\mu_o dWxI_o}{(3a^2-x^2)} \sin \omega t$$

At any instant, the amplitude is determined by x(t), but the phase remains independent of x(t), with the voltage leading the current by 90°. By design, the output signal is zero at x=0 and tends to be proportional to x over a range of x << a.

PROBLEM 5.3

Part a

The potential function which satisfies the boundary conditions along constant θ planes is

$$\phi = \frac{V\theta}{\psi} \tag{a}$$

where differentiation shows that Laplaces equation is satisfied. The constant has been set so that the potential is V on the upper electrode where $\theta = \psi$, and zero on the lower electrode where $\theta = 0$. Then, the electric field is

$$\bar{E} = -\nabla\phi = -\bar{i}_\theta \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\bar{i}_\theta \frac{V}{r\psi} \tag{b}$$

Part b

The charge on the upper electrode can be written as a function of (V,ψ) by writing

$$q = D\varepsilon_o \int_a^b \frac{V}{r\psi} dr = \frac{D\varepsilon_o V}{\psi} \ln(\frac{b}{d}) \tag{c}$$

PROBLEM 5.3 (Continued)

Part c

Then, the energy stored in the electromechanical coupling follows as

$$W = \int V dq = \int \frac{q\psi dq}{D\varepsilon_o \ln(\frac{b}{a})} = \frac{1}{2} q^2 \frac{\psi}{D\varepsilon_o \ln(\frac{b}{a})} \tag{d}$$

and hence

$$T^e = -\frac{\partial W}{\partial \psi} = -\frac{1}{2}\frac{q^2}{D\varepsilon_o \ln(\frac{b}{a})} \tag{e}$$

Part d

The mechanical torque equation for the movable plate requires that the inertial torque be balanced by that due to the torsion spring and the electric field

$$\frac{Jd^2\psi}{dt^2} = \alpha(\psi_o - \psi) - \frac{1}{2}\frac{q^2}{D\varepsilon_o \ln(\frac{b}{a})} \tag{f}$$

The electrical equation requires that currents sum to zero at the current node, and makes use of the terminal equation (c).

$$\frac{dQ}{dt} = \frac{dq}{dt} + \frac{Gd}{dt}\left[\frac{q\psi}{D\varepsilon_o \ln(\frac{b}{a})}\right] \tag{g}$$

Part e

With $G = 0$, $Q(t) = q(t)$. (This is true to within a constant, corresponding to charge placed on the upper plate initially. We will assume that this constant is zero.) Then, (f) reduces to

$$\frac{d^2\psi}{dt^2} + \frac{\alpha}{J} = \frac{\alpha}{J}\psi_o - \frac{1}{4}\frac{Q_o^2}{JD\varepsilon_o \ln(\frac{b}{a})}(1+\cos 2\omega t) \tag{h}$$

where we have used the identity $\cos^2\omega t = \frac{1}{2}(1 + \cos 2\omega t)$. This equation has a solution with a constant part

$$\psi_1 = \psi_o - \frac{1}{4}\frac{Q_o^2}{\alpha D\varepsilon_o \ln(\frac{b}{a})} \tag{i}$$

and a sinusoidal steady state part

$$\psi' = -\frac{Q_o^2 \cos 2\omega t}{J4D\varepsilon_o \ln(\frac{b}{a})[\frac{\alpha}{J} - (2\omega)^2]} \tag{j}$$

PROBLEM 5.3 (Continued)

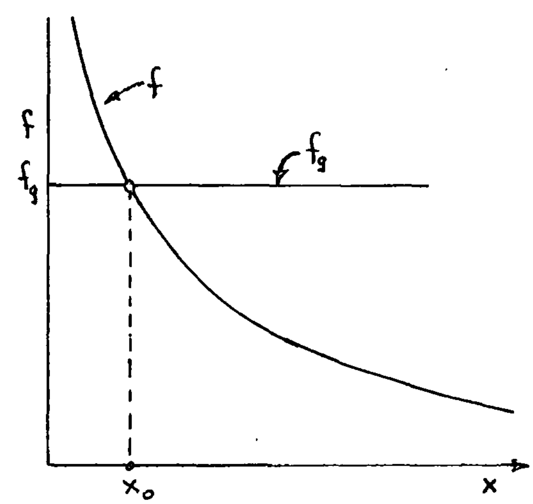
as can be seen by direct substitution. The plate responds with a d-c part and a part which has twice the frequency of the drive. As can be seen from the mathematical description itself, this is because regardless of whether the upper plate is positive or negative, it will be attracted toward the opposite plate where the image charges reside. The plates always attract. Hence, if we wish to obtain a mechanical response that is proportional to the driving signal, we must bias the system with an additional source and used the drive to simply increase and decrease the amount of this force.

PROBLEM 5.4

Part a

The equation of motion is found from (d) and (h) with $i=I_o$, as given in the solution to Prob. 3.4.

$$M\frac{d^2x}{dt^2} = Mg - \frac{1}{2}I_o^2\frac{(N^2\mu_o aw)}{(\frac{da}{b}+x)^2} \qquad \text{(a)}$$



Part b

The mass M can be in static equilibrium if the forces due to the field and gravity just balance,

$$f_g = f$$

or

$$Mg = \frac{1}{2}I_o^2\frac{(N^2\mu_o aw)}{(\frac{da}{b}+x)^2} \qquad \text{(b)}$$

A solution to this equation is shown graphically in the figure. The equilibrium is statically unstable because if the mass moves in the positive x direction from x_o, the gravitational force exceeds the magnetic force and tends to carry it further from equilibrium.

Part c

Because small perturbations from equilibrium are being considered it is appropriate to linearize. We assume $x = x_o + x'(t)$ and expand the last term in (a) to obtain

PROBLEM 5.4 (Continued)

$$-\frac{1}{2} I_o^2 \frac{(N^2\mu_o aw)}{(\frac{da}{b} + x_o)^2} + I_o^2 \frac{(N^2\mu_o aw)}{(\frac{da}{b} + x_o)^3} x' + \ldots \tag{c}$$

(see Sec. 5.1.2a). The constant terms in the equation of motion cancel out by virtue of (b) and the equation of motion is

$$\frac{d^2x'}{dt^2} - \alpha^2 x' = 0; \; \alpha = \sqrt{\frac{I_o^2(N^2\mu_o aw)}{(\frac{da}{b} + x_o)^3 M}} \tag{d}$$

Solutions are exp $\pm\,\alpha t$, and the linear combination which satisfies the given initial conditions is

$$x' = \frac{v_o}{\alpha}\,[e^{\alpha t} - e^{-\alpha t}] \tag{e}$$

PROBLEM 5.5

Part a

For small values of x relative to d, the equation of motion is

$$M\,\frac{d^2x}{dt^2} = \frac{Q_oQ}{4\pi\varepsilon}\,[\frac{1}{d^2} - \frac{2x}{d^3} - \frac{1}{d^2} - \frac{2x}{d^3}] \tag{a}$$

which reduces to

$$\frac{d^2x}{dt^2} + \omega_o^2\, x = 0 \text{ where } \omega_o^2 = \frac{Q_oQ_1}{M\pi\varepsilon d^3} \tag{b}$$

The equivalent spring constant will be positive if

$$\frac{Q_oQ_1}{\pi\varepsilon d^3} > 0 \tag{c}$$

and hence this is the condition for stability. The system is stable if the charges have like signs.

Part b

The solution to (b) has the form

$$x = A\cos\omega_o t + B\sin\omega_o t \tag{d}$$

and in view of the initial conditions, B = 0 and A = x_o.

PROBLEM 5.6

Part a

Questions of equilibrium and stability are of interest. Therefore, the equation of motion is written in the standard form

$$M\frac{d^2x}{dt^2} = -\frac{\partial V}{\partial x} \tag{a}$$

where

$$V = Mgx - W' \tag{b}$$

Here the contribution of W' to the potential is negative because $F^e = \partial W'/\partial x$. The separate potentials are shown in the figure, together with the total potential. From this plot it is clear that there will be one point of static equilibrium as indicated.

Part b

An analytical expression for the point of equilibrium follows by setting the force equal to zero

$$\frac{\partial V}{\partial x} = Mg + \frac{2L_oX^3}{b^4}I^2 \tag{c}$$

Solving for X, we have

$$X = -\left[\frac{Mgb^4}{2L_oI^2}\right]^{1/3} \tag{d}$$

Part c

It is clear from the potential plot that the equilibrium is stable.

PROBLEM 5.7

From Prob. 3.15 the equation of motion is, for small θ

$$J\frac{d^2\theta}{dt^2} = -K\theta + \frac{1}{2}\mu_oDN^2\ \ln\left(\frac{b}{a}\right)I_o^2\left[\frac{4\theta}{(\beta-\alpha)^3}\right] \tag{a}$$

Thus, the system will have a stable static equilibrium at $\theta = 0$ if the effective spring constant is positive, or if

$$K > \frac{2\mu_oDN^2}{(\beta-\alpha)^3}\ln\left(\frac{b}{a}\right)I_o^2 \tag{b}$$

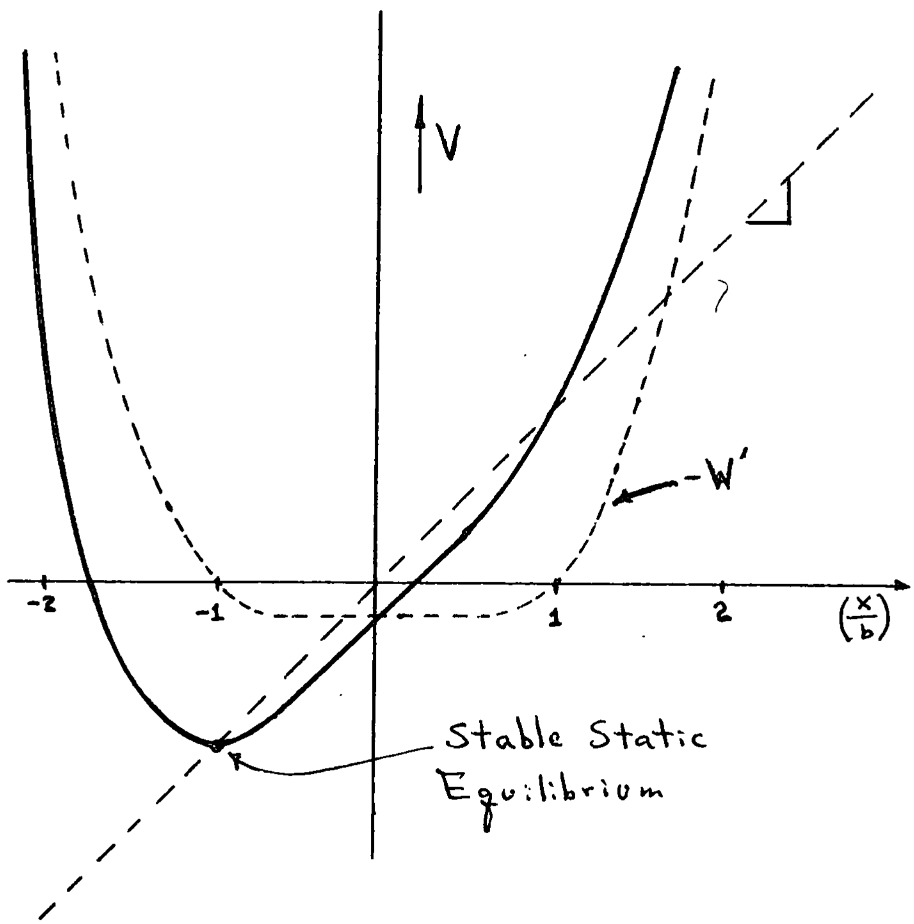


Figure for Prob. 5.6

PROBLEM 5.8

Part a

The coenergy is

$$W' = \int_o^{i_1} \lambda_1(i_1', 0, x)\,di_1' + \int_o^{i_2} \lambda_2(i_1, i_2', x)\,di_2' \tag{a}$$

which can be evaluated using the given terminal relations

$$W' = [\tfrac{1}{2}L_1\, i_1^2 + Mi_1 i_2 + \tfrac{1}{2}\, L_2 i_2^2]/(1 + \tfrac{x}{a})^3 \tag{b}$$

If follows that the force of electrical origin is

$$f^e = \frac{\partial W'}{\partial x} = -\frac{3}{2a}[L_1 i_1^2 + 2Mi_1 i_2 + L_2 i_2^2]/(1 + \tfrac{x}{a})^4 \tag{c}$$

Part b

The static force equation takes the form

$$-f^e = Mg \tag{d}$$

or, with $i_2=0$ and $i_1=I$,

$$\frac{3}{2a}\,\frac{L_1\, I^2}{[1 + \frac{X_o}{a}]^4} = Mg \tag{e}$$

Solution of this equation gives the required equilibrium position X_o

$$\frac{X_o}{a} = [\frac{3}{2a}\,\frac{L_1 I^2}{Mg}]^{1/4} - 1 \tag{f}$$

Part c

For small perturbations from the equilibrium defined by (e),

$$M_o\,\frac{d^2x'}{dt^2} - \frac{6L_1 I^2 x'}{a^2(1 + \frac{X_o}{a})^5} = f(t) \tag{g}$$

where $f(t)$ is an external force acting in the x direction on M.

