6

Systems Represented by Differential and Difference Equations

An important class of linear, time-invariant systems consists of systems represented by linear constant-coefficient differential equations in continuous time and linear constant-coefficient difference equations in discrete time. Continuous-time linear, time-invariant systems that satisfy differential equations are very common; they include electrical circuits composed of resistors, inductors, and capacitors and mechanical systems composed of masses, springs, and dashpots. In discrete time a wide variety of data filtering, time series analysis, and digital filtering systems and algorithms are described by difference equations.

In this lecture, we review the time-domain solution for linear constant-coefficient differential equations and show how the same basic strategy applies to difference equations. While this review is presented somewhat quickly, it is assumed that you have had some prior exposure to differential equations and their time-domain solution, perhaps in the context of circuits or mechanical systems. In any case, in Lecture 9 after we have developed the Fourier transform, we will see some more efficient (at least mathematically) ways of obtaining the solution.

In considering the time-domain solution to linear constant-coefficient differential and difference equations, we should recognize a number of important features. Foremost is the fact that the differential or difference equation by itself specifies a family of responses only for a given input $x(t)$. In particular we can always add to any solution another solution that satisfies the homogeneous equation corresponding to $x(t)$ or $x(n)$ being zero. Thus, for unique specification of a system, in addition to the differential or difference equation some auxiliary conditions (for example, a set of initial conditions) are needed that will specify the arbitrary constants present in the homogeneous solution.

In Lecture 5 we showed that a linear, time-invariant system has the property that if the input is zero for all time, then the output will also be zero for all time. Consequently, a linear, time-invariant system specified by a linear constant-coefficient differential or difference equation must have its auxiliary conditions consistent with that property. If, in fact, the system is causal in addition to being linear and time-invariant, then the auxiliary conditions will correspond to the requirement of initial rest; that is, if the input is zero prior to some time, then the output must be zero at least until the same time. In the context of *RLC* circuits, for example, this would correspond to an assumption of no initial capacitor voltages or inductor currents prior to the time at which the input becomes nonzero.

An important distinction between linear constant-coefficient differential equations associated with continuous-time systems and linear constant-coefficient difference equations associated with discrete-time systems is that for causal systems the difference equation can be reformulated as an explicit relationship that states how successive values of the output can be computed from previously computed output values and the input. This recursive procedure for calculating the response of a difference equation is extremely useful in implementing causal systems. However, it is important to recognize that either in algebraic terms or in terms of block diagrams, a difference equation can often be rearranged into other forms leading to implementations that may have particular advantages. For example, as illustrated in the lecture, the most direct representation of a difference equation in terms of a block diagram or algorithm is often not the most efficient. Since the order in which linear, time-invariant systems are cascaded is not important to the overall input-output response, the most direct representation can be rearranged so that its implementation requires significantly less memory or, equivalently, delay registers. In addition, there are many other rearrangements, each having particular advantages and disadvantages. Similar kinds of rearrangements of the block diagrams also apply to the block diagram realizations of linear constant-coefficient differential equations for continuous-time systems.

Suggested Reading

Section 3.5, Systems Described by Differential and Difference Equations, pages 101–111

Section 3.6, Block-Diagram Representations of LTI Systems Described by Differential and Difference Equations, pages 111–119

LINEAR CONSTANT-COEFFICIENT DIFFERENTIAL EQUATION

N^{th} Order: $$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

LINEAR CONSTANT-COEFFICIENT DIFFERENCE EQUATION

N^{th} Order: $$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$$

TRANSPARENCY 6.1
*N*th-order linear constant-coefficient differential and difference equations.

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \qquad (1)$$

$$\sum_{k=0}^{N} a_k \frac{d^k y_h(t)}{dt^k} = 0 \qquad \text{(Homogeneous Equation)} \qquad (2)$$

Given $x(t)$, if $y_p(t)$ satisfies (1) then so does

$y_p(t) + y_h(t)$ where $y_h(t)$ satisfies (2)

$y_p(t) \triangleq$ Particular solution

$y_h(t) \triangleq$ Homogeneous solution

TRANSPARENCY 6.2
Family of solutions for a linear constant-coefficient differential equation.

TRANSPARENCY 6.3
Form of the homogeneous solution.

HOMOGENEOUS SOLUTION

$$\sum_{k=0}^{N} a_k \frac{d^k y_h(t)}{dt^k} = 0$$

"guess" solution of the form

$$y_h(t) = Ae^{st}$$

$$\sum_{k=0}^{N} a_k A s^k e^{st} = 0$$

$$\sum_{k=0}^{N} a_k s^k = 0 \quad \text{N roots } s_i \quad i = 1, \ldots, N$$

$$y_h(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + \ldots + A_N e^{s_N t}$$

TRANSPARENCY 6.4
Requirements on the auxiliary conditions for a linear constant-coefficient differential equation to correspond to a linear system or to a causal LTI system.

Need N auxiliary conditions, e.g.

