16 Sampling

Solutions to Recommended Problems

S16.1

If $\omega_0 = \pi \times 10^3$, then

$$\cos(\omega_0 n \times 10^{-3}) = \cos(\pi n) = (-1)^n$$

Similarly, for $\omega_0 = 3\pi \times 10^{-3}$ and $\omega_0 = 5\pi \times 10^{-3}$,

$$\cos(\omega_0 n \times 10^{-3}) = (-1)^n$$

S16.2

The sampling function

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \qquad T = \tfrac{1}{3},$$

has a spectrum given by

$$P(\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$
$$= 6\pi \sum_{n=-\infty}^{\infty} \delta(\omega - 6\pi k),$$

shown in Figure S16.2-1.

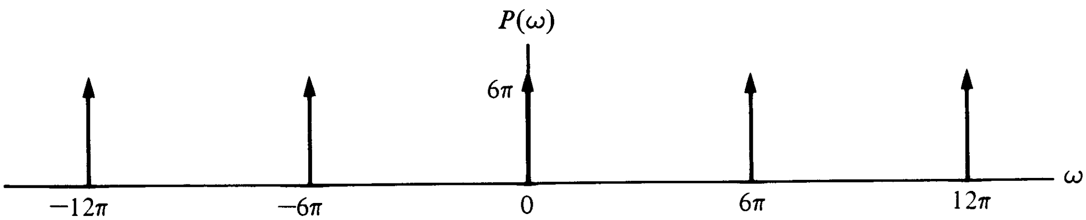

Figure S16.2-1

$\cos(\omega_0 t)$ has a spectrum given by $\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$, shown in Figure S16.2-2.

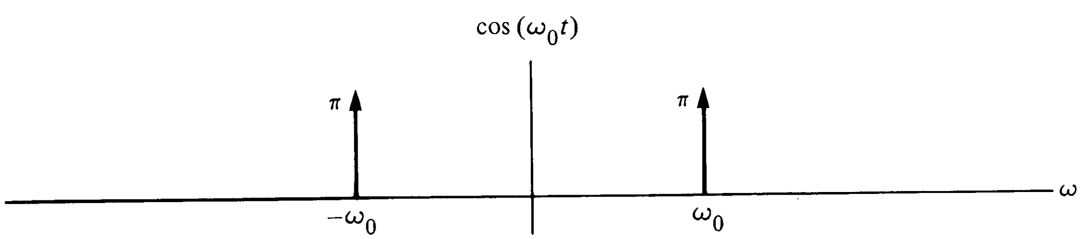

Figure S16.2-2

From the convolution theorem

$$X_p(\omega) = \frac{1}{2\pi} P(\omega) * [\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)]$$

Hence, it is straightforward to find $X_p(\omega)$.

(a) (i) For $\omega_0 = \pi$:

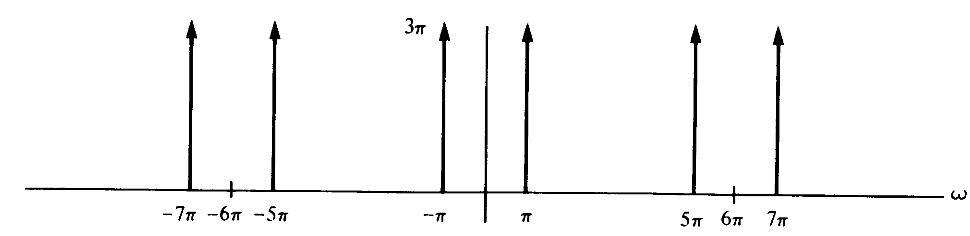

Figure S16.2-3

(ii) For $\omega_0 = 2\pi$:

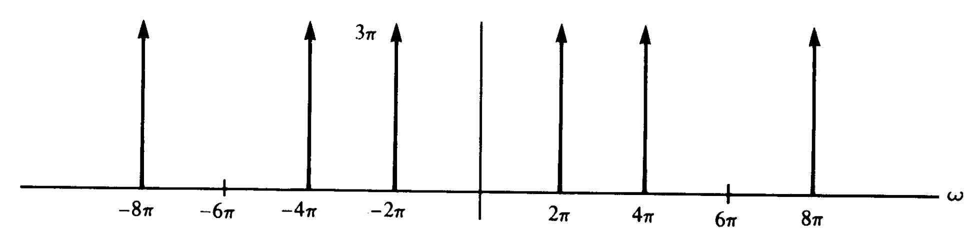

Figure S16.2-4

(iii) For $\omega_0 = 3\pi$:

Figure S16.2-5

(iv) For $\omega_0 = 5\pi$:

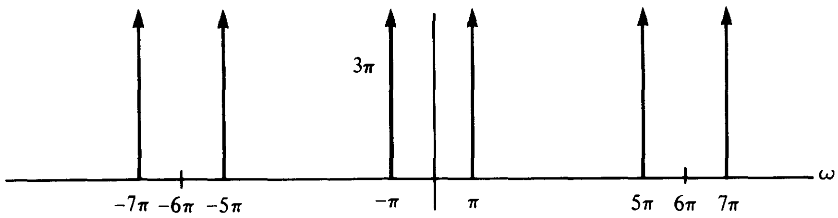

Figure S16.2-6

(b) From part (a), it is clear that (i) and (iv) are identical.

S16.3

The signal $x(t) = \cos(\omega_0 t + \theta)$, where $\omega_0 = 2\pi f_0$, can be written as

$$x(t) = \tfrac{1}{2}e^{j\theta}e^{j\omega_0 t} + \tfrac{1}{2}e^{-j\theta}e^{-j\omega_0 t}$$

and the spectrum of $x(t)$ is given by

$$X(\omega) = \pi e^{j\theta}\delta(\omega - \omega_0) + \pi e^{-j\theta}\delta(\omega + \omega_0)$$

The spectrum of $p(t)$ is given by

$$P(\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi k}{T}\right)$$

Therefore, the spectrum of $x_p(t)$ is

$$X_p(\omega) = \frac{1}{2\pi}\left(\frac{2\pi^2}{T}\right)\left[\sum_{k=-\infty}^{\infty} e^{j\theta}\delta\left(\omega - \frac{2\pi k}{T} - \omega_0\right) + e^{-j\theta}\delta\left(\omega - \frac{2\pi k}{T} + \omega_0\right)\right]$$

and the spectrum of $X_r(\omega)$ is given by

$$X_r(\omega) = H(\omega)X_p(\omega)$$

(a) $\omega_0 = 2\pi \times 250, \qquad \theta = \frac{\pi}{4}, \qquad T = 10^{-3},$

$$X_p(\omega) = \frac{\pi}{T}\sum_{k=-\infty}^{\infty}[e^{j\theta}\delta(\omega - 2\pi \times 10^3 k - 2\pi \times 250)$$
$$+ e^{-j\theta}\delta(\omega - 2\pi \times 10^3 k + 2\pi \times 250)]$$

Hence, only the $k = 0$ term is passed by the filter:

$$X_r(\omega) = \pi[e^{j\theta}\delta(\omega - 2\pi \times 250) + e^{-j\theta}\delta(\omega + 2\pi \times 250)]$$

and

$$x_r(t) = \frac{1}{2}e^{j\theta}e^{j2\pi\times 250t} + \frac{1}{2}e^{-j\theta}e^{-j2\pi\times 250t}$$
$$= \cos(2\pi \times 250t + \theta)$$
$$= \cos\left(2\pi \times 250t + \frac{\pi}{4}\right)$$

(b) $\omega_0 = 2\pi \times 750$ Hz, $\quad T = 10^{-3}$,

$$X_p(\omega) = \frac{\pi}{T}\sum_{k=-\infty}^{\infty} [e^{j\theta}\delta(\omega - 2\pi \times 10^3 k - 2\pi \times 750) + e^{-j\theta}\delta(\omega - 2\pi \times 10^3 k + 2\pi \times 750)]$$

Only the $k = \pm 1$ term has nonzero contribution:

$$X_r(\omega) = \frac{\pi}{T}[e^{j\theta}\delta(\omega + 2\pi \times 250) + e^{-j\theta}\delta(\omega - 2\pi \times 250)]$$

Hence,

$$\begin{aligned} x_r(t) &= \cos(2\pi \times 250t - \theta) \\ &= \cos\left(2\pi \times 250t - \frac{\pi}{2}\right) \end{aligned}$$

(c) $\omega_0 = 2\pi \times 500$, $\quad \theta = \frac{\pi}{2}$, $\quad T = 10^{-3}$,

$$X_p(\omega) = \frac{\pi}{T}\sum_{k=-\infty}^{\infty} [e^{j\theta}\delta(\omega - 2\pi \times 10^3 k - 2\pi \times 500) + e^{-j\theta}\delta(\omega - 2\pi \times 10^3 k + 2\pi \times 500)]$$

Since $H(\omega) = 0$ at $\omega = 2\pi \times 500$, the output is zero: $x_r(t) = 0$.