With the external force an impulse of magnitude I_o and the mass initially at rest, one initial condition is $x(0) = 0$. The second is given by integrating the equation of motion form 0^- to 0^+

$$\int_{0^-}^{0^+} \frac{d}{dt}(M_o\,\frac{dx}{dt})dt - \text{constant}\int_{0^-}^{0^+} x'dt = I_o\int_{0^-}^{0^+} u_{-1}(t)dt \tag{h}$$

The first term is the jump in momentum at $t=0$, while the second is zero if x is to remain continuous. By definition, the integral on the right is I_o. Hence, from (h) the second initial condition is

PROBLEM 5.8 (Continued)

$$M_o \frac{dx'}{dt}(0) = I_o\,, \quad x'(0) = 0 \tag{i}$$

In view of these conditions, the response is

$$x'(t) = \frac{I_o}{2\alpha M_o}\left(e^{\alpha t} - e^{-\alpha t}\right) \tag{j}$$

where

$$\alpha = \left[6L_1 I^2/a^2 M_o\left(1 + \frac{X_o}{a}\right)^5\right]^{1/2}$$

Part d

With proportional feedback through the current i_2, the mutual term in the force equation makes a linear contribution and the force equation becomes

$$M_o \frac{d^2x'}{dt^2} - \left[\frac{6L_1 I^2}{a^2\left(1 + \frac{X_o}{a}\right)^5} - \frac{3MI\alpha}{a\left(1+\frac{X_o}{a}\right)^4}\right]x' = f(t) \tag{k}$$

The effective spring constant is positive if

$$\alpha I > 2L_1 I^2/a\left(1 + \frac{X_o}{a}\right)^5 M \tag{l}$$

and hence this is the condition for stability. However, once initiated oscillations remain undamped according to this model.

Part e (see insert)

With a damping term introduced by the feedback, the mechanical equation becomes

$$M_o \frac{d^2x'}{dt^2} + \frac{3MI}{a\left(1+\frac{X_o}{a}\right)^4}\beta\frac{dx'}{dt} + K_e x' = f(t) \tag{m}$$

where

$$K_e = \frac{3MI\alpha}{a\left(1+\frac{X_o}{a}\right)^4} - \frac{6L_1 I^2}{a^2\left(1 + \frac{X_o}{a}\right)^5} = \frac{3MI\alpha}{a\left(1+\frac{X_o}{a}\right)^4} - \frac{6L_1I^2}{a^2\left(1+\frac{X_o}{a}\right)^5}$$

This equation has solutions of the form exp st, where substitution shows that

$$s = -\frac{3MI\beta}{2aM_o\left(1+\frac{X_o}{a}\right)^4} \pm \sqrt{\left(\frac{3MI\beta}{2aM_o}\right)^2\frac{1}{\left(1+\frac{X_o}{a}\right)^8} - \frac{K_e}{M_o}} \tag{n}$$

For the response to decay, K_e must be positive (the system must be stable without damping) and β must be positive.

$$\frac{3}{2aM_o}\frac{MI\beta}{\left(1+\frac{X_o}{a}\right)^4} > \sqrt{\left(\frac{3MI\beta}{2aM_o\left(1+\frac{X_o}{a}\right)^4}\right)^2 - \frac{K_e}{M_o}}$$

PROBLEM 5.9

Part a

The mechanical equation of motion is

$$M\frac{d^2x}{dt^2} = -K(x-\ell_o)-B\frac{dx}{dt}+f^e \tag{a}$$

Part b

where the force f^e is found from the coenergy function which is (because the system is electrically linear) $W' = \frac{1}{2}Li^2 = \frac{1}{2}Ax^3i^2$

$$f^e = \frac{\partial W'}{\partial x} = \frac{3}{2}Ax^2i^2 \tag{b}$$

Part c

We can both find the equilibrium points X_o and determine if they are stable by writing the linearized equation at the outset. Hence, we let $x(t)=X_o+x'(t)$ and (a) and (b) combine to give

$$M\frac{d^2x'}{dt^2} = -K(X_o-\ell_o)-Kx' - B\frac{dx'}{dt} + \frac{3}{2}AI_o^2\,(X_o^2 + 2X_ox') \tag{c}$$

With the given condition on I_o, the constant (equilibrium) part of this equation is

$$X_o - \ell_o = \frac{3X_o^2}{16\ell_o} \tag{d}$$

which can be solved for X_o/ℓ_o to obtain

$$\frac{X_o}{\ell_o} = \begin{bmatrix} 12/3 \\ 4/3 \end{bmatrix} \tag{e}$$

That is, there are two possible equilibrium positions. The perturbation part of (c) tells whether or not these are stable. That equation, upon substitution of X_o and the given value of I_o, becomes

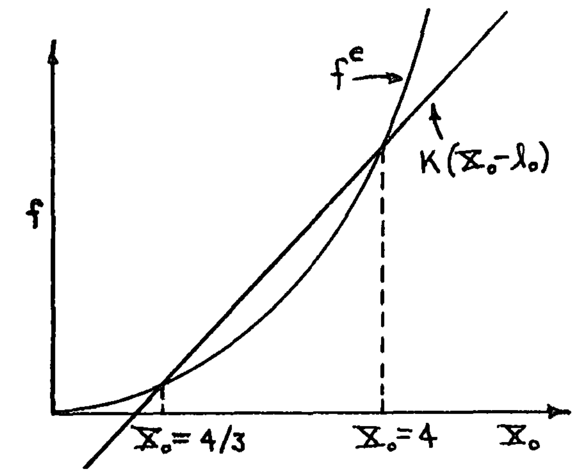
$$M\frac{d^2x'}{dt^2} = -K[1-\left(\begin{matrix}3/2\\1/2\end{matrix}\right)]x' - B\frac{dx'}{dt} \tag{f}$$

where the two possibilities correspond to the two equilibrium points. Hence, we conclude that the effective spring constant is positive (and the system is stable) at $X_o/\ell_o = 4/3$ and the effective spring constant is negative (and hence the equilibrium is unstable) at $X_o/\ell_o = 4$.

Part d

The same conclusions as to the stability of the equilibrium points can be made from the figure.

PROBLEM 5.9 (Continued)



Consider the equilibrium at $X_o = 4$. A small displacement to the right makes the force f^e dominate the spring force, and this tends to carry the mass further in the x direction. Hence, this point is unstable. Similar arguments show that the other point is stable.

PROBLEM 5.10

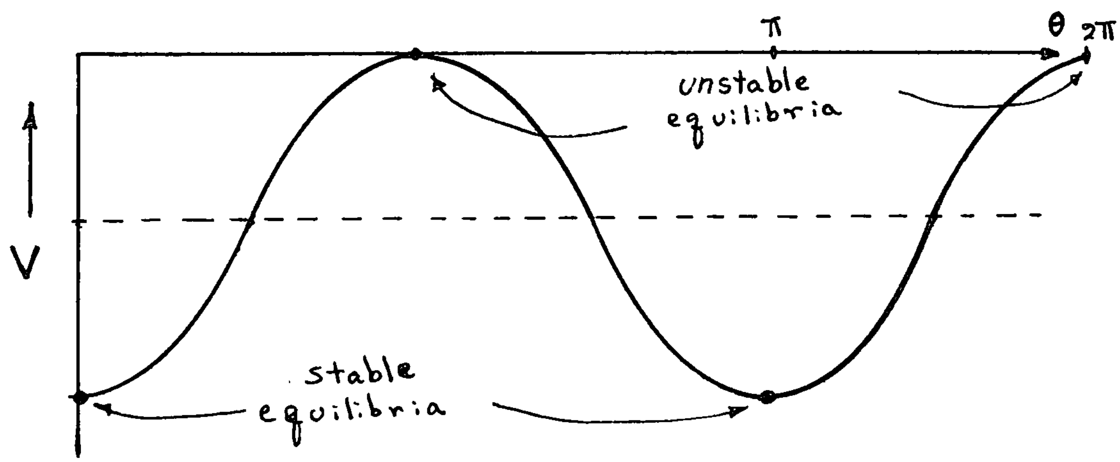
Part a

The terminals are constrained to constant potential, so use coenergy found from terminal equation as

$$W' = \int q dv = \frac{1}{2} C_o(1 + \cos 2\theta)V_o^2 \qquad \text{(a)}$$

Then, since $T^e = \partial W'/\partial\theta$ and there are no other torques acting on the shaft, the total torque can be found by taking the negative derivative of a potential $V = -W'$, where V is the potential well. A sketch of this well is as shown in the figure.

PROBLEM 5.10 (Continued)



Here it is clear that there are points of zero slope (and hence zero torque and possible static equilibrium) at

$$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots \tag{b}$$

Part b

From the potential well it is clear that the first and third equilibria are stable, while the second and fourth are unstable.

PROBLEM 5.11

Part a

From the terminal pair relation, the coenergy is given by

$$W_m'(i_1,i_2,\theta) = \frac{1}{2}(L_o+M\cos 2\theta)i_1^2 + \frac{1}{2}(L_o-M\cos 2\theta)i_2^2 + M\sin 2\theta\, i_1 i_2 \tag{a}$$

so that the torque of electrical origin is

$$T^e = M[\sin 2\theta(i_2^2-i_1^2) + 2\cos 2\theta\, i_1 i_2] \tag{b}$$

Part b

For the two phase currents, as given,

$$i_2^2 - i_1^2 = -I^2\cos 2\omega_s t$$

$$i_1 i_2 = I^2\,\frac{1}{2}\sin 2\omega_s t \tag{c}$$

so that the torque T^e becomes

PROBLEM 5.11 (Continued)

$$T^e = MI^2[-\sin 2\theta \cos 2\omega_s t + \sin 2\omega_s t \cos 2\theta] \qquad (d)$$

or

$$T^e = MI^2 \sin(2\omega_s t - 2\theta) \qquad (e)$$

Substitution of $\theta = \omega_m t + \delta$ obtains

$$T^e = - MI^2 \sin[2(\omega_m - \omega_s)t + 2\delta] \qquad (f)$$

and for this torque to be constant, we must have the frequency condition

$$\omega_m = \omega_s \qquad (g)$$

under which condition, the torque can be written as

$$T^e = - MI^2 \sin 2\delta \qquad (h)$$

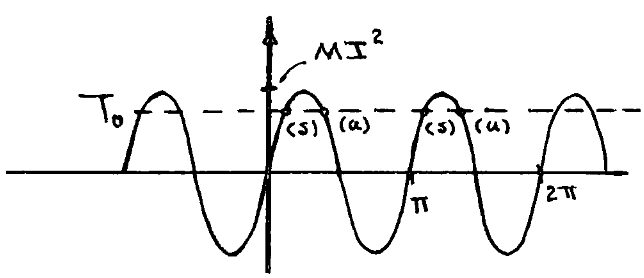
Part c

To determine the possible equilibrium angles δ_o, the perturbations and time derivatives are set to zero in the mechanical equations of motion.

$$T_o = MI^2 \sin 2\delta_o \qquad (i)$$

Here, we have written the time dependence in a form that is convenient if $\cos 2\delta_o > 0$, as it is at the points marked (s) in the figure. Hence, these points are stable. At the points marked (u), the argument of the sin function and the denominator are imaginary, and the response takes the form of a sinh function. Hence, the equilibrium points indicated by (u) are unstable.

