20 The Laplace Transform

Solutions to Recommended Problems

S20.1

(a) The Fourier transform of the signal does not exist because of the presence of growing exponentials. In other words, $x(t)$ is not absolutely integrable.

(b) (i) For the case $\sigma = 1$, we have that

$$x(t)e^{-\sigma t} = 3e^{t}u(t) + 4e^{2t}u(t)$$

Although the growth rate has been slowed, the Fourier transform still does not converge.

(ii) For the case $\sigma = 2.5$, we have that

$$x(t)e^{-\sigma t} = 3e^{-0.5t}u(t) + 4e^{0.5t}u(t)$$

The first term has now been sufficiently weighted that it decays to 0 as t goes to infinity. However, since the second term is still growing exponentially, the Fourier transform does not converge.

(iii) For the case $\sigma = 3.5$, we have that

$$x(t)e^{-\sigma t} = 3e^{-1.5t}u(t) + 4e^{-0.5t}u(t)$$

Both terms do decay as t goes to infinity, and the Fourier transform converges. We note that for any value of $\sigma > 3.0$, the signal $x(t)e^{-\sigma t}$ decays exponentially, and the Fourier transform converges.

(c) The Laplace transform of $x(t)$ is

$$X(s) = \frac{3}{s-2} + \frac{4}{s-3} = \frac{7(s - \frac{17}{7})}{(s-2)(s-3)},$$

and its pole-zero plot and ROC are as shown in Figure S20.1.

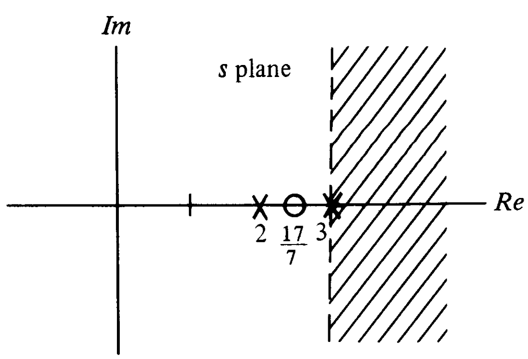

Figure S20.1

Note that if $\sigma > 3.0$, $s = \sigma + j\omega$ is in the region of convergence because, as we showed in part (b)(iii), the Fourier transform converges.

S20.2

(a) $X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}\,dt = \dfrac{1}{s+a}$

The Laplace transform converges for $Re\{s\} + a > 0$, so

$$\sigma + a > 0, \qquad \text{or} \qquad \sigma > -a,$$

as shown in Figure S20.2-1.

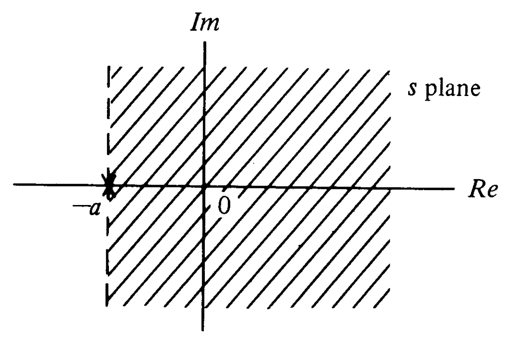

Figure S20.2-1

(b) $X(s) = \dfrac{1}{s+a}$

The Laplace transform converges for $Re\{s\} + a > 0$, so

$$\sigma + a > 0, \qquad \text{or} \qquad \sigma > -a,$$

as shown in Figure S20.2-2.

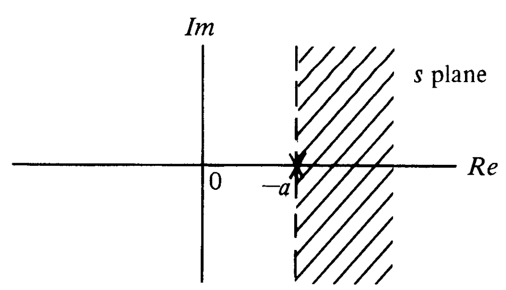

Figure S20.2-2

(c) $X(s) = \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}\,dt = \int_{-\infty}^{0} -e^{-(s+a)t}\,dt = \left.\dfrac{e^{-(s+a)t}}{s+a}\right|_{-\infty}^{0}$

$= \dfrac{1}{s+a}$

if $Re\{s\} + a < 0, \sigma + a < 0, \sigma < -a$.