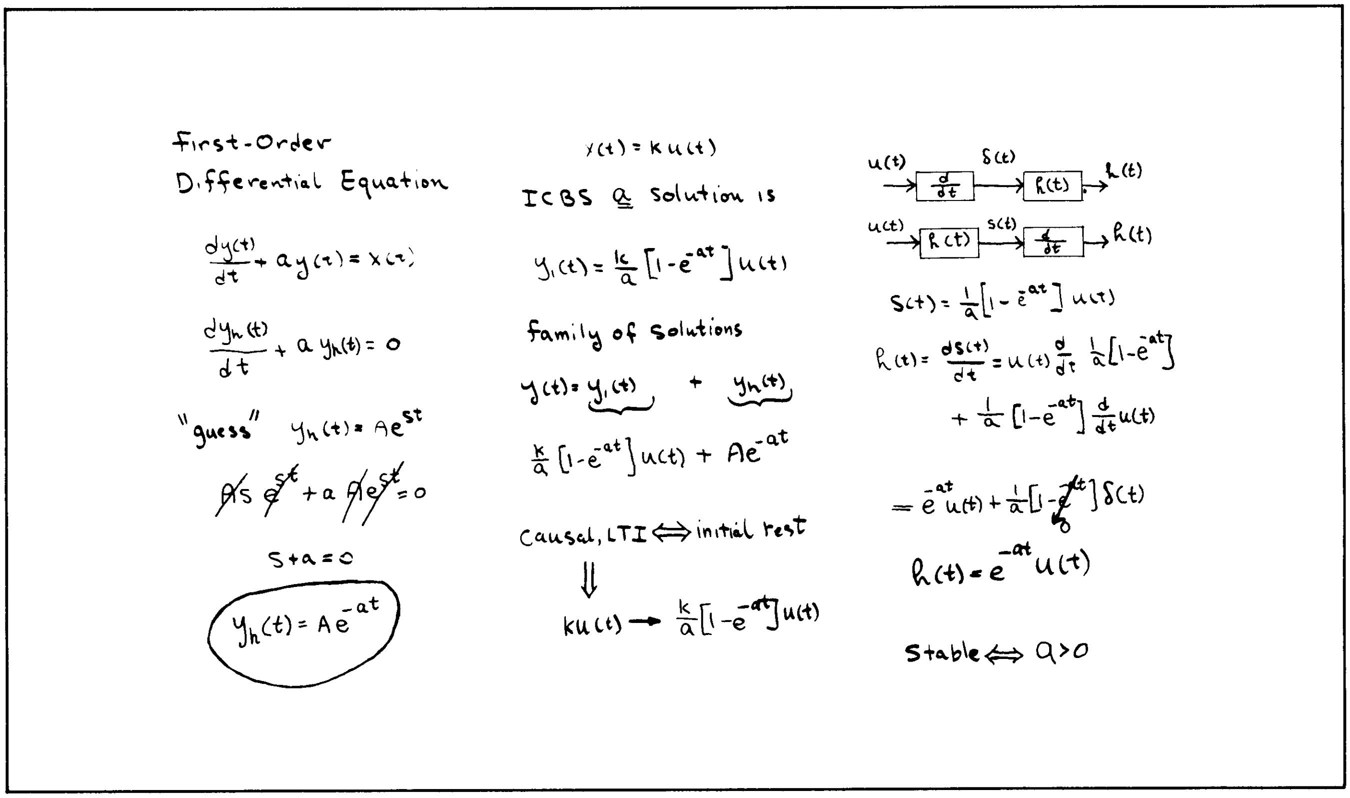
$$y(t), \frac{dy(t)}{dt}, \ldots, \frac{d^{N-1}y(t)}{dt^{N-1}} \quad \text{at } t = t_o$$

- Linear system <=> auxiliary conditions = 0
- Causal, LTI <=> initial rest:

  if $x(t) = 0 \quad t < t_o$

  then $y(t) = 0 \quad t < t_o$

MARKERBOARD 6.1



$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k] \qquad (1)$$

$$\sum_{k=0}^{N} a_k\, y_h[n-k] = 0 \quad \text{(Homogeneous Equation)} \qquad (2)$$

If $y_p[n]$ satisfies (1) then so does

$y_p[n] + y_h[n]$ where $y_h[n]$ satisfies (2)

$y_p[n] \triangleq$ Particular solution

$y_h[n] \triangleq$ Homogeneous solution

TRANSPARENCY 6.5
Family of solutions for a *N*th-order linear constant-coefficient difference equation.

TRANSPARENCY 6.6
Form of the homogeneous solution.

HOMOGENEOUS SOLUTION

$$\sum_{k=0}^{N} a_k \, y_h[n-k] = 0$$

"guess" solution of the form

$$y_h[n] = Az^n$$

$$\sum_{k=0}^{N} a_k \, A z^n z^{-k} = 0$$

$$\sum_{k=0}^{N} a_k \, z^{-k} = 0 \quad \text{N roots } z_1, z_2, \ldots, z_N$$

$$y_h[n] = A_1 z_1^n + \ldots + A_N z_N^n$$

TRANSPARENCY 6.7
Requirements on the auxiliary conditions for a linear constant-coefficient difference equation to correspond to a linear system or to a causal LTI system.

$$\sum_{k=0}^{N} a_k \, y[n-k] = \sum_{k=0}^{M} b_k \, x[n-k]$$

$$y[n] = \underbrace{A_1 z_1^n + \ldots + A_N z_N^n}_{y_h[n]} + y_p[n]$$

Need N auxiliary conditions, e.g.

$$y[n_o], y[n_o - 1], \ldots, y[n_o - N + 1]$$

- Linear system <=> auxiliary conditions = 0
- Causal, LTI <=> initial rest:
  if $x[n] = 0 \;\; n < n_o$
  then $y[n] = 0 \;\; n < n_o$

EXPLICIT SOLUTION TO DIFFERENCE EQUATION

Assume causality

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$$

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k\, x[n-k] - \sum_{k=1}^{N} a_k\, y[n-k]\right\}$$

Given $x[n]$ and $y[n_0-1], y[n_0-2], \ldots y[n_0-N]$

– can calculate $y[n_0]$

– then $y[n_0+1]$ etc.

TRANSPARENCY 6.8
Recursive causal solution to a linear constant-coefficient difference equation.