Graphical solutions of this expression are shown in the figure. For there to be equilibrium values of δ the currents must be large enough that the torque can be maintained with the rotor in synchronism with the rotating field. $(MI^2 > T_o)$



PROBLEM 5.11 (Continued)

Returning to the perturbation part of the equation of motion with $\omega_m = \omega_s$,

$$J \frac{d^2}{dt^2}(\omega_m t + \delta_o + \delta') = T_o + T' - MI^2 \sin(2\delta_o + 2\delta') \tag{j}$$

linearization gives

$$J \frac{d^2\delta'}{dt^2} + (2MI^2 \cos 2\delta_o)\delta' = T' \tag{k}$$

where the constant terms cancel out by virtue of (i). With $T' = \tau_o u_o(t)$ and initial rest conditions, the initial conditions are

$$\frac{d\delta'}{dt}(0^+) = \frac{\tau_o}{J} \tag{l}$$

$$\delta'(0^+) = 0 \tag{m}$$

and hence the solution for $\delta'(t)$ is

$$\delta'(t) = \frac{\tau_o}{J\sqrt{\dfrac{2MI^2\cos 2\delta_o}{J}}} \sin\left[\sqrt{\frac{2MI^2\cos 2\delta_o}{J}}\, t\right] \tag{n}$$

PROBLEM 5.12

Part a

The magnitude of the field intensity (H) in the gaps is the same. Hence, from Ampere's law,

$$H = Ni/2x \tag{a}$$

and the flux linked by the terminals is N times that passing across either of the gaps.

$$\lambda = \frac{\mu_o a d N^2}{2x} i = L(x)i \tag{b}$$

Because the system is electrically linear, $W'(i,x) = \frac{1}{2} Li^2$, and we have

$$f^e = \frac{\partial W'}{\partial x} = -\frac{N^2 a d \mu_o}{4x^2} i^2 \tag{c}$$

as the required force of electrical origin acting in the x direction.

Part b

Taking into account the forces due to the springs, gravity and the magnetic field, the force equation becomes

PROBLEM 5.12 (Continued)

$$M \frac{d^2x}{dt^2} = -2Kx + Mg - \frac{N^2 ad\mu_o}{4x^2} i^2 + f(t) \qquad \text{(d)}$$

where the last term accounts for the driving force.

The electrical equation requires that the currents sum to zero at the electrical node, where the voltage is $d\lambda/dt$, with λ given by (b).

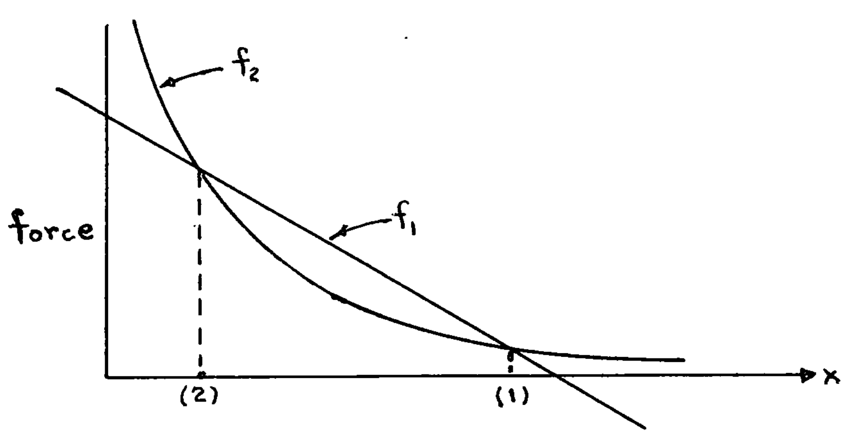
$$I = \frac{1}{R}\frac{d}{dt}\left[\frac{\mu_o adN^2}{2x} i\right] + i \qquad \text{(e)}$$

Part c

In static equilibrium, the electrical equation reduces to $i=I$, while the mechanical equation which takes the form $f_1 = f_2$ is satisfied if

$$-2KX + Mg = \frac{N^2 ad\mu_o I^2}{4X^2} \qquad \text{(f)}$$

Here, f_2 is the negative of the force of electrical origin and therefore (if positive) acts in the $-x$ direction. The respective sides of (f) are shown in the sketch, where the points of possible static equilibrium are indicated. Point (1) is stable, because a small excursion to the right makes f_2 dominate over f_1 and this tends to return the mass in the minus x direction toward the equilibrium point. By contrast, equilibrium point (2) is characterized by having a larger force f_2 and f_1 for small excursions to the left. Hence, the dominate force tends to carry the mass even further from the point of equilibrium and the situation is unstable. In what follows, $x = X$ will be used to indicate the position of stable static equilibrium (1).



PROBLEM 5.12 (Continued)

Part d

If R is very large, then

$$i \approx I$$

even under dynamic conditions. This approximation allows the removal of the characteristic time L/R from the analysis as reflected in the reduction in the order of differential equation required to define the dynamics. The mechanical response is determined by the mechanical equation ($x = X + x'$)

$$M \frac{d^2x'}{dt^2} = -2Kx' + \frac{N^2 ad\mu_o}{2X^3} I^2 x' + f(t) \tag{g}$$

where the constant terms have been balanced out and small perturbations are assumed. In view of the form taken by the excitation, assume $x = \text{Re}\,\hat{x}\,e^{j\omega t}$ and define $K_e \equiv 2K - N^2 ad\mu_o I^2/2X^3$. Then, (g) shows that

$$\hat{x} = \hat{f}/(K_e - \omega^2 M) \tag{h}$$

To compute the output voltage

$$v_o \simeq \frac{d\lambda}{dt}\bigg|_{i=I} = -\frac{\mu_o adN^2 I}{2X^2}\frac{dx'}{dt} \tag{i}$$

or

$$\hat{v}_o = -j\frac{\omega\mu_o adN^2 I}{2X^2}\hat{x} \tag{j}$$

Then, from (h), the transfer function is

$$\frac{\hat{v}_o}{\hat{f}} = -j\frac{\omega\mu_o adN^2 I}{2X^2(K_e - \omega^2 M)} \tag{k}$$

PROBLEM 5.13

Part a

The system is electrically linear. Hence, the coenergy takes the standard form

$$W' = \frac{1}{2}L_{11}i_1^2 + L_{12}i_1i_2 + \frac{1}{2}L_{22}i_2^2 \tag{a}$$

and it follows that the force of electrical origin on the plunger is

$$f^e = \frac{\partial W'}{\partial x} = \frac{1}{2}i_1^2\frac{\partial L_{11}}{\partial x} + i_1i_2\frac{\partial L_{12}}{\partial x} + \frac{1}{2}i_2^2\frac{\partial L_{22}}{\partial x} \tag{b}$$

PROBLEM 5.13 (Continued)

which, for the particular terminal relations of this problem becomes

$$f^e = L_o \left\{ \frac{-i_1^2}{d}\left(1+\frac{x}{d}\right) - \frac{i_1 i_2}{d}\frac{2x}{d} + \frac{i_2^2}{d}\left(1-\frac{x}{d}\right)\right\} \tag{c}$$

Finally, in terms of this force, the mechanical equation of motion is

$$M\frac{d^2x}{dt^2} = -Kx - B\frac{dx}{dt} + f^e \tag{d}$$

The circuit connections show that the currents i_1 and i_2 are related to the source currents by

$$\begin{aligned} i_1 &= I_o + i \\ i_2 &= I_o - i \end{aligned} \tag{e}$$

Part b

If we use (e) in (b) and linearize, it follows that

$$f^e = -\frac{4L_oI_o}{d}\,i - \frac{4L_oI_o^2}{d}\,\frac{x}{d} \tag{f}$$

and the equation of motion is

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + \omega_o^2 x = -\,Ci \tag{g}$$

where

$$\omega_o = \sqrt{\left[K + \frac{4L_oI_o^2}{d^2}\right]/M}$$

$$\alpha = B/M$$

$$C = 4L_oI_o/dM$$

Part c

Both the spring constant and damping in the equation of motion are positive, and hence the system is always stable.

Part d

The homogeneous equation has solutions of the form e^{pt} where

$$p^2 + \alpha p + \omega_o^2 = 0 \tag{h}$$

or, since the system is underdamped

$$p = -\frac{\alpha}{2} \pm j\sqrt{\omega_o^2 - \left(\frac{\alpha}{2}\right)^2} = -\frac{\alpha}{2} \pm j\omega_p \tag{i}$$

PROBLEM 5.13 (Continued)

The general solution is

$$x(t) = -\frac{CI_o}{\omega_o^2} + e^{-\frac{\alpha}{2}t}[A \sin \omega_p t + D \cos \omega_p t] \quad (j)$$

where the constants are determined by the initial conditions $x(0) = 0$ and $dx/dt(0) = 0$

$$D = \frac{CI_o}{\omega_o^2}\ ; \quad A = \frac{\alpha CI_o}{2\omega_p\omega_o^2} \quad (k)$$

Part e

With a sinusoidal steady state condition, assume $x = \text{Re}\ \hat{x}\ e^{j\omega t}$ and write $i(t) = \text{Re}(-jI_o)e^{j\omega t}$ and (g) becomes

$$\hat{x}(-\omega^2 + j\omega\alpha + \omega_o^2) = CjI_o \quad (l)$$

Thus, the required solution is

$$x(t) = \frac{\text{Re}jCI_oe^{j\omega t}}{(\omega_o^2-\omega^2)+j\omega\alpha} \quad (m)$$

PROBLEM 5.14

Part a

From the terminal equations, the current i_1 is determined by Kirchhoff's current law

$$G\ L_1 \frac{di_1}{dt} + i_1 = I + GMI_2\Omega \sin \Omega t \quad (a)$$

The first term in this expression is the current which flows through G because of the voltage developed across the self inductance of the coil, while the last is a current through G induced by the rotational motion. The terms on the right are known functions of time, and constitute a driving function for the linear equation.

Part b

We can divide the solution into particular solutions due to the two driving terms and a homogeneous solution. From the constant drive I we have the solution

$$i_1 = I \quad (b)$$

Because $\sin \Omega t = \text{Re}(-je^{j\Omega t})$, if we assume a particular solution for the sinusoidal drive of the form $i_1 = \text{Re}(\hat{I}_1e^{j\Omega t})$, we have

PROBLEM 5.14 (Continued)

$$\hat{I}_1\ (j\Omega GL_1 + 1) = -\ j\Omega GMI_2 \tag{c}$$

or, rearranging

$$\hat{I}_1 = \frac{-\Omega GMI_2(\Omega GL_1 + j)}{1 + (\Omega GL_1)^2} \tag{d}$$

We now multiply this complex amplitude by $e^{j\Omega t}$ and take the real part to obtain the particular solution due to the sinusoidal drive

$$i_1 = \frac{-GMI_2\Omega}{1+(\Omega GL_1)^2}\,(\Omega GL_1 \cos \Omega t - \sin \Omega t) \tag{e}$$

The homogeneous solution is

$$i_1 = A\, e^{-t/GL_1} \tag{f}$$

and the total solution is the sum of (b), (e) and (f) with the constant A determined by the initial conditions.

In view of the initial conditions, the complete solution for i_1, normalized to the value necessary to produce a flux equal to the maximum mutual flux, is then

$$\frac{L_1 i_1}{MI_2} = \left[\frac{(\Omega GL_1)^2}{1+(\Omega GL_1)^2} - \frac{L_1 I}{MI_2}\right] e^{-\frac{\Omega t}{\Omega GL_1}}$$

$$+ \frac{GL_1\Omega}{1+(\Omega GL_1)^2}\,(\sin \Omega t - \Omega GL_1 \cos \Omega t) + \frac{L_1 I}{MI_2} \tag{g}$$

Part c

The terminal relation is used to find the flux linking coil 1

$$\frac{\lambda_1}{MI_2} = \left[\frac{(\Omega GL_1)^2}{1+(\Omega GL_1)^2} - \frac{L_1 I}{MI_2}\right] e^{-\frac{\Omega t}{\Omega GL_1}}$$

$$+ \frac{GL_1\Omega}{1+(\Omega GL_1)^2}\sin \Omega t + \frac{\cos \Omega t}{1+(\Omega GL_1)^2} + \frac{L_1 I}{MI_2} \tag{h}$$

The flux has been normalized with respect to the maximum mutual flux (MI_2).

PROBLEM 5.14 (Continued)

Part d

In order to identify the limiting cases and the appropriate approximations it is useful to plot (g) and (h) as functions of time. These equations contain two constants, ΩGL_1 and L_1I/MI_2. The time required for one rotation is $2\pi/\Omega$ and GL_1 is the time constant of the inductance L_1 and conductance G in series. Thus, ΩGL_1 is essentially the ratio of an electrical time constant to the time required for the coil to traverse the applied field one time. The quantity MI_2 is the maximum flux of the externally applied field that can link the rotatable coil and L_1I is the self flux of the coil due to current I acting alone. Thus, L_1I/MI_2 is the ratio of self excitation to mutual excitation.