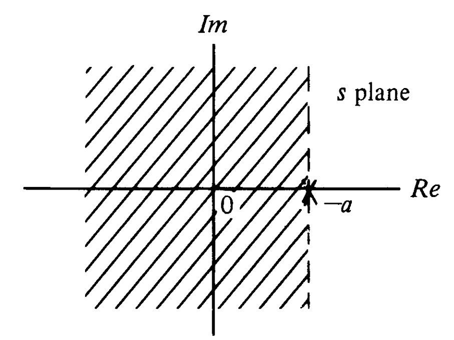

Figure S20.2-3

S20.3

(a) (i) Since the Fourier transform of $x(t)e^{-t}$ exists, $\sigma = 1$ must be in the ROC. Therefore only one possible ROC exists, shown in Figure S20.3-1.

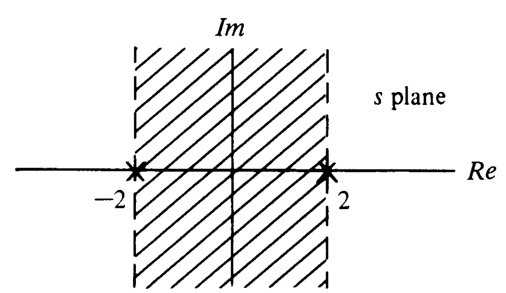

Figure S20.3-1

(ii) We are specifying a left-sided signal. The corresponding ROC is as given in Figure S20.3-2.

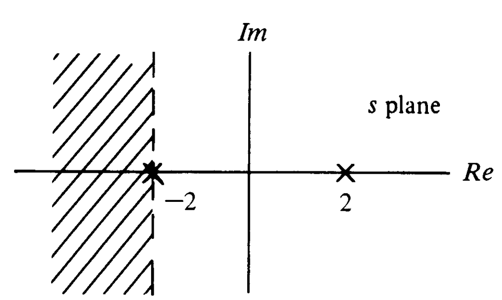

Figure S20.3-2

(iii) We are specifying a right-sided signal. The corresponding ROC is as given in Figure S20.3-3.

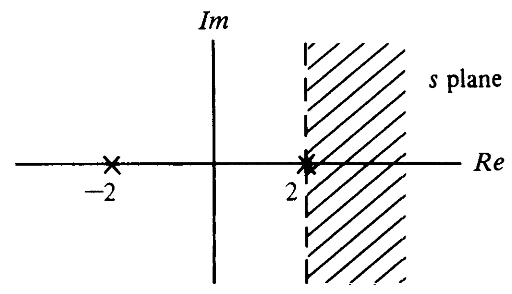

Figure S20.3-3

(b) Since there are no poles present, the ROC exists everywhere in the s plane.

(c) (i) $\sigma = 1$ must be in the ROC. Therefore, the only possible ROC is that shown in Figure S20.3-4.

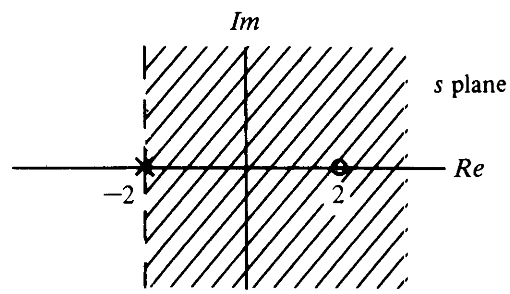

Figure S20.3-4

(ii) We are specifying a left-sided signal. The corresponding ROC is as shown in Figure S20.3-5.

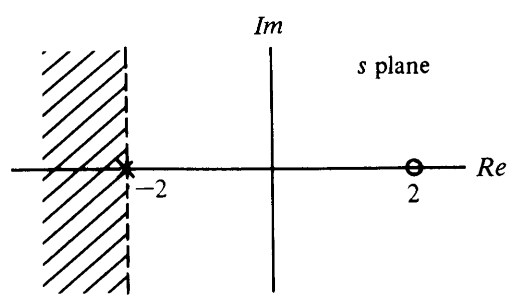

Figure S20.3-5

(iii) We are specifying a right-sided signal. The corresponding ROC is as given in Figure S20.3-6.

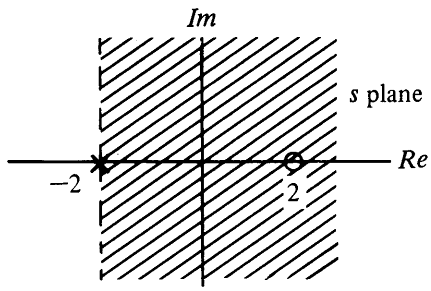

Figure S20.3-6

(d) (i) $\sigma = 1$ must be in the ROC. Therefore, the only possible ROC is as shown in Figure S20.3-7.