S16.4

(a)
$$\begin{aligned} x_p(t) &= \sum_{n=-\infty}^{\infty} x(t)\,\delta(t - 2\Delta n) - \sum_{n=-\infty}^{\infty} x(t)\,\delta(t - \Delta - 2\Delta n) \\ &= x(t)\left[\sum_{n=-\infty}^{\infty} \delta(t - 2\Delta n) - \sum_{n=-\infty}^{\infty} \delta(t - \Delta - 2\Delta n)\right] \end{aligned}$$

By the convolution theorem,

$$\begin{aligned} X_p(\omega) &= \frac{1}{2\pi}X(\omega) * \frac{2\pi}{2\Delta}\sum_{n=-\infty}^{\infty}\delta\left(\omega - n\frac{2\pi}{2\Delta}\right) \\ &\quad - \frac{1}{2\pi}X(\omega) * \frac{2\pi}{2\Delta}\sum_{n=-\infty}^{\infty}\delta\left(\omega - n\frac{2\pi}{2\Delta}\right)e^{-j\omega\Delta} \\ &= X(\omega) * \left[\frac{1}{2\Delta}\sum_{n=-\infty}^{\infty}(1 - e^{-j\pi n})\,\delta\left(\omega - \frac{n\pi}{\Delta}\right)\right] \\ &= X(\omega) * \left[\frac{1}{2\Delta}\sum_{n=-\infty}^{\infty}(1 - (-1)^n)\,\delta\left(\omega - \frac{n\pi}{\Delta}\right)\right] \end{aligned}$$

$X_p(\omega)$ is sketched in Figure S16.4-1 and $Y(\omega)$ is sketched in Figure S16.4-2.

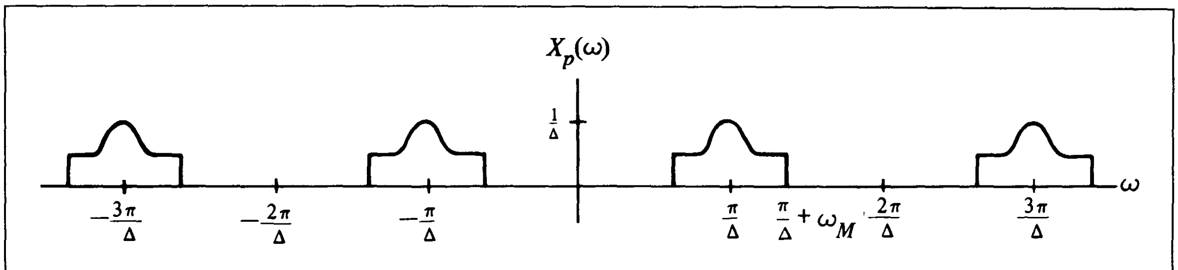

Figure S16.4-1

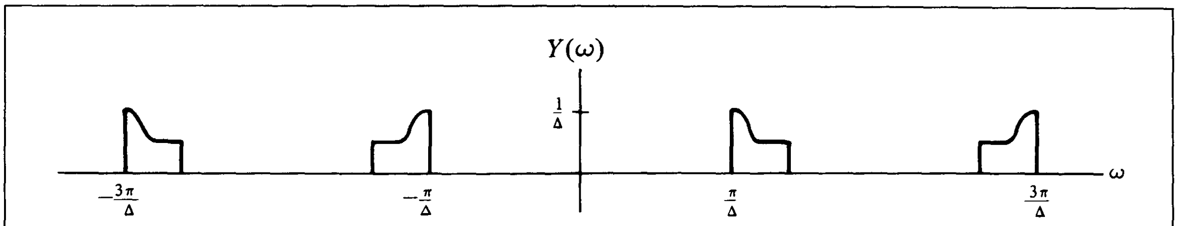

Figure S16.4-2

(b)