To first consider the limiting case that can be approximated by a current source we require that

$$\Omega GL_1 << 1 \text{ and } \Omega GL_1 << \frac{L_1I}{MI_2} \quad \text{(i)}$$

To demonstrate this set

$$\Omega GL_1 = 0.1 \text{ and } \frac{L_1I}{MI_2} = 1 \quad \text{(j)}$$

and plot current and flux as shown in Fig. (a). We note first that the transient dies out very quickly compared to the time of one rotation. Furthermore, the flux varies appreciably while the current varies very little compared to its average value. In the ideal limit ($G\to 0$) the transient would die out instantaneously and the current would be constant. Thus the approximation of the situation by an ideal current-source excitation would involve a small error; however, the saving in analytical time is often well worth the decrease in accuracy resulting from the approximation.

Part e

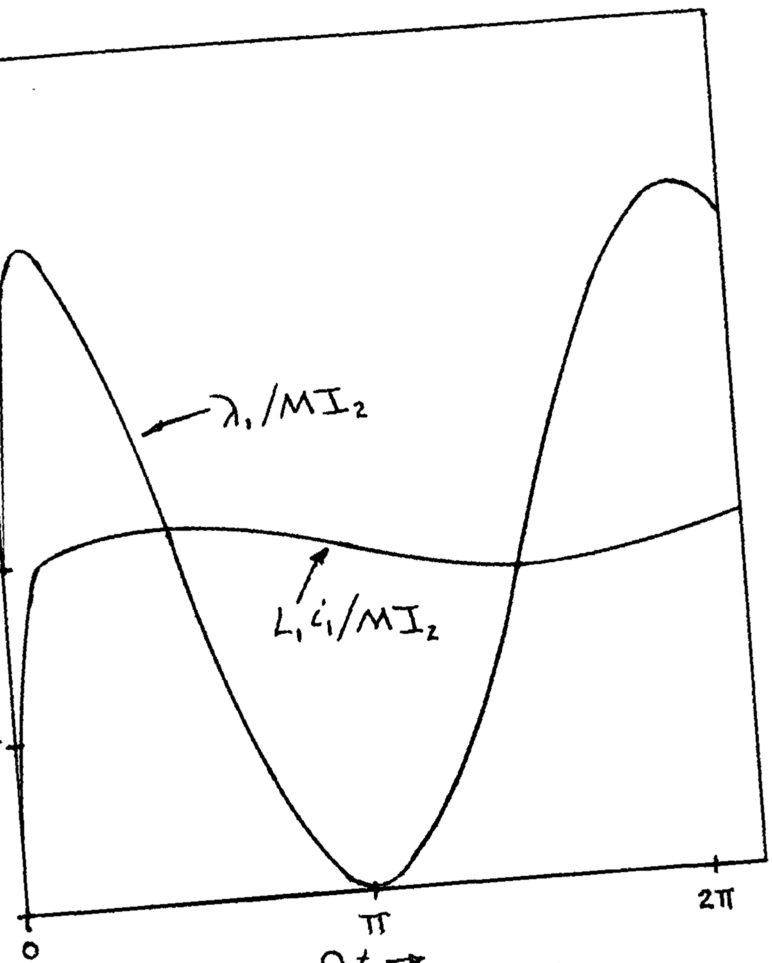
We next consider the limiting case that can be approximated by a constant-flux constraint. This requires that

$$\Omega GL_1 >> 1 \quad \text{(k)}$$

To study this case, set

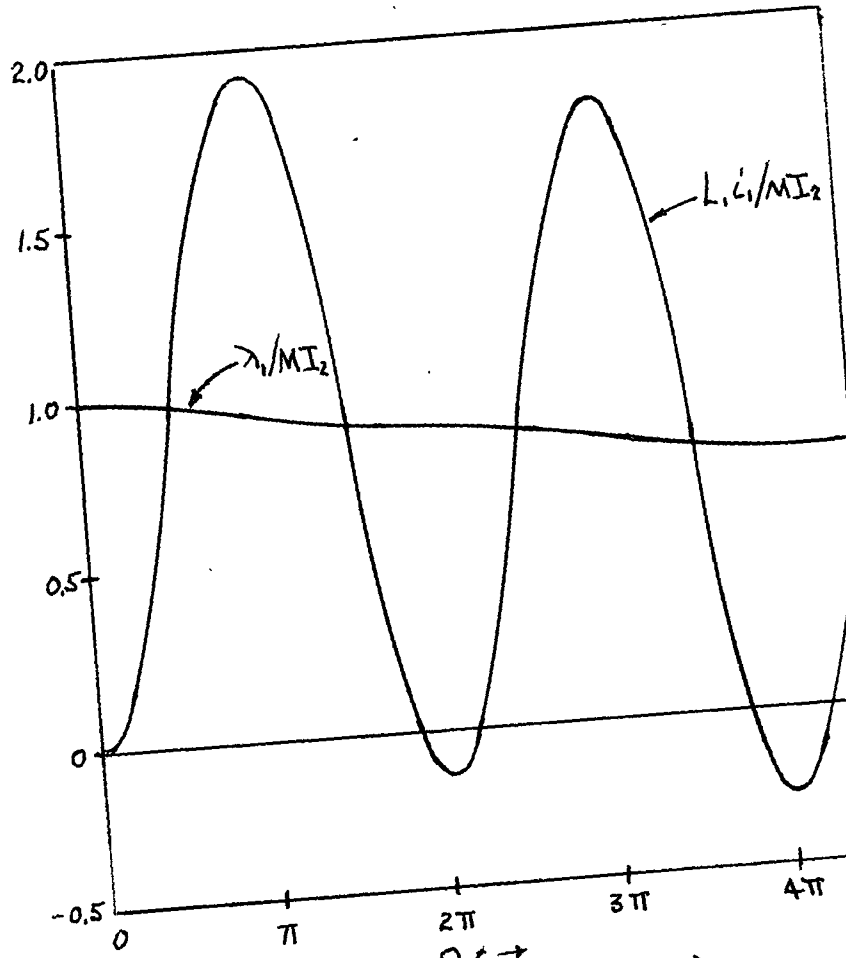
$$\Omega GL_1 = 50 \text{ and } I = 0 \quad \text{(l)}$$

The resulting curves of flux and current are shown plotted in Fig. (b). Note that with this constraint the current varies drastically but the flux pulsates only slightly about a value that decays slowly compared to a rotational period. Thus, when considering events that occur in a time interval comparable



λ_1/MI_2
$L_1 i_1/MI_2$
0
π
2π
$\Omega t \rightarrow$

(a)



2.0
1.5
1.0
0.5
0
-0.5
$L_1 i_1/MI_2$
λ_1/MI_2
0
π
2π
3π
4π
$\Omega t \rightarrow$

(b)

PROBLEM 5.14 (Continued)

with the rotational period, we can approximate this system with a constant-flux constraint. In the ideal, limiting case, which can be approached with superconductors, $G\to\infty$ and λ_1 stays constant at its initial value. This initial value is the flux that links the coil at the instant the switch S is closed.

In the limiting cases of constant-current and constant flux constraints the losses in the electrical circuit go to zero. This fact allows us to take advantage of the conservative character of lossless systems, as discussed in Sec. 5.2.1.

Part f

Between the two limiting cases of constant-current and constant flux constraints the conductance G is finite and provides electrical damping on the mechanical system. We can show this by demonstrating that mechanical power supplied by the speed source is dissipated in the conductance G. For this purpose we need to evaluate the torque supplied by the speed source. Because the rotational velocity is constant, we have

$$T^m = - T^e \tag{m}$$

The torque of electrical origin T^e is in turn

$$T^e = \frac{\partial W'(i_1, i_2, \theta)}{\partial\theta} \tag{n}$$

Because the system is electrically linear, the coenergy W' is

$$W' = \frac{1}{2} L_1 i_1^2 + M\, i_1 i_2 \cos\theta + \frac{1}{2} L_2\, i_2^2 \tag{o}$$

and therefore,

$$T^e = - M\, i_1\, I_2 \sin\theta \tag{p}$$

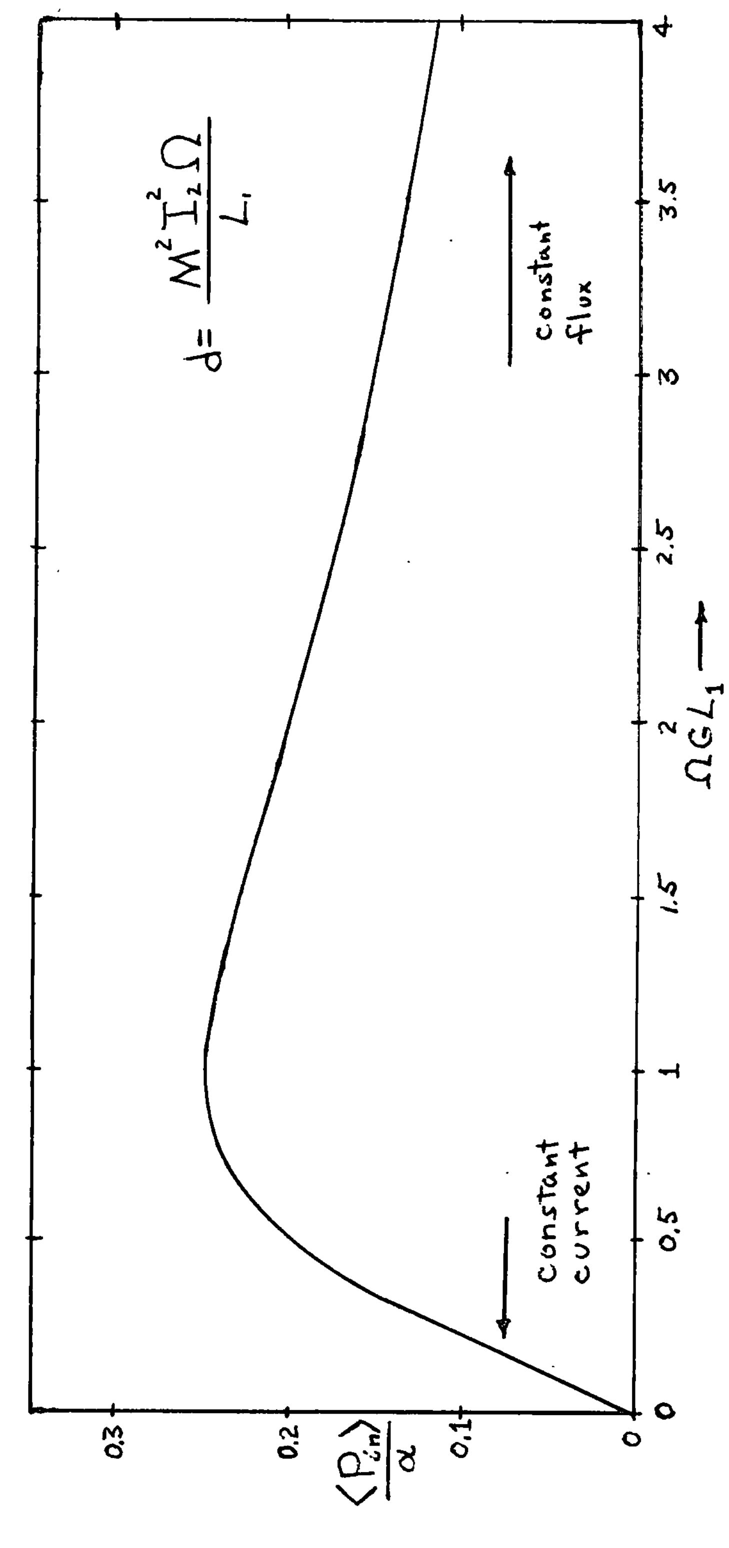
The power supplied by the torque T^m to rotate the coil is

$$P_{in} = - T^e \frac{d\theta}{dt} = M\Omega i_1 I_2 \sin\Omega t \tag{q}$$

Part g

Hence, from (p) and (q), it follows that in the sinusoidal steady state the average power $<P_{in}>$ supplied by the external torque is

$$<P_{in}> = \frac{1}{2}\left[\frac{G\, M^2 I_2^2\, \Omega^2}{1+(\Omega G L_1)^2}\right] \tag{r}$$



$\alpha = \frac{M^2 I_2^2 \Omega}{L_1}$
constant flux
constant current
$\frac{\langle P_{in} \rangle}{\alpha}$
0.3
0.2
0.1
0
0
0.5
1
1.5
2
2.5
3
3.5
4
$\Omega G L_1$

PROBLEM 5.14 (Continued)

This power, which is dissipated in the conductance G, is plotted as a function of ΩGL_1 in Fig. (c). Note that because Ω and L_1 are used as normalizing constants, ΩGL_1 can only be varied by varying G. Note that for both large and small values of ΩGL_1 the average mechanical power dissipated in G becomes small. The maximum in $\langle P_{in}\rangle$ occurs at $\Omega GL_1 = 1$.