MARKERBOARD 6.2

First-Order Difference Equation

$y[n] - a y[n-1] = x[n]$

causal, LTI ⟺ Initial Rest

$y[n] = x[n] + a\, y[n-1]$

$x[n] = \delta[n]$

$h[n] = \delta[n] + a\, h[n-1]$

$h[n] = 0 \quad n<0$

$h[0] = 1 \qquad a^n u[n]$

$h[1] = a$

$h[2] = a^2 \qquad$ stable ⟺ $|a|<1$

$\vdots$

causal, LTI

$\delta[n] \rightarrow a^n u[n]$

family of Solutions

$\delta[n] \rightarrow a^n u[n] + y_h[n]$

$y_h[n] - a y_h[n-1] = 0$

$y_h[n] = A z^n$

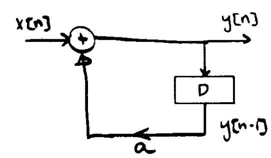
$A z^n - a A z^{n-1} = 0$

$1 - a z^{-1} = 0$

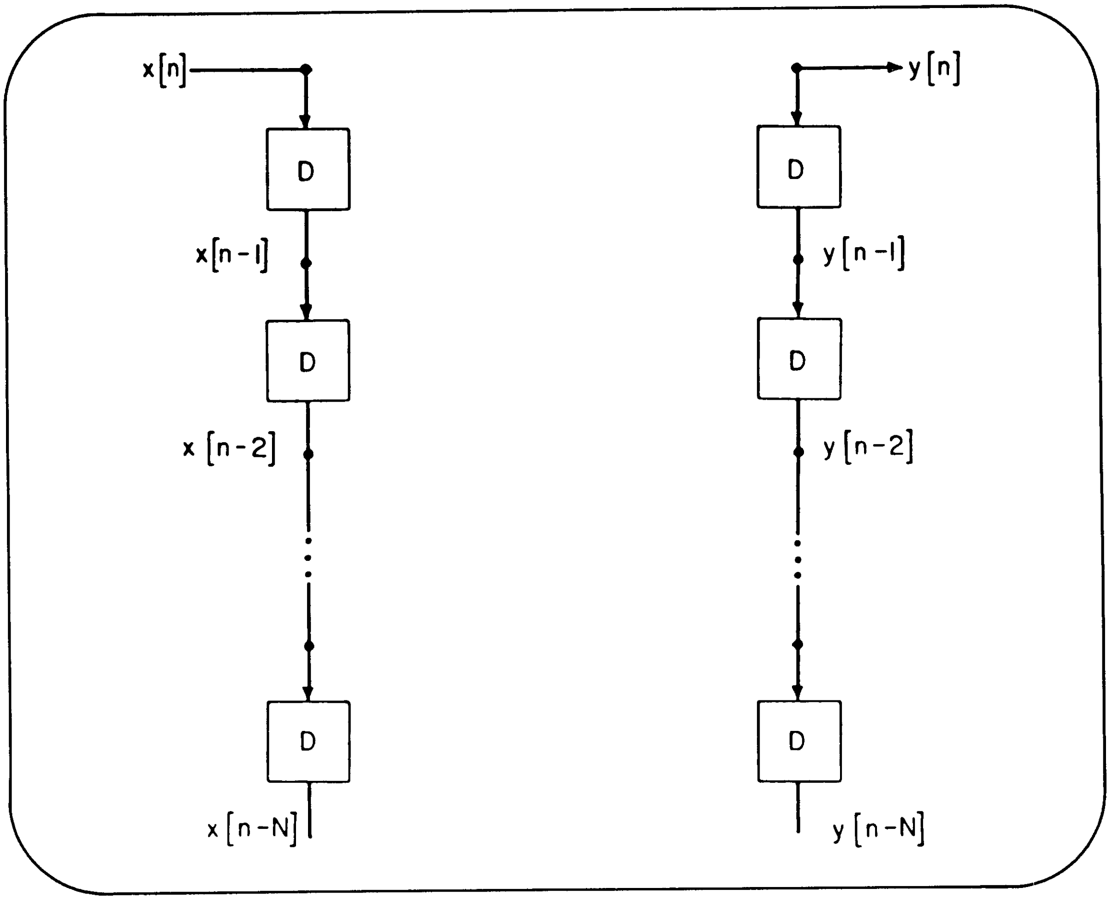
$y_h[n] = A a^n$

$\delta[n] \rightarrow a^n u[n] + A a^n$

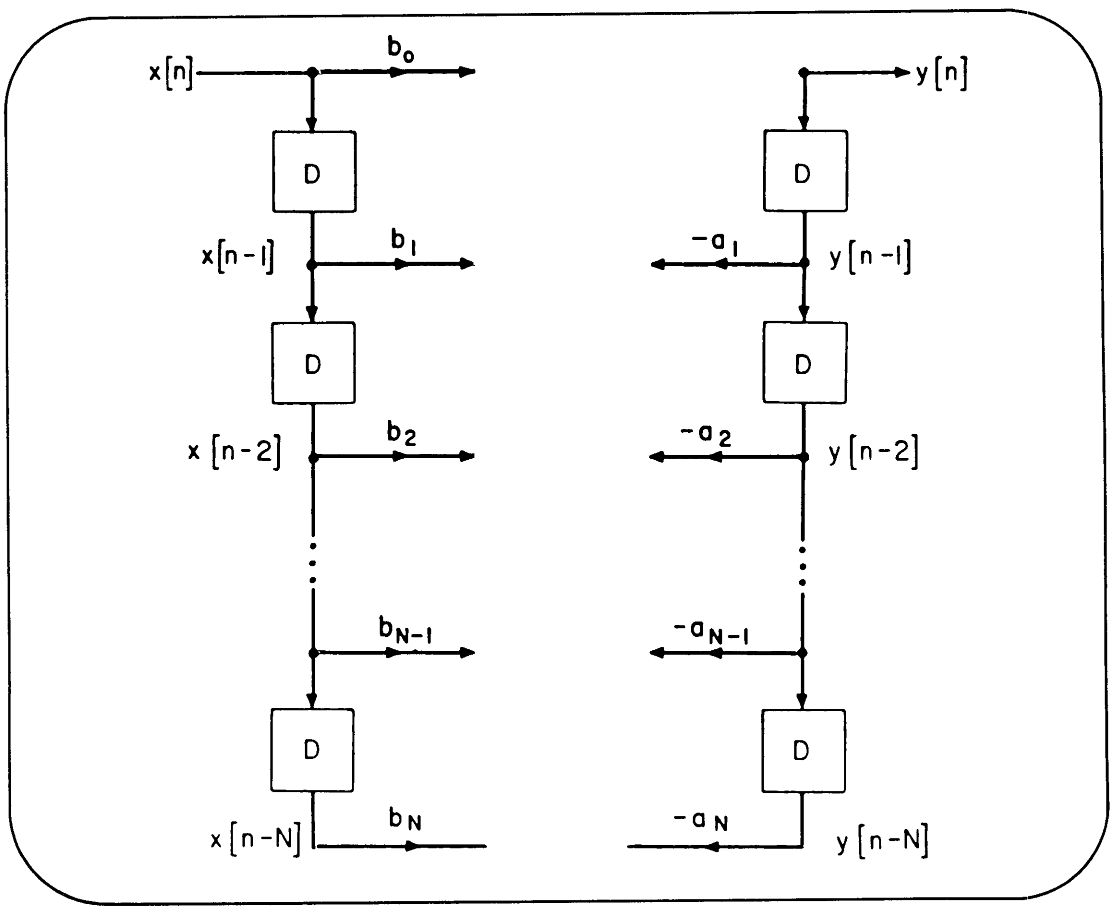
$y[n] = x[n] + a\, y[n-1]$

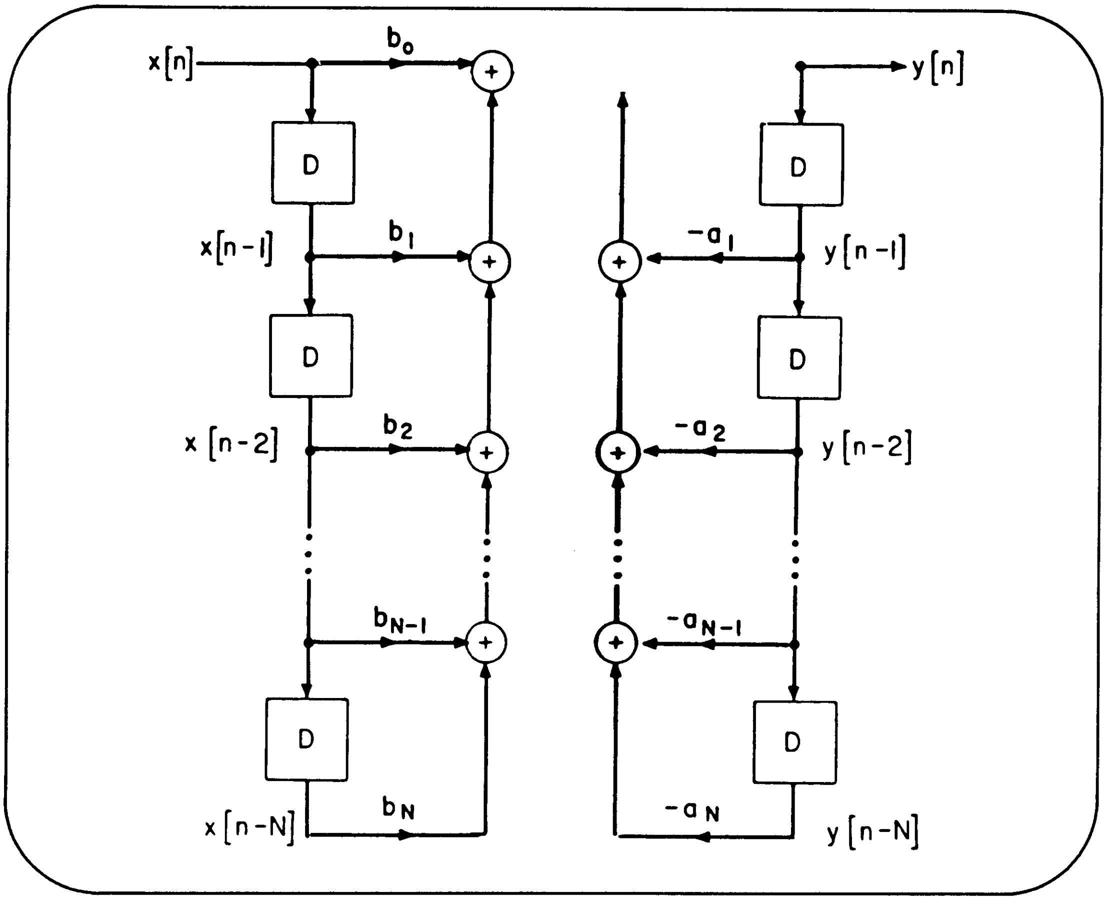


TRANSPARENCY 6.9
Development of the block diagram representation of the recursive solution. Delays associated with the input and output.