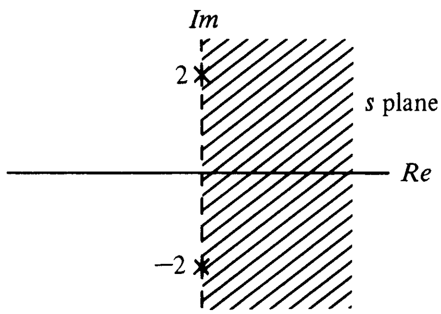

Figure S20.3-7

(ii) We are specifying a left-sided signal. The corresponding ROC is as shown in Figure S20.3-8.

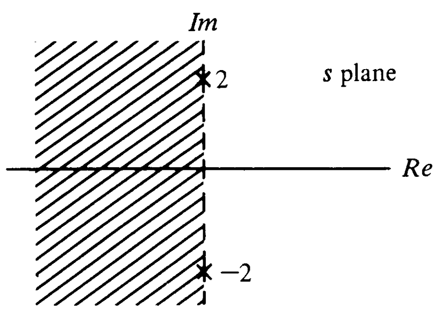

Figure S20.3-8

(iii) We are specifying a right-sided signal. The corresponding ROC is as shown in Figure S20.3-9.

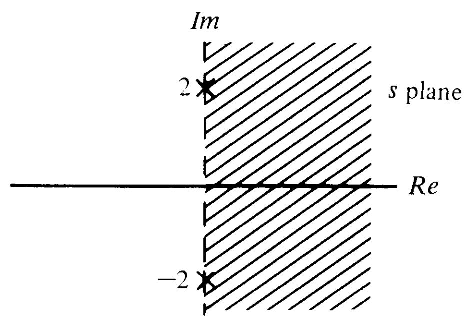

Figure S20.3-9

$x(t)$	Constraint on ROC for Pole-Zero Pattern (a)	(b)	(c)	(d)
(i) Fourier transform of $x(t)e^{-t}$ converges	$-2 < \sigma < 2$	Entire s plane	$\sigma > -2$	$\sigma > 0$
(ii) $x(t) = 0$, $t > 10$	$\sigma < -2$	Entire s plane	$\sigma < -2$	$\sigma < 0$
(iii) $x(t) = 0$, $t < 0$	$\sigma > 2$	Entire s plane	$\sigma > -2$	$\sigma > 0$

Table S20.3

S20.4

(a) For $x(t)$ right-sided, the ROC is to the right of the rightmost pole, as shown in Figure S20.4-1.

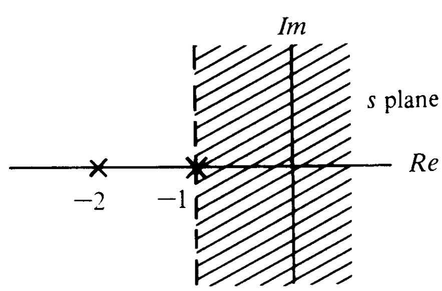

Figure S20.4-1

Using partial fractions,

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2},$$

so, by inspection,

$$x(t) = e^{-t}u(t) - e^{-2t}u(t)$$

(b) For $x(t)$ left-sided, the ROC is to the left of the leftmost pole, as shown in Figure S20.4-2.

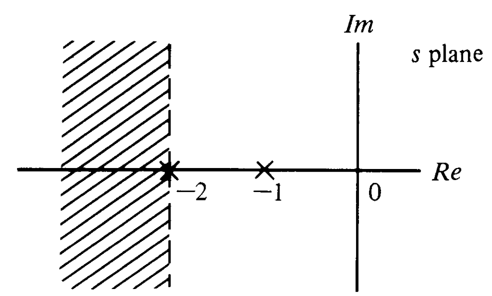

Figure S20.4-2

Since

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2},$$

we conclude that

$$x(t) = -e^{-t}u(-t) - (-e^{-2t}u(-t))$$

(c) For the two-sided assumption, we know that $x(t)$ will have the form

$$f_1(t)u(-t) + f_2(t)u(t)$$

We know the inverse Laplace transforms of the following:

$$\frac{1}{s+1} = \begin{cases} e^{-t}u(t), & \text{assuming right-sided,} \\ -e^{-t}u(-t), & \text{assuming left-sided,} \end{cases}$$

$$\frac{1}{s+2} = \begin{cases} e^{-2t}u(t), & \text{assuming right-sided,} \\ -e^{-2t}u(-t), & \text{assuming left-sided} \end{cases}$$