PROBLEM 5.15

Part a

The coenergy of the capacitor is

$$W_e' = \frac{1}{2} C(x) V^2 = \frac{1}{2} (\varepsilon_o \frac{A}{x}) V^2$$

The electric force in the x direction is

$$f_e = \frac{\partial W_e'}{\partial x} = -\frac{1}{2} \frac{\varepsilon_o A}{x^2} V^2$$

If this force is linearized around $x = x_o$, $V = V_o$

$$f_e(x) = -\frac{1}{2} \frac{\varepsilon_o A V_o^2}{x_o^2} - \frac{\varepsilon_o A V_o v}{x_o^2} + \frac{\varepsilon_o A V_o^2 x}{x_o^3}$$

The linearized equation of motion is then

$$B \frac{dx}{dt} + (K - \frac{\varepsilon_o A V_o^2}{x_o^3}) x = -\frac{\varepsilon_o A}{x_o^2} V_o v + f(t)$$

The equation for the electric circuit is

$$V + R \frac{d}{dt} (C(x)V) = V_o$$

Part b

We can keep the voltage constant if

$$R \longrightarrow 0$$

In this case

$$B \frac{dx}{dt} + K'x = f(t) = F\, u_{-1}(t); \; K' = K - \frac{\varepsilon_o A V_o^2}{x_o^3}$$

The particular solution is

$$x(t) = F/K'$$

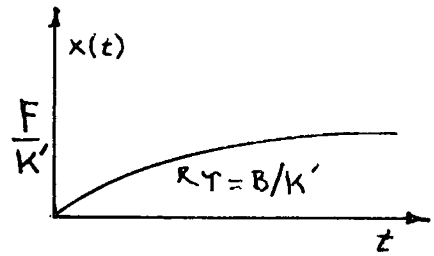
PROBLEM 5.15 (Continued)

The natural frequency S is the solution to

$$sB + K'x = 0 \qquad s = -K'/B$$

Notice that since

$$K'/B = (K - \frac{\varepsilon_o A V_o^2}{x_o^3})/B$$



there is voltage V_o above which the system is unstable. Assuming V_o is less than this voltage

$$x(t) = F/K' \, (1 - e^{-(K'/B)t})$$

Now we can be more specific about the size of R. We want the time constant of the RC circuit to be small compared to the "action time" of the mechanical system

$$RC(x_o) << B/K'$$

$$R << \frac{B}{K'C(x_o)}$$

Part c

From part a we suspect that

$$RC(x_o) >> \tau_{mech}$$

where τ_{mech} can be found by letting $R \to \infty$. Since the charge will be constant

$$\frac{dq}{dt} = 0 \qquad q = C(x_o)V_o = C(x_o + x)(V_o + v)$$

$$\approx C(x_o)V_o + C(x_o)v + V_o \frac{dC}{dx}(x_o)x$$

$$v = -\frac{V_o}{C(x_o)} \frac{dC}{dx}(x_o)x = +\frac{V_o x}{\varepsilon_o A} \frac{\varepsilon_o A}{x_o^2} x = V_o \frac{x}{x_o}$$

Using this expression for induced v, the linearized equation of motion becomes

$$B\frac{dx}{dt} + (K - \frac{\varepsilon_o A V_o^2}{x_o^3})x = -\frac{\varepsilon_o A}{x_o^3} V_o^2 x + f(t)$$

$$B\frac{dx}{dt} + Kx = f(t)$$

PROBLEM 5.15 (Continued)

The electric effect disappears because the force of a capacitor with constant charge is independent of the plate separation. The solutions are the same as part (a) except that $K' = K$. The constraint on the resistor is then

$$R >> \frac{1}{C(x_o)} \quad B/K$$

PROBLEM 5.16

We wish to write the sum of the forces in the form

$$f = f_1 + f_2 = -\frac{\partial V}{\partial x} \qquad \text{(a)}$$

For $x > 0$, this is done by making

$$V = -\frac{1}{2} Kx^2 + F_o x \qquad \text{(b)}$$

as shown in the figure. The potential is symmetric about the origin. The largest value of v_o that can be contained by the potential well is determined by the peak value of potential which, from (b), comes at

$$x = F_o/K \qquad \text{(c)}$$

where the potential is

$$V = \frac{1}{2} F_o^2/K \qquad \text{(d)}$$

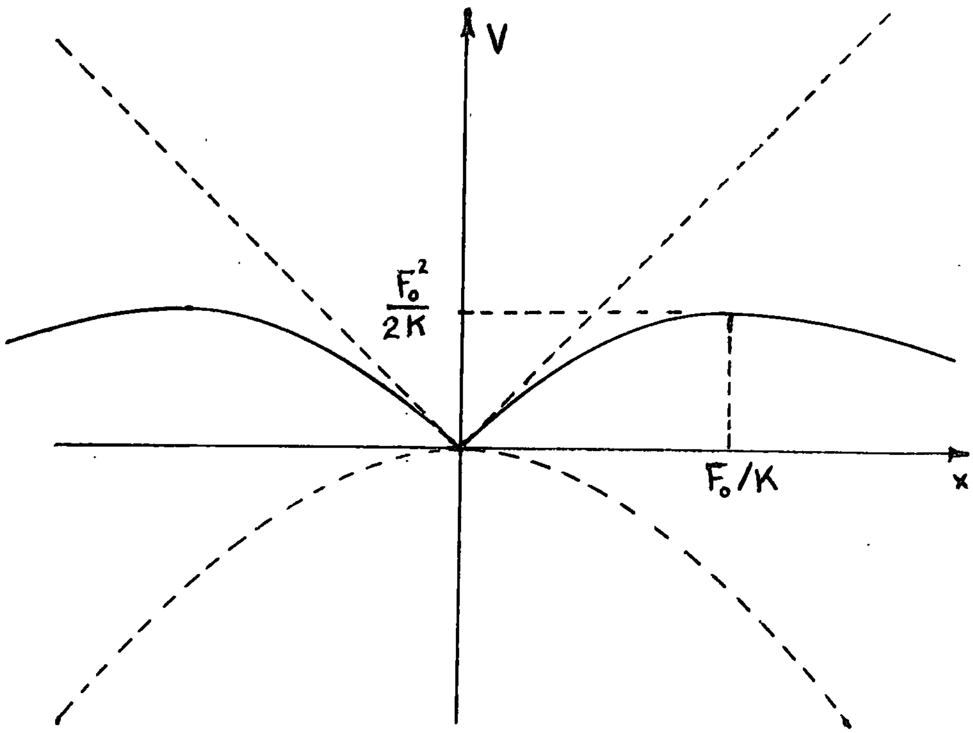
Because the minimum value of the potential is zero, this means that the kinetic energy must exceed this peak value to surmount the barrier. Hence,

$$\frac{1}{2} Mv_o^2 = \frac{1}{2} F_o^2/K \qquad \text{(e)}$$

or

$$v_o = \sqrt{\frac{F_o^2}{KM}} \qquad \text{(f)}$$

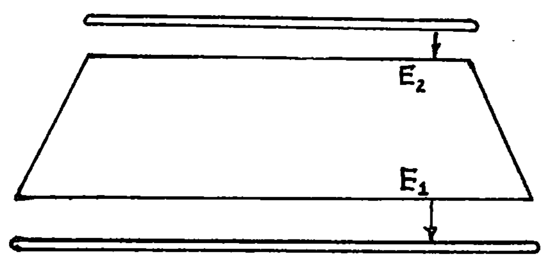
PROBLEM 5.16 (Continued)



PROBLEM 5.17

Part a

The electric field intensities defined in the figure are



$$E_2 = (v_2 - v_1)/(d-x) \tag{a}$$

$$E_1 = v_1/(d+x) \tag{b}$$

Hence, the total charge on the respective electrodes is

$$q_1 = v_1\left[\frac{A_2\varepsilon_o}{d+x} + \frac{A_1\varepsilon_o}{d-x}\right] - \frac{v_2 A_1 \varepsilon_o}{d-x} \tag{c}$$

$$q_2 = \frac{A_1\varepsilon_o(v_2-v_1)}{d-x} \tag{d}$$

Part b

Conservation of energy requires

$$v_1 dq_1 + v_2 dq_2 = dW + f^e dx \tag{e}$$

and since the charge q_1 and voltage v_2 are constrained, we make the transformation $v_2dq_2 = d(v_2q_2)-q_2dv_2$ to obtain

$$v_1dq_1-q_2dv_2 = dW'' + f^e dx \tag{f}$$

It follows from this form of the conservation of energy equation that $f^e = -\frac{\partial W''}{\partial x}$ and hence $W'' \equiv U$. To find the desired function we integrate (f) using the terminal relations.

$$U = W'' = \int v_1dq_1 - q_2dv_2 \tag{g}$$

The integration on q_1 makes no contribution since q_1 is constrained to be zero. We require $v_2(q_1=0,v_2)$ to evaluate the remaining integral

$$q_2(q_1=0,v_2) = \frac{v_2A_1\varepsilon_o}{d-x}\left[1 - \frac{1}{[\frac{A_2(d-x)}{A_1(d+x)} + 1]}\right] \tag{h}$$

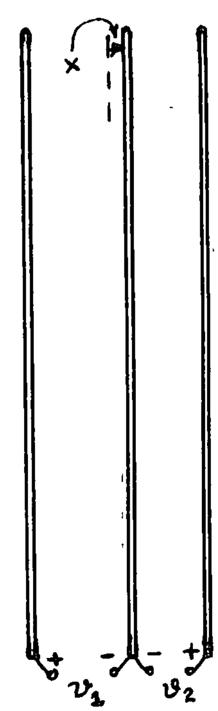
Then, from (g),

$$U = -\frac{1}{2}\frac{V_o^2A_1\varepsilon_o}{d-x}\left[1 - \frac{1}{[\frac{A_2(d-x)}{A_1(d+x)} + 1]}\right] \tag{i}$$

PROBLEM 5.18

Part a

Because the two outer plates are constrained differently once the switch is opened, it is convenient to work in terms of two electrical terminal pairs, defined as shown in the figure. The plane parallel geometry makes it straightforward to compute the terminal relations as being those for simple parallel plate capacitors, with no mutual capacitance.



$$q_1 = v_1\varepsilon_o A/a + x \tag{a}$$

$$q_2 = v_2\varepsilon_o A/a-x \tag{b}$$

PROBLEM 5.18 (Continued)

Conservation of energy for the electromechanical coupling requires

$$v_1 dq_1 + v_2 dq_2 = dW + f^e dx \tag{c}$$

This is written in a form where q_1 and v_2 are the independent variables by using the transformation $v_2 dq_2 = d(v_2 q_2) - q_2 dv_2$ and defining $W'' = W - v_2 q_2$

$$v_1 dq_1 - q_2 dv_2 = dW'' + f^e dx \tag{d}$$

This is done because after the switch is opened it is these variables that are conserved. In fact, for $t > 0$,

$$v_2 = V_o \text{ and (from (a))} q_1 = V_o \varepsilon_o A/a \tag{e}$$

The energy function W" follows from (d) and the terminal conditions, as

$$W'' = \int v_1 dq_1 - \int q_2 dv_2 \tag{f}$$

or

$$\frac{1}{2}\frac{(a+x)}{\varepsilon_o A} q_1^2 - \frac{1}{2}\frac{\varepsilon_o A v_2^2}{a-x} \tag{g}$$

Hence, for $t > 0$, we have (from (e))

$$W'' = \frac{1}{2}\frac{(a+x)}{a^2}\varepsilon_o A V_o^2 - \frac{1}{2}\frac{\varepsilon_o A V_o^2}{a-x} \tag{h}$$

Part b

The electrical force on the plate is $f^e = -\frac{\partial W''}{\partial x}$. Hence, the force equation is (assuming a mass M for the plate)

$$M\frac{d^2x}{dt^2} = -Kx - \frac{1}{2}\frac{\varepsilon_o A V_o^2}{a^2} + \frac{1}{2}\frac{\varepsilon_o A V_o^2}{(a-x)^2} \tag{i}$$