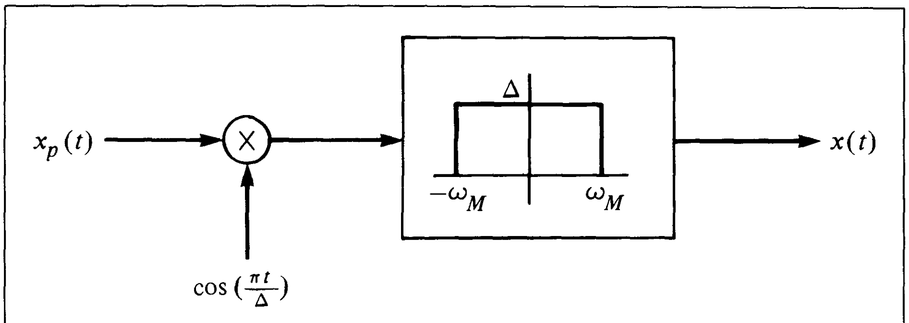

Figure S16.4-3

(c)

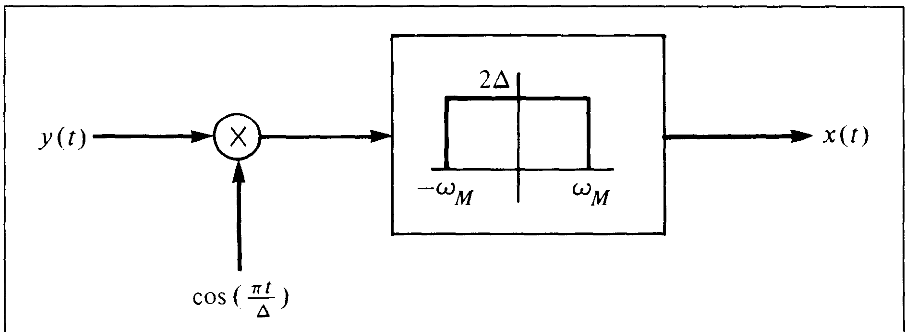

Figure S16.4-4

(d) Δ is maximum when π/Δ is minimum. From part (a) we see that aliasing is avoided in $X_p(\omega)$ if $\omega_M \le \pi/\Delta$. Hence, $\Delta_{max} = \pi/\omega_M$.

S16.5

(a) The transform of the sampled function appears as in Figure S16.5-1.

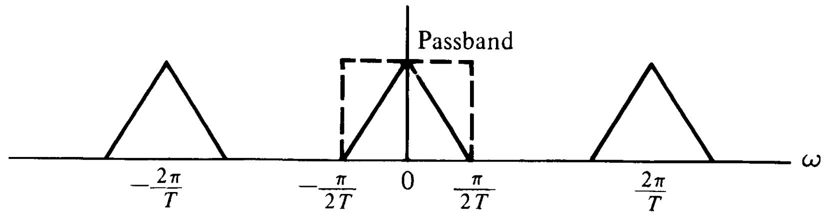

Figure S16.5-1

Hence, (a) matches (i).

(b)

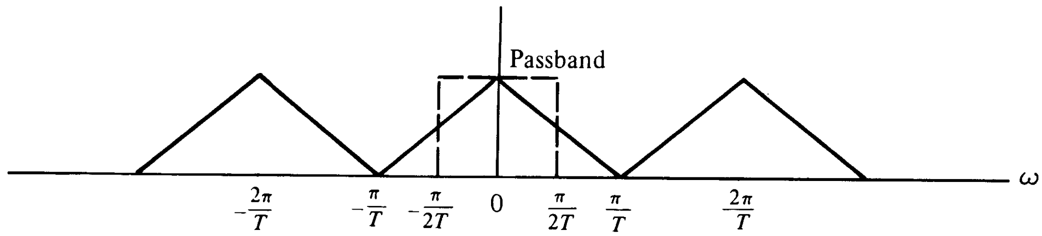

Figure S16.5-2

Hence, (b) does not match any.

(c) Matches (ii).

(d)

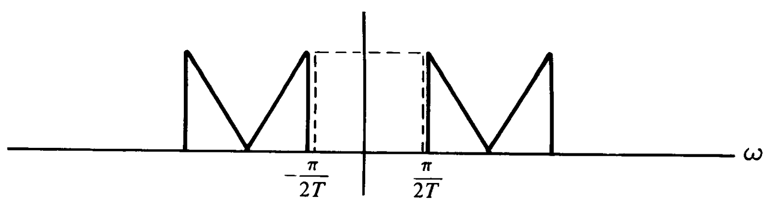

Figure S16.5-3

Hence, (d) does not match any.