Which of the combinations should we choose for the two-sided case? Suppose we choose

$$x(t) = e^{-t}u(t) + (-e^{-2t})u(-t)$$

We ask, For what values of σ does $x(t)e^{-\sigma t}$ have a Fourier transform? And we see that there are no values. That is, suppose we choose $\sigma > -1$, so that the first term has a Fourier transform. For $\sigma > -1$, $e^{-2t}e^{-\sigma t}$ is a growing exponential as t goes to negative infinity, so the second term does not have a Fourier transform. If we increase σ, the first term decays faster as t goes to infinity, but the second term grows faster as t goes to negative infinity. Therefore, choosing $\sigma > -1$ will not yield a Fourier transform of $x(t)e^{-\sigma t}$. If we choose $\sigma \leq -1$, we note that the first term will not have a Fourier transform. Therefore, we conclude that our choice of the two-sided sequence was wrong. It corresponds to the *invalid* region of convergence shown in Figure S20.4-3.

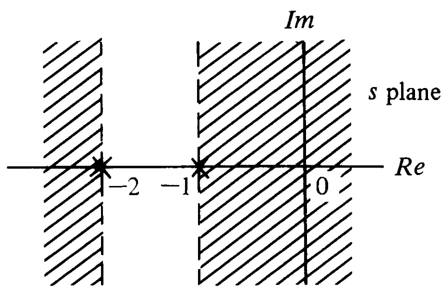

Figure S20.4-3

If we choose the other possibility,

$$x(t) = -e^{-t}u(-t) - e^{-2t}u(t),$$

we see that the valid region of convergence is as given in Figure S20.4-4.

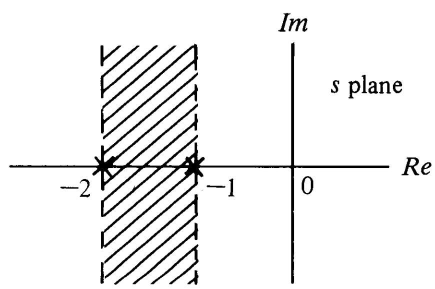

Figure S20.4-4

S20.5

There are two ways to solve this problem.

Method 1

This method is based on recognizing that the system input is a superposition of eigenfunctions. Specifically, the eigenfunction property follows from the convolution integral

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t - \tau)\, d\tau$$

Now suppose $x(t) = e^{at}$. Then

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{a(t-\tau)}\, d\tau = e^{at} \int_{-\infty}^{\infty} h(\tau)e^{-a\tau}\, d\tau$$

Now we recognize that

$$\int_{-\infty}^{\infty} h(\tau)e^{-a\tau}\, d\tau = H(s)\Big|_{s=a},$$

so that if $x(t) = e^{at}$, then

$$y(t) = \left[H(s)\Big|_{s=a}\right]e^{at},$$

i.e., e^{at} is an eigenfunction of the system.

Using linearity and superposition, we recognize that if

$$x(t) = e^{-t/2} + 2e^{-t/3},$$

then

$$y(t) = e^{-t/2}H(s)\Big|_{s=-1/2} + 2e^{-t/3}H(s)\Big|_{s=-1/3}$$

so that

$$y(t) = 2e^{-t/2} + 3e^{-t/3} \qquad \text{for all } t.$$

Method 2

We consider the solution of this problem as the superposition of the response to two signals $x_1(t)$, $x_2(t)$, where $x_1(t)$ is the noncausal part of $x(t)$ and $x_2(t)$ is the causal part of $x(t)$. That is,

$$x_1(t) = e^{-t/2}u(-t) + 2e^{-t/3}u(-t),$$
$$x_2(t) = e^{-t/2}u(t) + 2e^{-t/3}u(t)$$

This allows us to use Laplace transforms, but we must be careful about the ROCs.

Now consider $\mathcal{L}\{x_1(t)\}$, where $\mathcal{L}\{\cdot\}$ denotes the Laplace transform:

$$\mathcal{L}\{x_1(t)\} = X_1(s) = -\frac{1}{s+\frac{1}{2}} - \frac{2}{s+\frac{1}{3}}, \qquad Re\{s\} < -\frac{1}{2}$$

Now since the response to $x_1(t)$ is

$$y_1(t) = \mathcal{L}^{-1}\{X_1(s)H(s)\},$$

then

$$\begin{aligned} Y_1(s) &= -\frac{1}{(s+1)(s+\frac{1}{2})} - \frac{2}{(s+\frac{1}{3})(s+1)}, \qquad -1 < Re\{s\} < -\frac{1}{2}, \\ &= \frac{2}{s+1} + \frac{-2}{s+\frac{1}{2}} + \frac{-3}{s+\frac{1}{3}} + \frac{3}{s+1}, \\ &= \frac{5}{s+1} - \frac{2}{s+\frac{1}{2}} - \frac{3}{s+\frac{1}{3}}, \end{aligned}$$

so

$$y_1(t) = 5e^{-t}u(t) + 2e^{-t/2}u(-t) + 3e^{-t/3}u(-t)$$

The pole-zero plot and associated ROC for $Y_1(s)$ is shown in Figure S20.5-1.