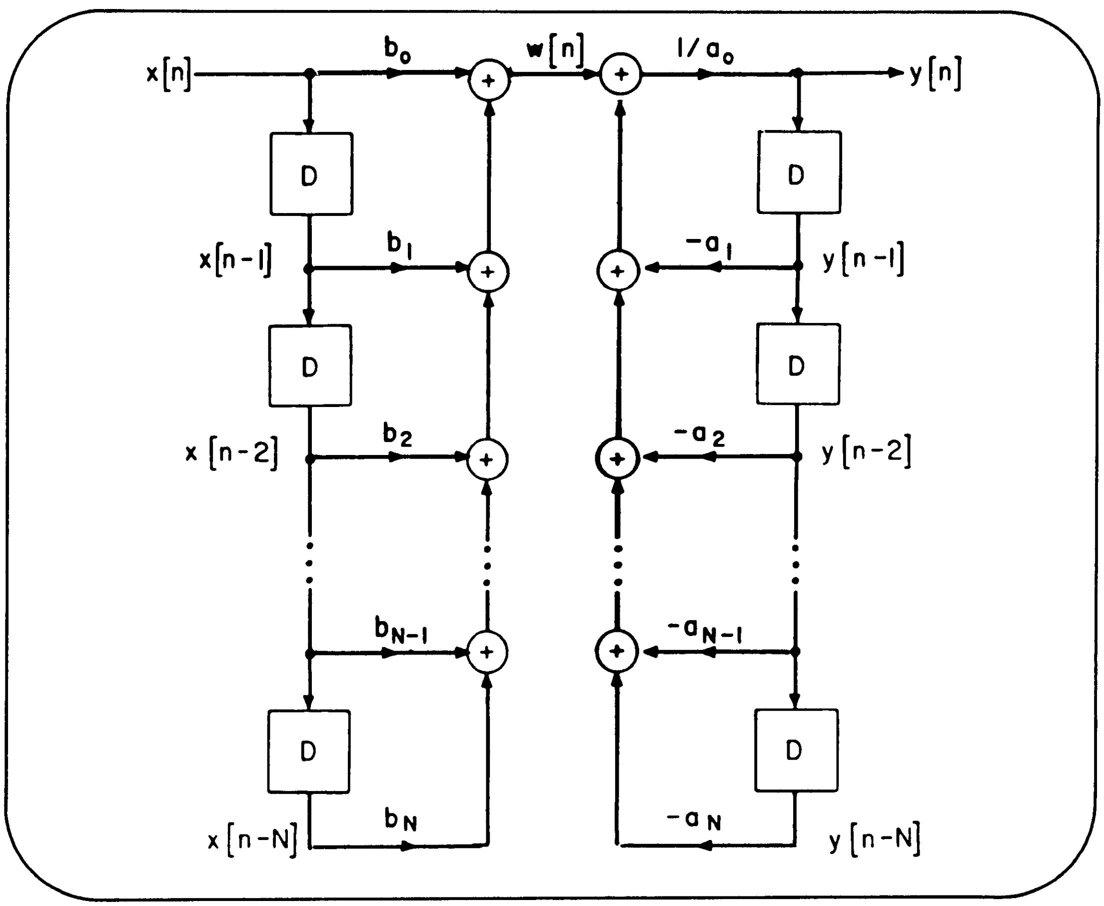


TRANSPARENCY 6.10
Incorporation of the coefficient multiplication.