S16.6

Since the input $x_p(t)$ cannot be distinguished for certain values of ω, the output also should not be distinguishable for certain values of ω. Hence, $Q(\omega)$ must be periodic in ω. Therefore, Figure P16.6-3 is a possible candidate, but Figure P16.6-2 is not.

Solutions to Optional Problems

S16.7

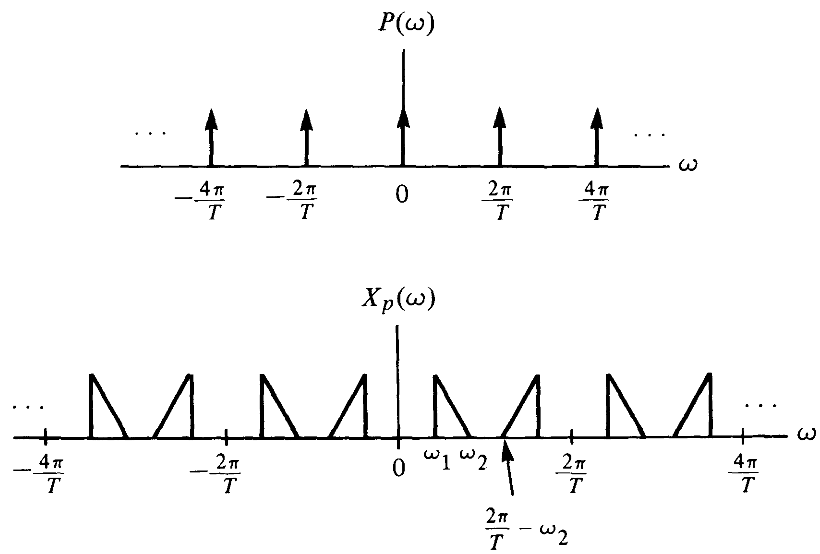

Figure S16.7-1

Note that as T increases, $(2\pi/T) - \omega_2$ approaches $\omega = 0$. Also, there is aliasing when $2\omega_1 - \omega_2 < (2\pi/T) - \omega_2 < \omega_2$. If $2\omega_1 - \omega_2 \geq 0$ (as given in the problem), then it is easy to see that there is no aliasing when $0 \leq (2\pi/T) - \omega_2 \leq 2\omega_1 - \omega_2$. For maximum T, we choose a minimum allowable value of $2\pi/T$:

$$\frac{2\pi}{T_{\max}} = \omega_2 \rightarrow T_{\max} = \frac{2\pi}{\omega_2},$$

which is sampling at half the Nyquist rate. $X_p(\omega)$ for this case is given in Figure S16.7-2.

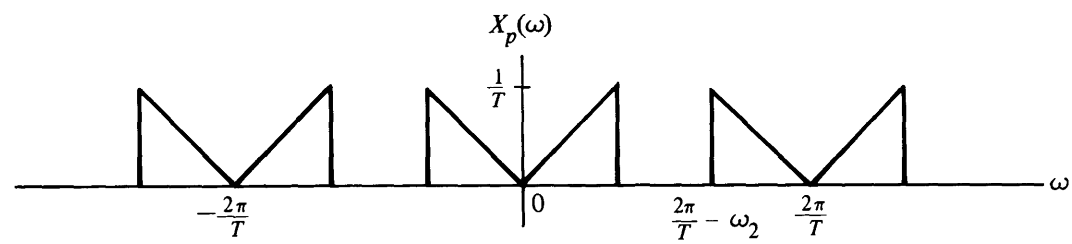

Figure S16.7-2

Hence,

$$A = T, \qquad \omega_b = 2\pi/T, \qquad \omega_a = \omega_1$$

S16.8

We are given the system shown in Figure S16.8-1.

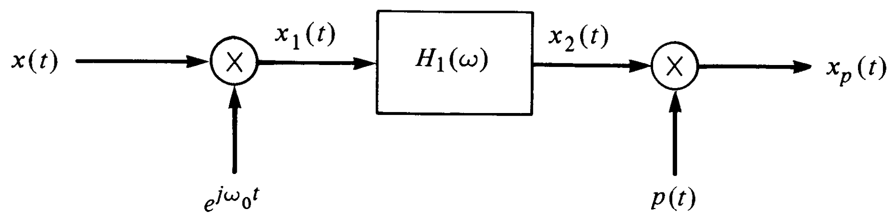

Figure S16.8-1

(a) $X(\omega)$ and $X_1(\omega)$ are as shown in Figure S16.8-2. $X_2(\omega)$ is as shown in Figure S16.8-3, and $X_p(\omega)$ is therefore as given in Figure S16.8-4.