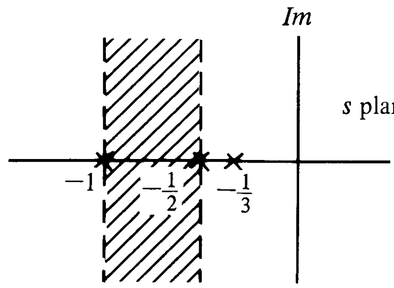

Figure S20.5-1

Next consider the response $y_2(t)$ to $x_2(t)$:

$$x_2(t) = e^{-t/2}u(t) + 2e^{-t/3}u(t),$$

$$X_2(s) = \frac{1}{s+\frac{1}{2}} + \frac{2}{s+\frac{1}{3}}, \qquad Re\{s\} > -\frac{1}{3},$$

$$Y_2(s) = X_2(s)H(s) = \frac{1}{(s+\frac{1}{2})(s+1)} + \frac{2}{(s+\frac{1}{3})(s+1)},$$

$$Y_2(s) = \frac{2}{s+\frac{1}{2}} + \frac{-2}{s+1} + \frac{3}{s+\frac{1}{3}} + \frac{-3}{s+1},$$

so

$$y_2(t) = -5e^{-t}u(t) + 2e^{-t/2}u(t) + 3e^{-t/3}u(t)$$

The pole-zero plot and associated ROC for $Y_2(s)$ is shown in Figure S20.5-2.

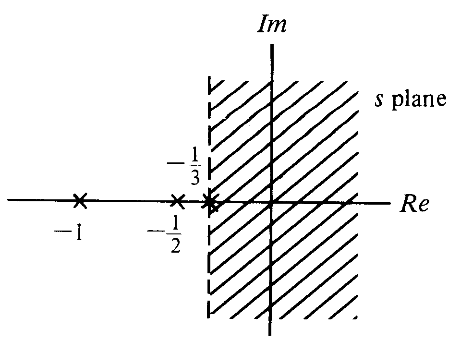

Figure S20.5-2

Since $y(t) = y_1(t) + y_2(t)$, then

$$y(t) = 2e^{-t/2} + 3e^{-t/3} \qquad \text{for all } t$$

S20.6

(a) Since

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

and $s = \sigma + j\omega$, then

$$X(s)\Big|_{s=\sigma+j\omega} = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}\,dt$$

We see that the Laplace transform is the Fourier transform of $x(t)e^{-\sigma t}$ from the definition of the Fourier analysis formula.

(b) $x(t)e^{-\sigma t} = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\left[X(s)\Big|_{\sigma+j\omega}\right]e^{j\omega t}\,d\omega$

This result is the inverse Fourier transform, or synthesis equation. So

$$\begin{aligned} x(t) &= e^{\sigma t}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[X(s)\Big|_{\sigma+j\omega}\right]e^{j\omega t}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[X(s)\Big|_{\sigma+j\omega}\right]e^{(\sigma+j\omega)t}\,d\omega, \end{aligned}$$

and letting $s = \sigma + j\omega$ yields $ds = j\,d\omega$:

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$$

Solutions to Optional Problems

S20.7

(a) $X(s) = \dfrac{1}{s+1}, \qquad Re\{s\} > -1$

Therefore, $x(t)$ is right-sided, and specifically

$$x(t) = e^{-t}u(t)$$

(b) $X(s) = \dfrac{1}{s+1}, \qquad Re\{s\} < -1$

Therefore,

$$x(t) = -e^{-t}u(-t)$$

(c) $X(s) = \dfrac{s}{s^2+4}, \qquad Re\{s\} > 0$

Since

$$\begin{aligned} e^{j\omega_0 t} &\overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s-j\omega_0} \\ e^{-j\omega_0 t} &\overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+j\omega_0} \\ \mathcal{L}\{\cos(\omega_0 t)u(t)\} &= \mathcal{L}\left\{\frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}\right\} = \frac{1}{2}\left(\frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0}\right) \\ \mathcal{L}\{\cos(\omega_0 t)u(t)\} &= \frac{s}{s^2+\omega_0^2} \end{aligned}$$

so

$$\text{if} \quad X(s) = \frac{s}{s^2+4}, \qquad \text{then} \quad x(t) = \cos(2t)u(t)$$