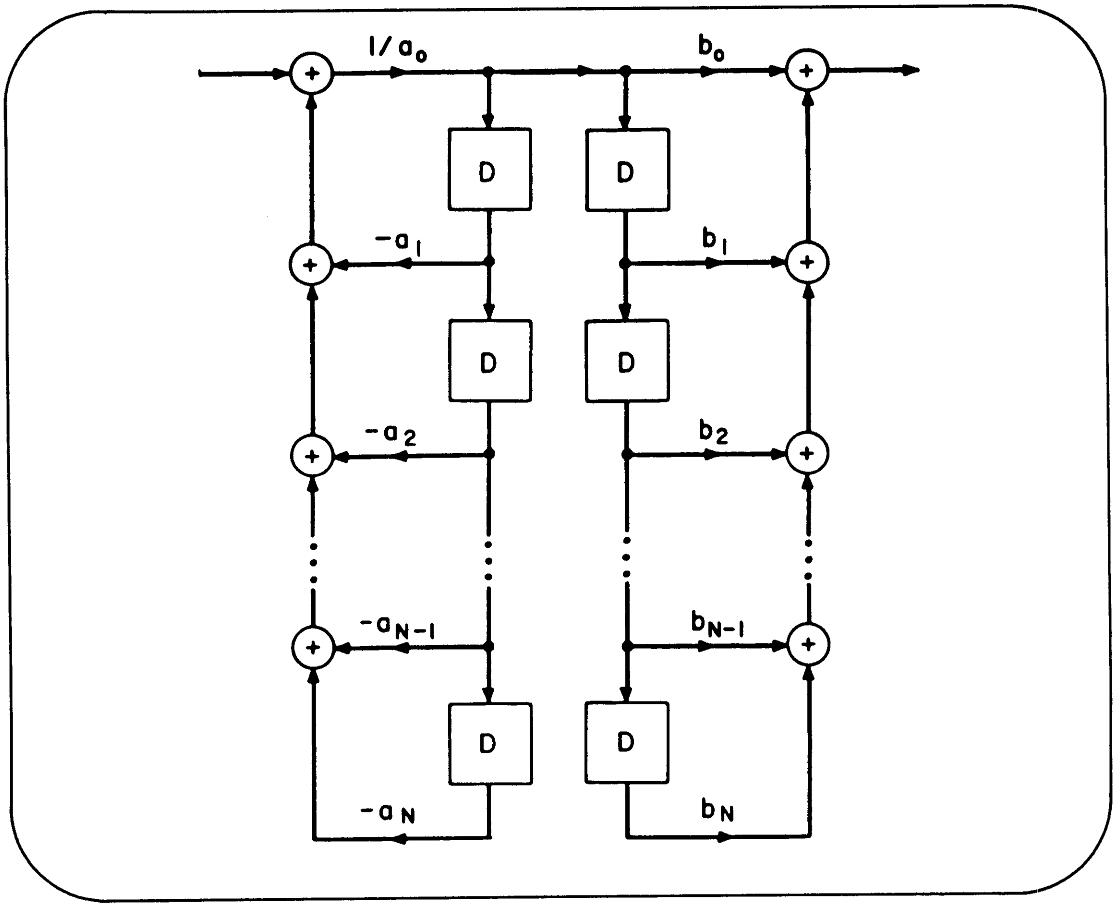




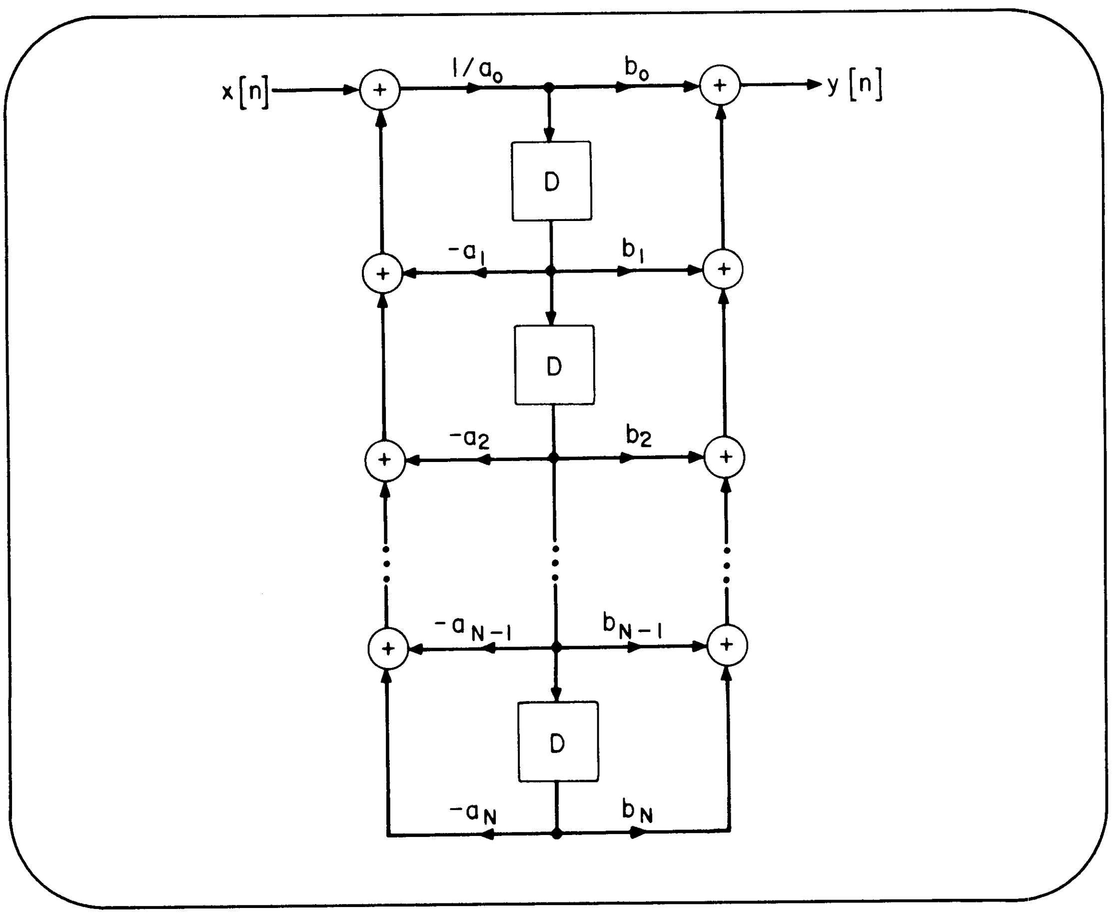
TRANSPARENCY 6.11
Forming the sums of the weighted, delayed sequences.



TRANSPARENCY 6.12
Complete block diagram. This form is often referred to as the direct form I implementation.