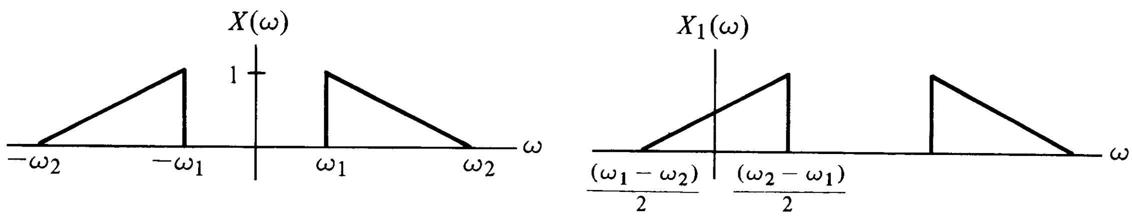

Figure S16.8-2

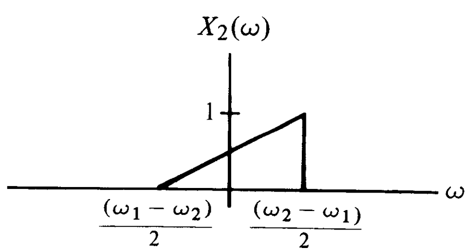

Figure S16.8-3

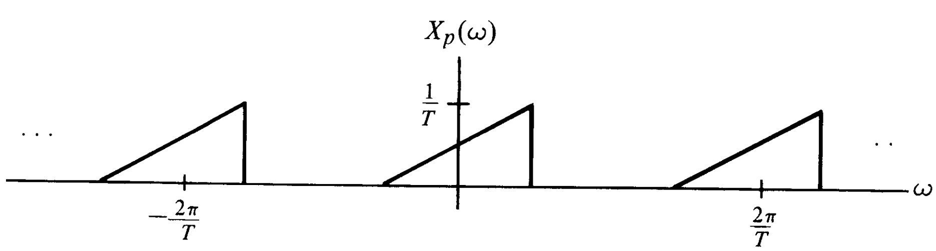

Figure S16.8-4

(b) $2\pi/T_{max}$ equals the Nyquist rate for $X_2(\omega)$:

$$\frac{2(\omega_2 - \omega_1)}{2} = \omega_2 - \omega_1$$

Hence,

$$T_{max} = \frac{2\pi}{(\omega_2 - \omega_1)}$$

(c)

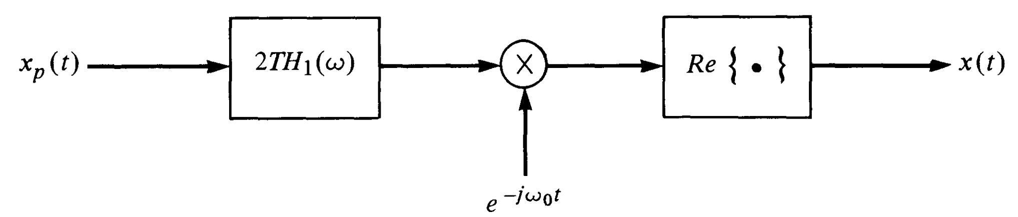

Figure S16.8-5

S16.9

The composite waveform spectrum is given in Figure S16.9-1.

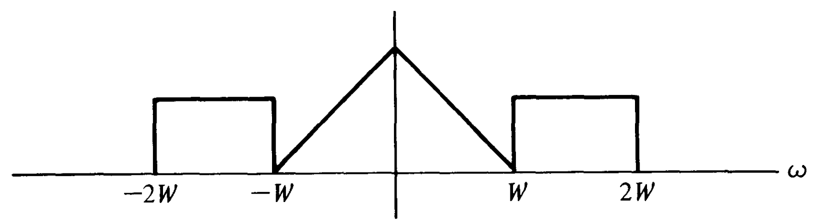

Figure S16.9-1

We can alias the noise region to get maximum T. This corresponds to the aliased spectrum, shown in Figure S16.9-2.