For small excursions about the origin, this can be written as

$$M\frac{d^2x}{dt^2} = -Kx - \frac{1}{2}\frac{\varepsilon_o A V_o^2}{a^2} + \frac{1}{2}\frac{\varepsilon_o A V_o^2}{a^2} + \frac{\varepsilon_o A V_o^2}{a^3}x \tag{j}$$

The constant terms balance, showing that a static equilibrium at the origin is possible. Then, the system is stable if the effective spring constant is positive.

$$K > \varepsilon_o A V_o^2/a^3 \tag{k}$$

Part c

The total potential V(x) for the system is the sum of W" and the potential energy stored in the springs. That is,

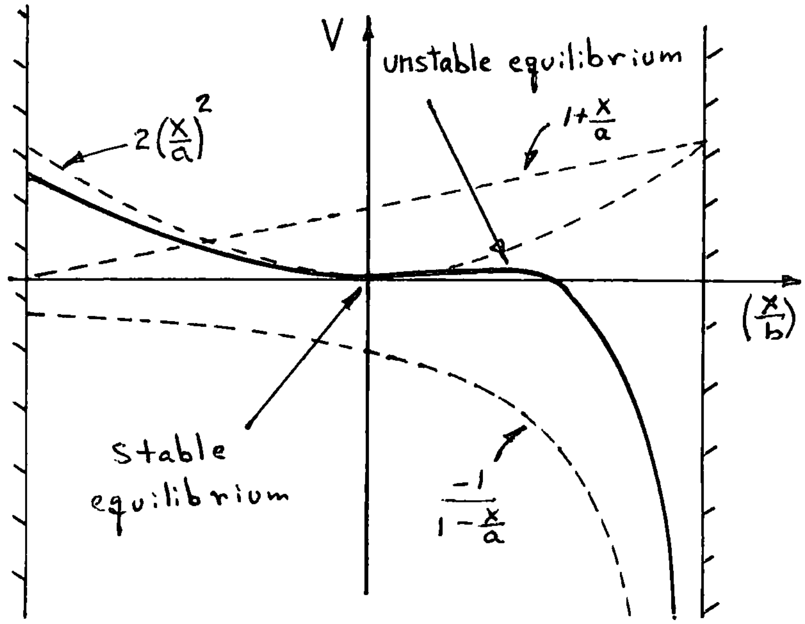
PROBLEM 5.18 (Continued)

$$V = \frac{1}{2} Kx^2 + \frac{1}{2} \frac{(a+x)}{a^2} \varepsilon_o AV_o^2 - \frac{1}{2} \frac{\varepsilon_o AV_o^2}{a-x} \tag{l}$$

or

$$\frac{a^2K}{2} \left(\frac{x}{a}\right)^2 + \frac{1}{2} \frac{\varepsilon_o AV_o^2}{a} \left[\left(1 + \frac{x}{a}\right) - \frac{1}{\left(1 - \frac{x}{a}\right)} \right] \tag{m}$$

This is sketched in the figure for $a^2K/2 = 2$ and $1/2\ \varepsilon_o AV_o^2/a = 1$. In addition to the point of stable equilibrium at the origin, there is also an unstable equilibrium point just to the right of the origin.



PROBLEM 5.19

Part a

The coenergy is

$$W' = \frac{1}{2} Li^2 = \frac{1}{2} L_o i^2/[1 - \frac{x}{a}]^4 \tag{a}$$

and hence the force of electrical origin is

$$f^e = \frac{\partial W'}{\partial x} = 2L_o i^2/a[1 - \frac{x}{a}]^5 \tag{b}$$

Hence, the mechanical equation of motion, written as a function of (i,x) is

$$M \frac{d^2x}{dt^2} = - Mg + \frac{2L_o i^2}{a[1 - \frac{x}{a}]^5} \tag{c}$$

PROBLEM 5.19 (Continued)

while the electrical loop equation, written in terms of these same variables (using the terminal relation for λ) is

$$V_o + v = Ri + \frac{d}{dt}\left[\frac{L_o i}{(1-\frac{x}{a})^4}\right] \tag{d}$$

These last two expressions are the equations of motion for the mass.

Part b

In static equilibrium, the above equations are satisfied by (x,v,i) having the respective values (X_o,V_o,I_o). Hence, we assume that

$$x = X_o + x'(t): \quad v = V_o + v(t): \quad i = I_o + i'(t) \tag{e}$$

The equilibrium part of (c) is then

$$0 = -Mg + \frac{2L_oI_o^2}{a}/(1-\frac{X_o}{a})^5 \tag{f}$$

while the perturbations from this equilibrium are governed by

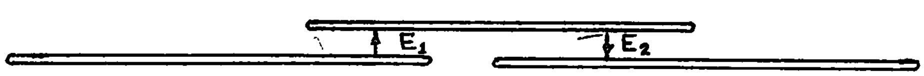
$$M\frac{d^2x'}{dt^2} = +\frac{10\,L_oI_o^2\,x'}{a^2(1-\frac{X_o}{a})^6} + \frac{4\,L_oI_o\,i'}{a(1-\frac{X_o}{a})^5} \tag{g}$$

The equilibrium part of (d) is simply $V_o = I_oR$, and the perturbation part is

$$v = Ri' + \frac{L_o}{[1-\frac{X_o}{a}]^4}\frac{di'}{dt} + \frac{4\,L_oI_o}{a[1-\frac{X_o}{a}]^5}\frac{dx'}{dt} \tag{h}$$

Equations (g) and (h) are the linearized equations of motion for the system which can be solved given the driving function $v(t)$ and (if the transient is of interest) the initial conditions.

PROBLEM 5.20



Part a

The electric field intensities, defined as shown, are

$$E_1 = (v_1-v_2)/s; \quad E_2 = v_2/s \tag{a}$$

PROBLEM 5.20 (Continued)

In terms of these quantities, the charges are

$$q_1 = \varepsilon_o(\frac{a}{2} - x)dE_1; \; q_2 = - \varepsilon_o(\frac{a}{2} - x)dE_1 + \varepsilon_o(\frac{a}{2} + x)dE_2 \qquad (b)$$

Combining (a) and (b), we have the required terminal relations

$$\begin{aligned} q_1 &= v_1 C_{11} - v_2 C_{12} \\ q_2 &= -v_1 C_{12} + v_2 C_{22} \end{aligned} \qquad (c)$$

where

$$C_{11} = \frac{\varepsilon_o d}{s}(\frac{a}{2} - x); \quad C_{22} = \frac{\varepsilon_o ad}{s}$$

$$C_{12} = \frac{\varepsilon_o d}{s}(\frac{a}{2} - x)$$

For the next part it is convenient to write these as $q_1(v_1,q_2)$ and $v_2(v_1,q_2)$.

$$\begin{aligned} q_1 &= v_1 [C_{11} - \frac{C_{12}^2}{C_{22}}] - q_2 \frac{C_{12}}{C_{22}} \\ v_2 &= \frac{q_2}{C_{22}} + v_1 \frac{C_{12}}{C_{22}} \end{aligned} \qquad (d)$$

Part b

Conservation of energy for the coupling requires

$$v_1 dq_1 + v_2 dq_2 = dW + f^e dx \qquad (e)$$

To treat v_1 and q_2 as independent variables (since they are constrained to be constant) we let $v_1 dq_1 = d(v_1 q_1) - q_1 dv_1$, and write (e) as

$$-q_1 \, dv_1 + v_2 dq_2 = - dW'' + f^e \, dx \qquad (f)$$

From this expression it is clear that $f^e = \partial W''/\partial x$ as required. In particular, the function W" is found by integrating (f)

$$W'' = \int_o^{V_o} q_1(v_1',0)dv_1' - \int_o^{Q} v_2(V_o,q_2')dq_2' \qquad (g)$$

to obtain

$$W'' = \frac{1}{2} V_o^2 [C_{11} - \frac{C_{12}^2}{C_{22}}] - \frac{Q^2}{2C_{22}} - \frac{V_o Q C_{12}}{C_{22}} \qquad (h)$$

Of course, C_{11}, C_{22} and C_{12} are functions of x as defined in (c).

PROBLEM 5.21

Part a

The equation of motion as developed in Prob. 3.8 but with $I(t)=I_o$=constant, is

$$J\frac{d^2\theta}{dt^2} = -\frac{I\,L_m^2\,I_o}{L_2}(1-\cos\theta)\sin\theta \tag{a}$$

This has the required form if we define

$$V = -\frac{I\,L_m^2}{L_2}I_o(\cos\theta + \frac{1}{2}\sin^2\theta) \tag{b}$$

as can be seen by differentiating (b) and recovering the equation of motion. This potential function could also have been obtained by starting directly with the thermodynamic energy equation and finding a hybred energy function (one having i_1, λ_2,θ as independent variables). See Example 5.2.2 for this more fundamental approach.

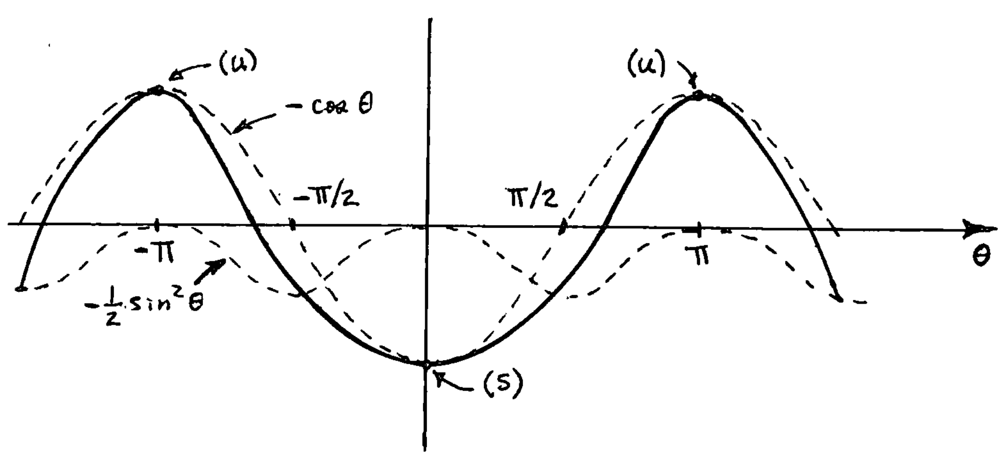
Part b

A sketch of the potential well is as shown below. The rotor can be in stable static equilibrium at $\theta = 0$ (s) and unstable static equilibrium at $\theta = \pi$(u).

Part c

For the rotor to execute continuous rotory motion from an initial rest position at $\theta = 0$, it must have sufficient kinetic energy to surmount the peak in potential at $\theta = \pi$. To do this,

$$\frac{1}{2}J(\frac{d\theta}{dt})^2 \geq \frac{2IL_m^2I_o}{L_2} \tag{c}$$



PROBLEM 5.22

Part a

The coenergy stored in the magnetic coupling is simply

$$W' = \frac{1}{2}L_o(1 + 0.2 \cos \theta + 0.05 \cos 2\theta)i^2 \qquad \text{(a)}$$

Since the gravitational field exerts a torque on the pendulum given by

$$T_p = \frac{\partial}{\partial\theta}(-Mg\,\ell\,\cos\theta) \qquad \text{(b)}$$

and the torque of electrical origin is $T^e = \partial W'/\partial\theta$, the mechanical equation of motion is

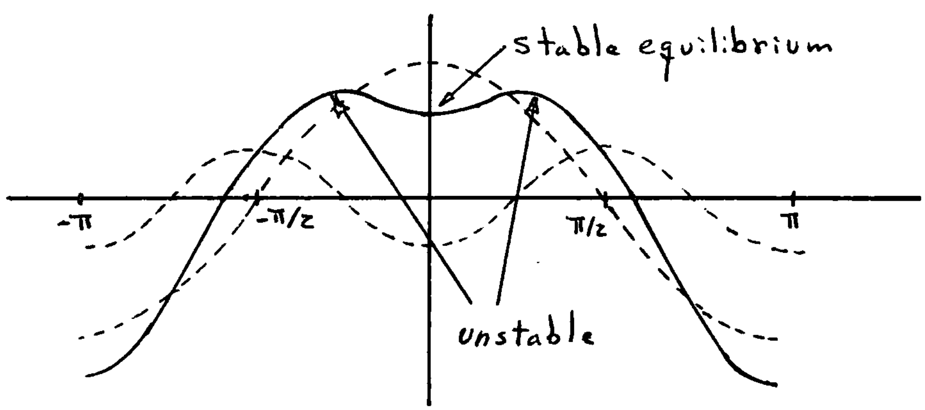
$$\frac{d}{dt}\left[\frac{1}{2}M\ell^2\left(\frac{d\theta}{dt}\right)^2 + V\right] = 0 \qquad \text{(c)}$$

where (because $I^2L_o = 6Mg\ell$)

$$V = Mg\ell[0.4 \cos\theta - 0.15 \cos 2\theta - 3]$$

Part b

The potential distribution V is plotted in the figure, where it is evident that there is a point of stable static equilibrium at $\theta = 0$ (the pendulum straight up) and two points of unstable static equilibrium to either side of center. The constant contribution has been ignored in the plot because it is arbitrary.