(d) $X(s) = \dfrac{s+1}{s^2+5s+6} = \dfrac{s+1}{(s+2)(s+3)} = \dfrac{-1}{s+2} + \dfrac{2}{s+3}$, so

$$x(t) = -e^{-2t}u(t) + 2e^{-3t}u(t)$$

(e) $X(s) = \dfrac{s+1}{(s+2)(s+3)} = \dfrac{-1}{s+2} + \dfrac{2}{s+3}$,

$x(t) = e^{-2t}u(-t) - 2e^{-3t}u(-t)$

(f)
$$\begin{aligned}
X(s) &= \frac{s^2-s+1}{s^2(s-1)}, \qquad 0 < Re\{s\} < 1 \\
&= \frac{1}{s-1} - \frac{1}{s(s-1)} + \frac{1}{s^2(s-1)} \\
&= \frac{1}{s-1} + \frac{1}{s} + \frac{-1}{s-1} + \frac{-1}{s^2} + \frac{-1}{s} + \frac{1}{s-1} \\
&= \frac{1}{s-1} - \frac{1}{s^2}, \\
x(t) &= -e^{t}u(-t) - tu(t)
\end{aligned}$$

(g)
$$\begin{aligned}
X(s) &= \frac{s^2-s+1}{(s+1)^2}, \qquad -1 < Re\{s\} \\
&= \frac{(s+1)^2-3s}{(s+1)^2} = 1 - \frac{3s}{(s+1)^2} \\
&= 1 - \frac{3(s+1)}{(s+1)^2} + \frac{3}{(s+1)^2}, \\
x(t) &= \delta(t) - 3e^{-t}u(t) + 3te^{-t}u(t)
\end{aligned}$$

(h) $X(s) = \dfrac{s+1}{(s+1)^2+4}$

Consider

$$Y(s) = \frac{s}{s^2+4} \rightarrow y(t) = \cos(2t)u(t) \qquad \text{from part (c)}$$

Now

$$f(t)e^{-at} \overset{\mathcal{L}}{\longleftrightarrow} F(s+a),$$

so

$$x(t) = e^{-t}\cos(2t)u(t)$$

S20.8

The Laplace transform of an impulse $a\delta(t)$ is a.Therefore, if we expand a rational Laplace transform by dividing the denominator into the numerator, we require a *constant* term in the expansion. This will occur only if the numerator has order greater than or equal to the order of the denominator. Therefore, a necessary condition on the number of zeros is that it be greater than or equal to the number of poles.

This is only a necessary and not a sufficient condition as it is possible to construct a rational Laplace transform that has a numerator order greater than the denominator order and that does not yield a constant term in the expansion. For example,

$$X(s) = \frac{s^2 + 1}{s} = s + \frac{1}{s},$$

which does not have a constant term. Therefore a *necessary* condition is that the number of zeros equal or exceed the number of poles.

S20.9

(a) $x(t) = e^{-at}u(t), \qquad a < 0,$

$$X(s) = \frac{1}{s + a},$$

and the ROC is shown in Figure S20.9-1.

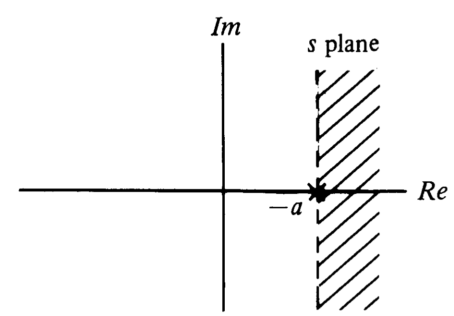

Figure S20.9-1

(b) $x(t) = -e^{at}u(-t), \qquad a > 0,$

$$X(s) = \frac{1}{s - a},$$

and the ROC is shown in Figure S20.9-2.

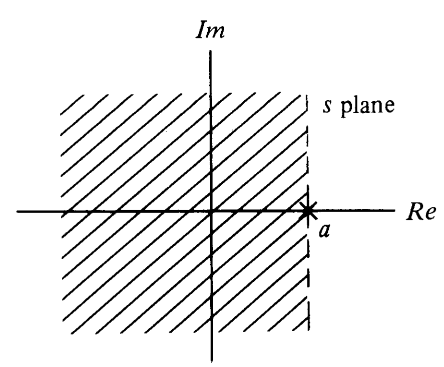

Figure S20.9-2

(c) $x(t) = e^{at}u(t), \qquad a > 0,$

$$X(s) = \frac{1}{s - a},$$

and the ROC is shown in Figure S20.9-3.