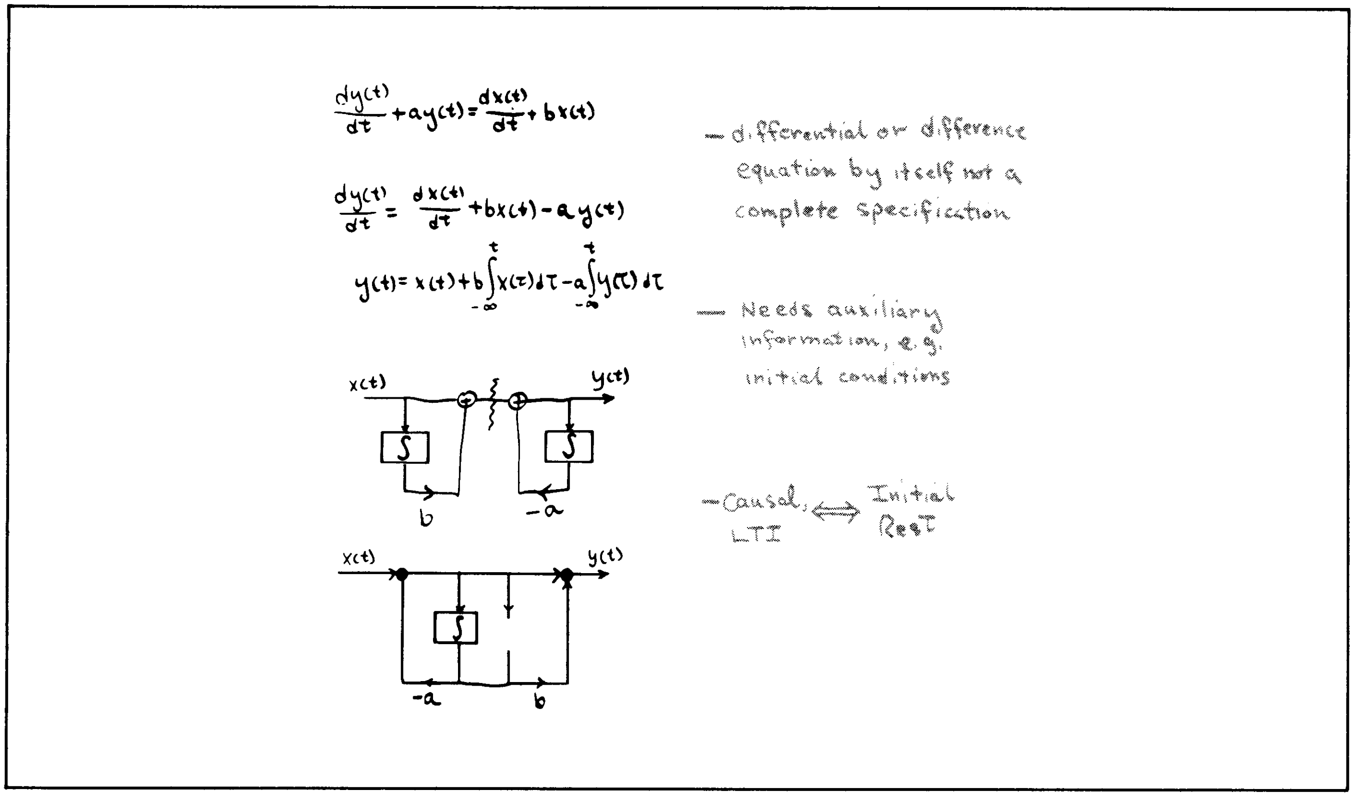


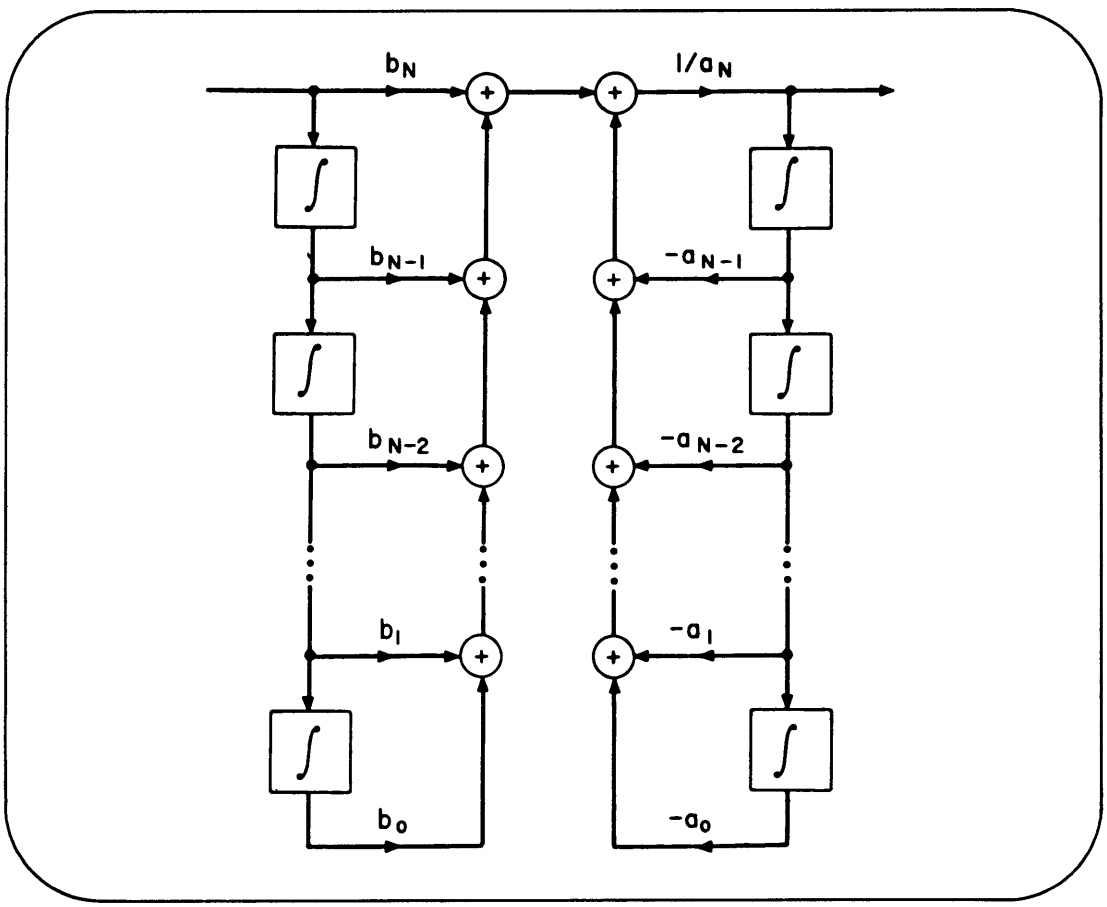
TRANSPARENCY 6.13
Interchanging the order in which the two segments of the block diagram of Transparency 6.12 are cascaded.



TRANSPARENCY 6.14
Result of combining the two chains of delays in Transparency 6.13. This form is often referred to as the direct form II implementation.

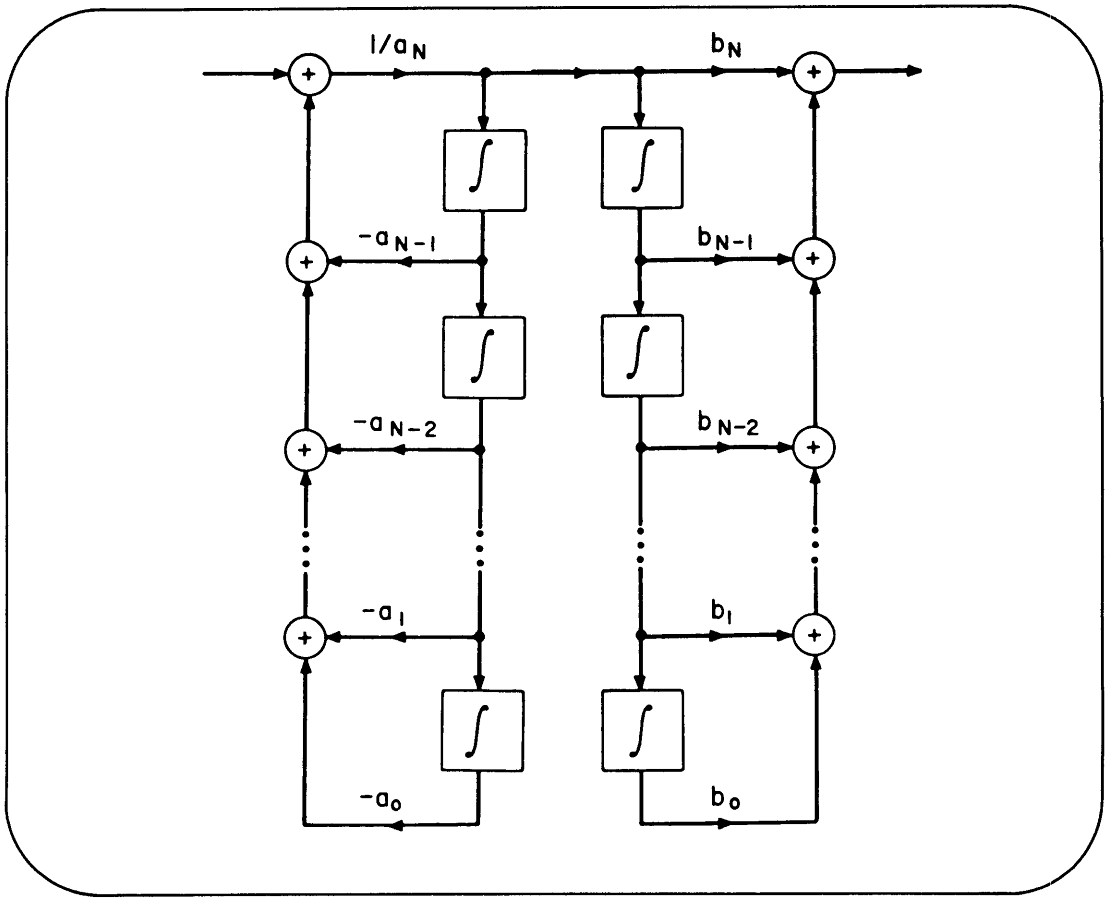
MARKERBOARD
6.3 (a)