PROBLEM 5.23

Part a

The magnetic field intensity is uniform over the cross section and equal to the surface current flowing around the circuit. Define H as into the paper and $H = i/D$. Then λ is H multiplied by μ_o and the area xd.

$$\lambda = \frac{\mu_o xd}{D} i \tag{a}$$

The system is electrically linear and so the energy is $W = \frac{1}{2}\lambda^2/L$. Then, since $f^e = -\partial W/\partial x$, the equation of motion is

$$M\frac{d^2x}{dt^2} = f = -Kx + \frac{1}{2}\frac{\Lambda^2 D}{\mu_o x^2 d} \tag{b}$$

Part b

Let $x = X_o + x'$ where x' is small and (b) becomes approximately

$$M\frac{d^2x'}{dt^2} = -KX_o - Kx' + \frac{1}{2}\frac{\Lambda^2 D}{\mu_o X_o^2 d} - \frac{\Lambda^2 Dx'}{\mu_o X_o^3 d} \tag{c}$$

The constant terms define the static equilibrium

$$X_o = \left[\frac{1}{2}\frac{\Lambda^2 D}{\mu_o dK}\right]^{1/3} \tag{d}$$

and if we use this expression for X_o, the perturbation equation becomes,

$$M\frac{d^2x'}{dt^2} = -Kx' - 2Kx' \tag{e}$$

Hence, the point of equilibrium at X_o as given by (d) is stable, and the magnetic field is equivalent to the spring constant 2K.

Part c

The total force is the negative derivative with respect to x of V where

$$V = \frac{1}{2}Kx^2 + \frac{1}{2}\frac{\Lambda^2 D}{\mu_o xd} \tag{f}$$

This makes it possible to integrate the equation of motion (b) once to obtain

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{M}(E-V)} \tag{g}$$

The potential well is as shown in figure (a). Here again it is apparent that the equilibrium point is one where the mass can be static and stable. The constant of integration E is established physically by releasing the mass from static

PROBLEM 5.23 (Continued)

positions such as (1) or (2) shown in Fig. (a). Then the bounded excursions of the mass can be pictured as having the level E shown in the diagram. The motions are periodic in nature regardless of the initial position or velocity.

Part d

The constant flux dynamics can be contrasted with those occurring at constant current simply by replacing the energy function with the coenergy function. That is, with the constant current constraint, it is appropriate to find the electrical force from $W' = \frac{1}{2} Li^2$, where $f^e = \partial W'/\partial x$. Hence, in this case

$$V = \frac{1}{2} Kx^2 - \frac{1}{2} \frac{\mu_o xd}{D} I^2 \tag{h}$$

A plot of this potential well is shown in Fig. (b). Once again there is a point X_o of stable static equilibrium given by

$$X_o = \frac{1}{2} \frac{\mu_o dI^2}{DK} \tag{i}$$

However, note that if oscillations of sufficiently large amplitude are initiated that it is now possible for the plate to hit the bottom of the parallel plate system at $x = 0$.

PROBLEM 5.25

Part a

Force on the capacitor plate is simply

$$f^e = \frac{\partial W'}{\partial x} = \frac{\partial}{\partial x} [\frac{1}{2} \frac{\pi a^2 \varepsilon_o v^2}{x}] \tag{a}$$

due to the electric field and a force f due to the attached string.

Part b

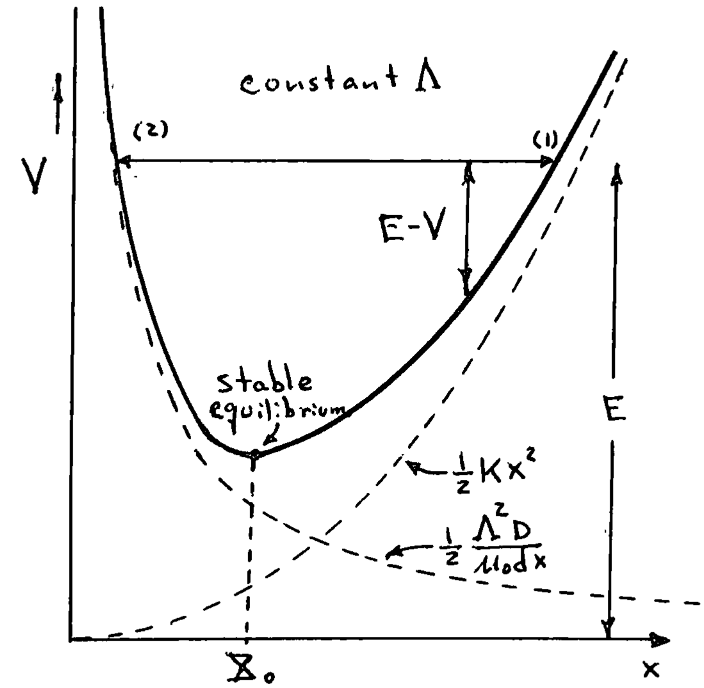
With the mass M_1 rotating at a constant angular velocity, the force f^e must balance the centrifugal force $\omega_m^2 rM_1$ transmitted to the capacitor plate by the string.

$$\frac{1}{2} \frac{\pi a^2 \varepsilon_o V_o^2}{\ell^2} = \omega_m^2 \ell M_1 \tag{b}$$

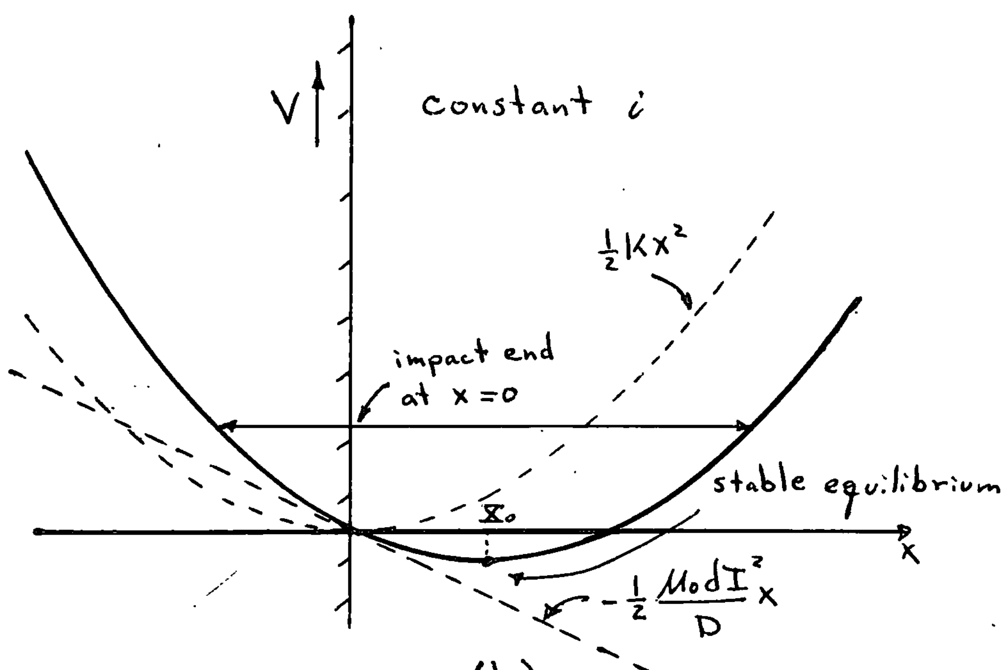
or

$$\omega_m = \sqrt{\frac{\pi a^2 \varepsilon_o V_o^2}{2 \ell^3 M_1}} \tag{c}$$

where ℓ is both the equilibrium spacing of the plates and the equilibrium radius of the trajectory for M_1.


constant Λ
V
(2)
(1)
E-V
E
stable equilibrium
$\frac{1}{2}Kx^2$
$\frac{1}{2}\frac{\Lambda^2 D}{\mu_o d x}$
X_o
x

(a)


V
constant i
$\frac{1}{2}Kx^2$
impact end at x = 0
stable equilibrium
X_o
x
$-\frac{1}{2}\frac{\mu_o d I^2}{D}x$

(b)

PROBLEM 5.25 (Continued)

Part c

The θ directed force equation is (see Prob. 2.8) for the accleration on a particle in circular coordinates)

$$M_1[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}] = 0 \tag{d}$$

which can be written as

$$\frac{d}{dt}[M_1 r^2 \frac{d\theta}{dt}] = 0 \tag{e}$$

This shows that the angular momentum is constant even as the mass M_1 moves in and out

$$M_1 r^2 \frac{d\theta}{dt} = M_1 \ell^2 \omega_m = \text{constant of the motion} \tag{f}$$

This result simply shows that if the radius increases, the angular velocity must decrease accordingly

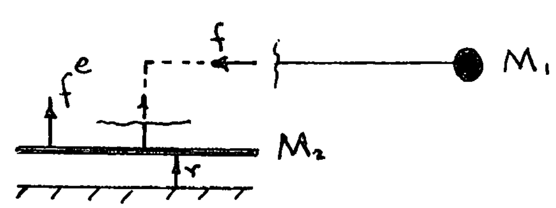
$$\frac{d\theta}{dt} = \frac{\ell^2}{r^2}\omega_m \tag{g}$$

Part d

The radial component of the force equation for M_1 is

$$M_1[\frac{d^2r}{dt^2} - r(\frac{d\theta}{dt})^2] = -f \tag{h}$$

where f is the force transmitted by the string, as shown in the figure.



The force equation for the capacitor plate is

$$M_2\frac{d^2r}{dt^2} = f^e + f \tag{i}$$

where f^e is supplied by (a) with $v = V_o$ = constant. Hence, these last two expressions can be added to eliminate f and obtain

PROBLEM 5.25 (Continued)

$$(M_1 + M_2)\frac{d^2r}{dt^2} - M_1 r\left(\frac{d\theta}{dt}\right)^2 = \frac{\partial}{\partial r}\left(\frac{1}{2}\frac{\pi a^2\varepsilon_o V_o^2}{r}\right) \qquad (j)$$

If we further use (g) to eliminate $d\theta/dt$, we obtain an expression for $r(t)$ that can be written in the standard form

$$(M_1 + M_2)\frac{d^2r}{dt^2} + \frac{\partial}{\partial r}V = 0 \qquad (k)$$

where

$$V = \frac{M_1\ \ell^4\ \omega_m^2}{2r^2} - \frac{1}{2}\frac{\pi a^2\varepsilon_o V_o^2}{r} \qquad (l)$$

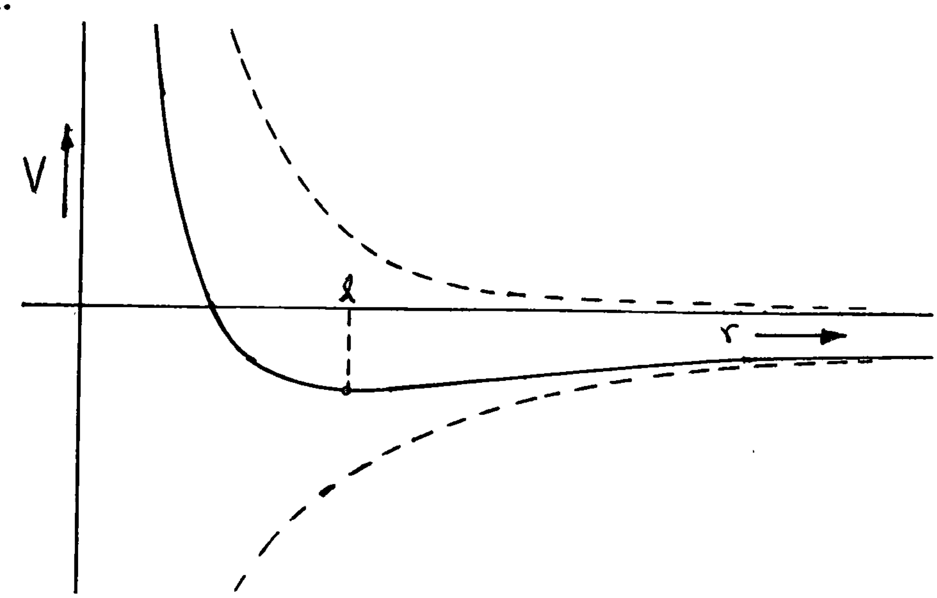
Of course, (k) can be multiplied by dr/dt and written in the form

$$\frac{d}{dt}\left[\frac{1}{2}(M_1 + M_2)\left(\frac{dr}{dt}\right)^2 + V\right] = 0 \qquad (m)$$

to show that V is a potential well for the combined mass of the rotating particle and the plate.