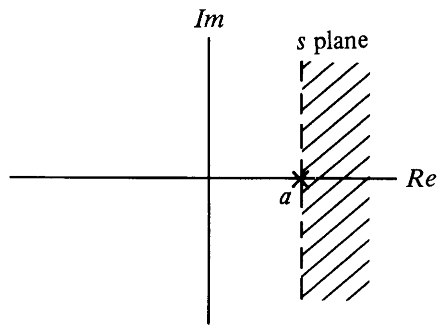

Figure S20.9-3

(d) $x(t) = e^{-a|t|}, \qquad a > 0,$

$$= e^{-at}u(t) + e^{at}u(-t),$$

$$X(s) = \frac{1}{s + a} + \frac{-1}{s - a},$$

and the ROC is shown in Figure S20.9-4.

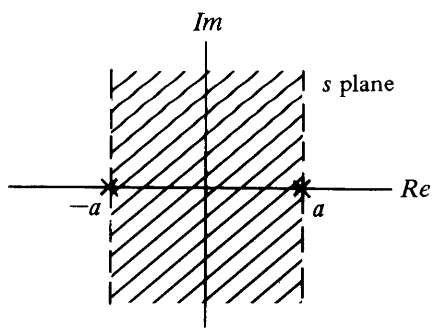

Figure S20.9-4

(e) $x(t) = u(t),$

$$X(s) = \int_0^{\infty} e^{-st}\, dt = \frac{1}{s},$$

and the ROC is shown in Figure S20.9-5.

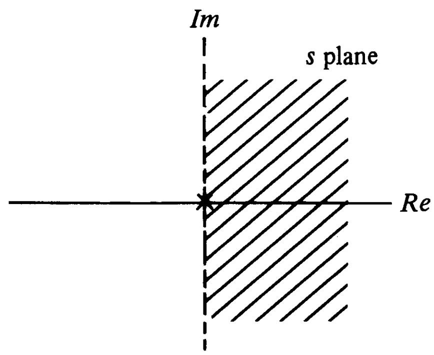

Figure S20.9-5

(f) $x(t) = \delta(t - t_0)$,

$$X(s) = \int_{-\infty}^{\infty} \delta(t - t_0)e^{-st}\,dt = e^{-st_0},$$

and the ROC is the entire s plane.

(g) $x(t) = \sum_{k=0}^{\infty} a^k \delta(t - kT)$,

$$X(s) = \sum_{k=0}^{\infty} a^k \int_{-\infty}^{\infty} \delta(t - kT)e^{-st}\,dt$$

$$= \sum_{k=0}^{\infty} a^k e^{-skT} = \frac{1}{1 - ae^{-sT}},$$

with ROC such that $|ae^{-sT}| < 1$. Now

$$a^2e^{-2sT} < 1 \to 2 \log a - 2sT < 0 \to s > \frac{1}{T} \log a$$

(h) $x(t) = \cos(\omega_0 t + b)u(t)$

Using the identity

$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$

we have that

$$x(t) = \cos b \cos(\omega_0 t)u(t) - \sin b \sin(\omega_0 t)u(t)$$

Using linearity and the transform pairs

$$\cos(\omega_0 t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2 + \omega_0^2},$$

$$\sin(\omega_0 t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{\omega_0}{s^2 + \omega_0^2},$$

we have

$$X(s) = \cos b \frac{s}{s^2 + \omega_0^2} - \sin b \frac{\omega_0}{s^2 + \omega_0^2},$$

$$X(s) = \cos b \frac{[s - (\tan b)\omega_0]}{s^2 + \omega_0^2},$$

and the ROC is shown in Figure S20.9-6.

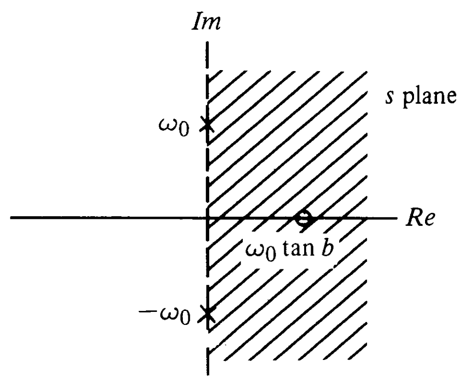

Figure S20.9-6

(i) Consider

$$\begin{aligned} x_1(t) &= \sin(\omega_0 t + b)u(t) \\ &= (\sin \omega_0 t \cos b + \cos \omega_0 t \sin b)u(t) \end{aligned}$$

Using linearity and the preceding $\sin \omega_0 t$, $\cos \omega_0 t$ pairs, we have

$$X_1(s) = \cos b \frac{\omega_0}{s^2 + \omega_0^2} + \sin b \frac{s}{s^2 + \omega_0^2},$$

$$X_1(s) = \sin b \frac{[s + (\cot b)\omega_0]}{s^2 + \omega_0^2}$$

Using the property that

$$f(t)e^{-at} \overset{\mathcal{L}}{\longleftrightarrow} F(s + a),$$

we have

$$X(s) = \sin b \frac{[s + a + (\cot b)\omega_0]}{(s + a)^2 + \omega_0^2},$$

with the ROC as given in Figure S20.9-7.