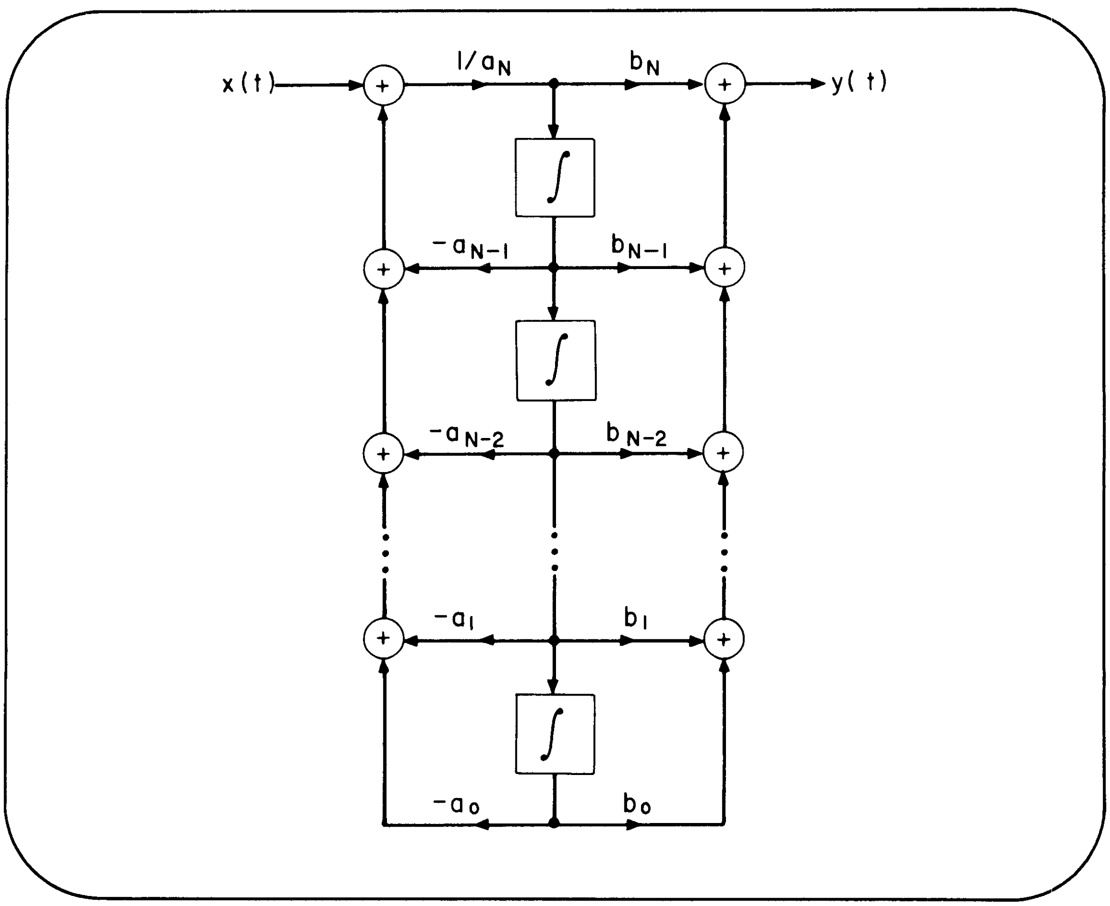


TRANSPARENCY 6.15
Direct form I block diagram representation of a linear constant-coefficient differential equation.

TRANSPARENCY 6.16
Resulting block diagram when the order of the two subsystems in Transparency 6.15 is reversed.



TRANSPARENCY 6.17
The result of combining the two chains of integrators in Transparency 6.16. This form is often referred to as the direct form II.

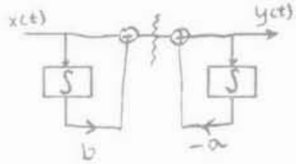


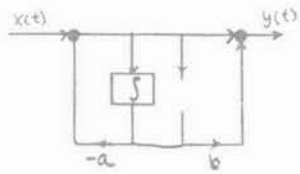
MARKERBOARD
6.3 (b)

$$\frac{dy(t)}{dt} + ay(t) = \frac{dx(t)}{dt} + bx(t)$$

$$\frac{dy(t)}{dt} = \frac{dx(t)}{dt} + bx(t) - ay(t)$$

$$y(t) = x(t) + b\int_{-\infty}^{t} x(\tau)d\tau - a\int_{-\infty}^{t} y(\tau)d\tau$$





— differential or difference equation by itself not a complete specification

— Needs auxiliary information, e.g. initial conditions

— Causal, LTI $\Longleftrightarrow$ Initial Rest