Part e

The potential well of (l) has the shape shown in the figure. The minimum represents the equilibrium position found in (c), as can be seen by differentiating (l) with respect to r, equating the expression to zero and solving for ω_m assuming that $r = \ell$. In this example, the potential well is the result of a combination of the negative coenergy for the electromechanical system, constrained to constant potential, and the dynamic system with angular momentum conserved.



PROBLEM 5.26

Part a

To begin the analysis we first write the Kirchhoff voltage equations for the two electric circuits with switch S closed

$$V = i_1 R_1 + \frac{d\lambda_1}{dt} \tag{a}$$

$$0 = i_2 R_2 + \frac{d\lambda_2}{dt} \tag{b}$$

To obtain the electrical terminal relations for the system we neglect fringing fields and assume infinite permeability for the magnetic material to obtain*

$$\lambda_1 = N_1 \phi \; , \quad \lambda_2 = N_2 \phi \tag{c}$$

where the flux ϕ through the coils is given by

$$\phi = \frac{2\mu_o \, wd \, (N_1 i_1 + N_2 i_2)}{g(1 + \frac{x}{g})} \tag{d}$$

We can also use (c) and (d) to calculate the stored magnetic energy as**

$$W_m = \frac{g(1 + \frac{x}{g})\phi^2}{4\mu_o \, wd} \tag{e}$$

We now multiply (a) by N_1/R_1 and (b) by N_2/R_2, add the results and use (c) and (d) to obtain

$$\frac{N_1 V}{R_1} = \frac{g(1 + \frac{x}{g})}{2\mu_o \, wd} \phi + \left(\frac{N_1^2}{R_1} + \frac{N_2^2}{R_2}\right) \frac{d\phi}{dt} \tag{f}$$

Note that we have only one electrical unknown, the flux ϕ, and if the plunger is at rest (x = constant) this equation has constant coefficients.

*The neglect of fringing fields makes the two windings unity coupled. In practice there will be small fringing fields that cause leakage inductances. However, these leakage inductances affect only the initial part of the transient and neglecting them causes negligible error when calculating the closing time of the relay.

**Here we have used the equation (System is electrically linear)

$$W_m = \frac{1}{2} L_1 i_1^2 + L_{12} i_1 i_2 + \frac{1}{2} L_2 i_2^2$$

PROBLEM 5.26 (Continued)

Part b

Use the given definitions to write (f) in the form

$$\phi_o = (1 + \frac{x}{g})\phi + \tau_o \frac{d\phi}{dt} \tag{g}$$

Part c

During interval 1 the flux is determined by (g) with $x = x_o$ and the initial condition is $\phi = 0$. Thus the flux undergoes the transient

$$\phi = \frac{\phi_o}{1 + \frac{x_o}{g}}\left[1 - e^{-(1 + \frac{x_o}{g})\frac{t}{\tau_o}}\right] \tag{h}$$

To determine the time at which interval 1 ends and to describe the dynamics of interval 2 we must write the equation of motion for the mechanical node. Neglecting inertia and damping forces this equation is

$$K(x - \ell) = f^e \tag{i}$$

In view of (c) (λ_1 and λ_2 are the independent variables implicit in ϕ) we can use (e) to evaluate the force f^e as

$$f^e = -\frac{\partial W_m(\lambda_1, \lambda_2, x)}{\partial x} = -\frac{\phi^2}{4\mu_o wd} \tag{j}$$

Thus, the mechanical equation of motion becomes

$$K(x - \ell) = -\frac{\phi^2}{4\mu_o wd} \tag{k}$$

The flux level ϕ_1 at which interval 1 ends is given by

$$K(x_o - \ell) = -\frac{\phi_1^2}{4\mu_o wd} \tag{l}$$

Part d

During interval 2, flux and displacement are related by (k), thus we eliminate x between (k) and (g) and obtain

$$\phi_o = \left[(1 + \frac{\ell}{g}) - (\frac{\ell - x_o}{g})\frac{\phi^2}{\phi_1^2}\right]\phi + \tau_o \frac{d\phi}{dt} \tag{m}$$

were we have used (ℓ) to write the equation in terms of ϕ_1. This is the nonlinear differential equation that must be solved to find the dynamical behavior during interval 2.

PROBLEM 5.26 (Continued)

To illustrate the solution of (m) it is convenient to normalize the equation as follows

$$\frac{d(\frac{\phi}{\phi_o})}{d(\frac{t}{\tau_o})} = (\frac{\ell - x_o}{g})(\frac{\phi_o}{\phi_1})^2 (\frac{\phi}{\phi_o})^3 - (1 + \frac{\ell}{g})\frac{\phi}{\phi_o} + 1 \tag{n}$$

We can now write the necessary integral formally as

$$\int_{\frac{\phi_1}{\phi_o}}^{\frac{\phi}{\phi_o}} \frac{d(\frac{\phi'}{\phi_o})}{\left[(\frac{\ell - x_o}{g})(\frac{\phi_o}{\phi_1})^2 (\frac{\phi'}{\phi_o})^3 - (1 + \frac{\ell}{g})\frac{\phi'}{\phi_o} + 1\right]} = \int_o^{\frac{t}{\tau_o}} d(\frac{t}{\tau_o}) \tag{o}$$

where we are measuring time t from the start of interval 2.

Using the given parameter values,

$$\int_{0.1}^{\frac{\phi}{\phi_o}} \frac{d(\frac{\phi'}{\phi_o})}{\left[400\,(\frac{\phi'}{\phi_o})^3 - 9\,\frac{\phi'}{\phi_o} + 1\right]} = \frac{t}{\tau_o} \tag{p}$$

We factor the cubic in the denominator into a first order and a quadratic factor and do a partial-fraction expansion* to obtain

$$\int_{0.1}^{\frac{\phi}{\phi_o}} \left[\frac{0.156}{5.29\,\frac{\phi'}{\phi_o} + 1} + \frac{(-2.23\,\frac{\phi'}{\phi_o} + 0.844)}{75.7\,(\frac{\phi'}{\phi_o})^2 - 14.3\,\frac{\phi'}{\phi_o} + 1}\right] d(\frac{\phi'}{\phi_o}) = \frac{t}{\tau_o} \tag{q}$$

Integraticn of this expression yields

*Phillips, H.B., Analytic Geometry and Calculus, second edition, John Wiley and Sons, New York, 1946, pp. 250-253.

PROBLEM 5.26 (Continued)

$$\frac{t}{\tau_o} = 0.0295 \ln [3.46 (\frac{\phi}{\phi_o}) + 0.654] - 0.0147 \ln [231 (\frac{\phi}{\phi_o})^2 - 43.5 (\frac{\phi}{\phi_o}) + 3.05]$$

$$+ 0.127 \tan^{-1} [15.1 (\frac{\phi}{\phi_o}) - 1.43] - 0.0108 \qquad \text{(r)}$$

Part e

During interval 3, the differential equation is (g) with x = 0, for which the solution is

$$\phi = \phi_2 + (\phi_o - \phi_2)(1 - e^{-\frac{t}{\tau_o}}) \qquad \text{(s)}$$

where t is measured from the start of interval 3 and where ϕ_2 is the value of flux at the start of interval 3 and is given by (k) with x = 0

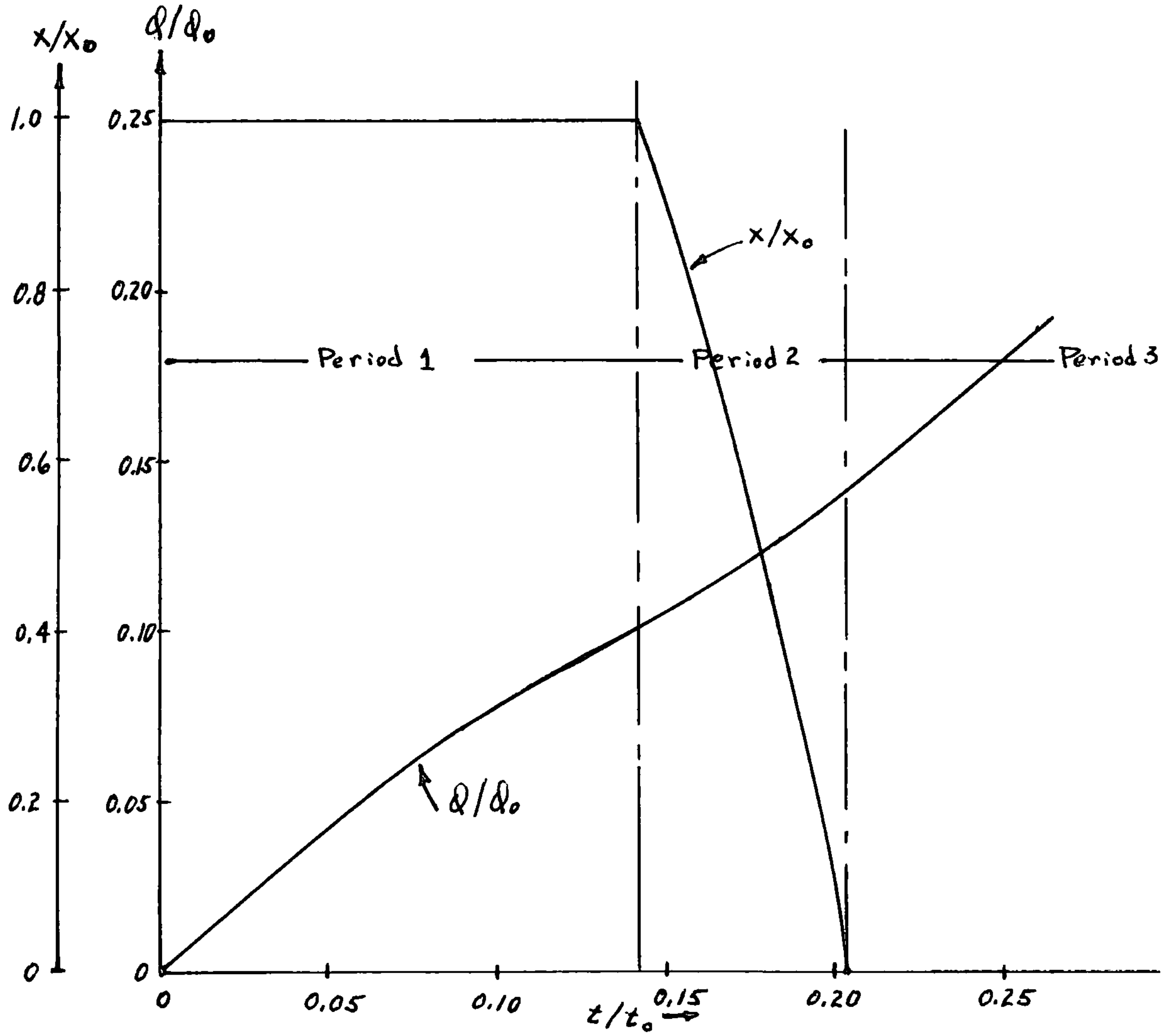
$$K\ell = \frac{\phi_2^2}{4\mu_o wd} \qquad \text{(t)}$$

Part f

For the assumed constants in this problem

$$\frac{\phi_2}{\phi_1} = \sqrt{2} \qquad \text{(u)}$$

The transients in flux and position are plotted in Fig. (a) as functions of time. Note that the mechanical transient occupies only a fraction of the time interval of the electrical transient. Thus, this example represents a case in which the electrical time constant is purposely made longer than the mechanical transient time.


x/x₀
Q/Q₀
1.0
0.8
0.6
0.4
0.2
0
0.25
0.20
0.15
0.10
0.05
0
x/x₀
Period 1
Period 2
Period 3
Q/Q₀
0
0.05
0.10
0.15
0.20
0.25
t/t₀