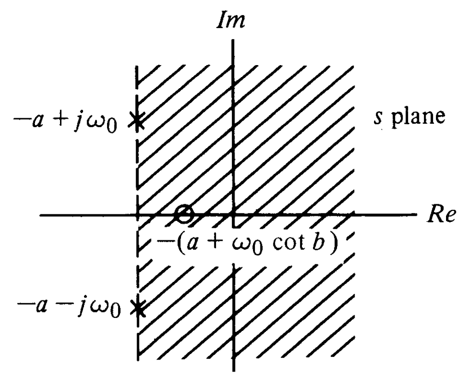

Figure S20.9-7

S20.10

(a) $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$

Consider

$$X_1(s) = \int_{-\infty}^{\infty} x(-t)e^{-st}\,dt$$

Letting $t = -t'$, we have

$$\begin{aligned} X_1(s) &= \int_{-\infty}^{\infty} x(t')e^{st'}\,dt' \\ &= X(-s), \end{aligned}$$

but $X_1(s) = X(s)$ since $x(t) = x(-t)$. Therefore, $X(s) = X(-s)$.

(b) $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$

Consider

$$\begin{aligned} X_1(s) &= \int_{-\infty}^{\infty} -x(-t)e^{-st}\,dt, \\ X_1(s) &= \int_{-\infty}^{\infty} -x(t')e^{st'}\,dt' \\ &= -X(s), \end{aligned}$$

but $X_1(s) = X(s)$ since $x(t) = -x(-t)$. Therefore, $X(s) = -X(-s)$.

(c) We note that if $X(s)$ has poles, then it must be two-sided in order for $x(t) = x(-t)$.

(i) $$X(s) = \frac{Ks}{(s+1)(s-1)},$$

$$X(-s) = \frac{-Ks}{(-s+1)(-s-1)} = \frac{-Ks}{(s-1)(s+1)} \neq X(s),$$

so $x(t) \neq x(-t)$.

(ii) $$X(s) = \frac{K(s+1)(s-1)}{s},$$
$$X(-s) = \frac{K(-s+1)(-s-1)}{-s} \neq X(s)$$

Also, this pole pattern cannot have a two-sided ROC.

(iii) $$X(s) = \frac{K(s+j)(s-j)}{(s+1)(s-1)},$$
$$X(-s) = \frac{K(-s+j)(-s-j)}{(-s+1)(-s-1)} = \frac{K(s-j)(s+j)}{(s-1)(s+1)} = X(s),$$

so this can correspond to an even $x(t)$. The corresponding ROC must be two-sided, as shown in Figure S20.10-1.

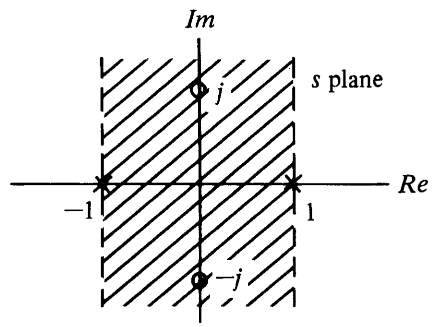

Figure S20.10-1

(iv) This does not have any possible two-sided ROCs.

(d) We see from the results in part (c)(i) that $X(s) = -X(-s)$, so the result in part (c)(i) corresponds to an odd $x(t)$ with an ROC as given in Figure S20.10-2.

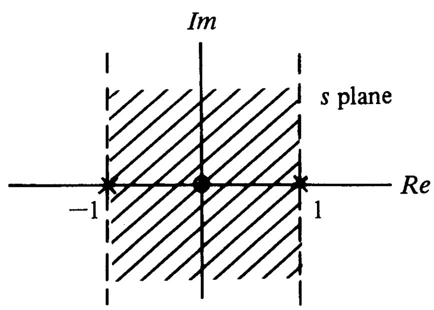

Figure S20.10-2

Parts (c)(ii) and (c)(iv) do not have any possible two-sided ROCs. Part (c)(iii) is even, as previously shown, and therefore cannot be odd.