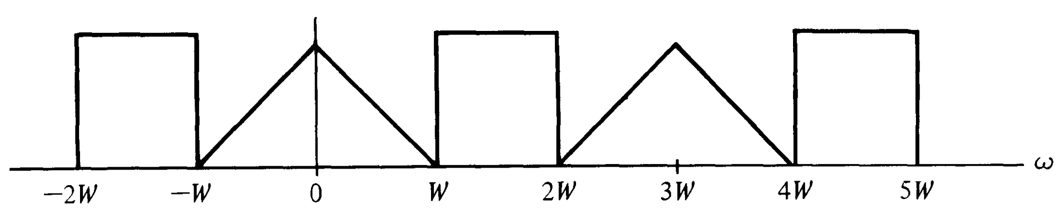

Figure S16.9-2

The value of T is given by

$$\frac{2\pi}{T} = 3W \rightarrow T_{max} = \frac{2\pi}{3W}$$

The value of A is T_{max} for $y(t) = x(t)$.

S16.10

The spectra of $x_{1,2}(t)$, where $T = \pi/W$, given in Figures S16.10-1 and S16.10-2, could have generated $x_r(t)$:

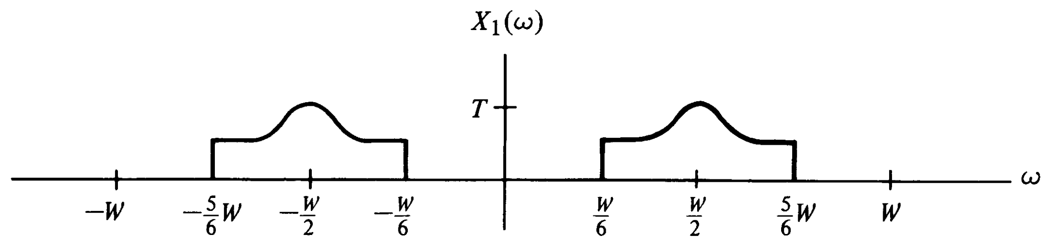

Figure S16.10-1

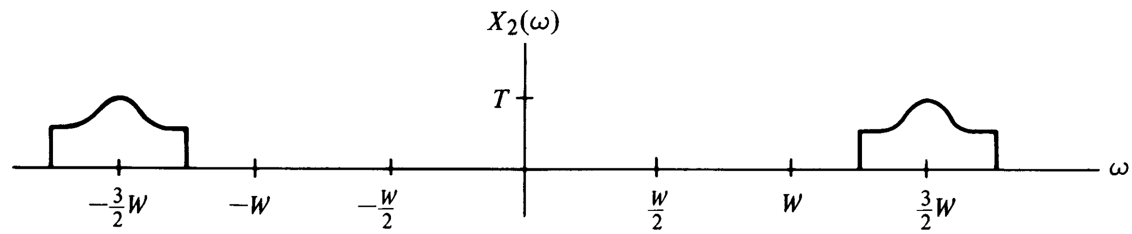

Figure S16.10-2

S16.11

(a) From the sampling theorem, $2\pi/T \geq 2W$. Hence,

$$T \leq \frac{\pi}{W} \rightarrow T_{\max} = \frac{\pi}{W}$$

Since

$$X_p(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\omega - k\frac{2\pi}{T}\right),$$

we require $A = T$ for $x_r(t) = x(t)$.

The minimum value of W_c is W so that we do not lose any information, and the maximum value of W_c is $(2\pi/T) - W$ to avoid periodic spectral contribution.

(b) (i) $X(\omega) = 0$ for $|\omega| > W$. Hence,

$$T_{\max} = \frac{\pi}{W}, \qquad A = T, \qquad W < W_c < \frac{2\pi}{T} - W$$

(ii) $X(\omega) = 0$ for $|\omega| > 2W$. Hence,

$$T_{\max} = \frac{\pi}{2W}, \qquad A = T, \qquad 2W < W_c < \frac{2\pi}{T} - 2W$$

(iii) $X(\omega) = 0$ for $|\omega| > 3W$. Hence,

$$T_{\max} = \frac{\pi}{3W}, \qquad A = T, \qquad 3W < W_c < \frac{2\pi}{T} - 3W$$

(iv) $X(\omega) = 0$ for $|\omega| > W/10$. Hence,

$$T_{\max} = \frac{10\pi}{W}, \qquad A = T, \qquad \frac{W}{10} < W_c < \frac{2\pi}{T} - \frac{W}{10}$$

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